LINEAR TRANSFORMATION ON MATRICES: THE INVARIANCE OF A CLASS OF GENERAL MATRIX FUNCTIONS. II

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1. Introduction. Let $M_m(F)$ be the vector space of m-square matrices $X = (x_{ij}), i, j = 1, \ldots, m$ over a field F; f a function on $M_m(F)$ to some set R. It is of interest to determine the structure of the linear maps $T: M_m(F) \rightarrow M_m(F)$ that preserve the values of the function f (i.e., $f(T(x)) = f(x)$ for all X). For example, if we take $f(x)$ to be the rank of X, we are asking for a determination of the types of linear operations on matrices that preserve rank **(6)**. Other classical invariants that may be taken for f are the determinant, the set of eigenvalues, and the rth elementary symmetric function of the eigenvalues. Dieudonné **(2)**, Hua **(3)**, Marcus **(4; 5; 6)** and others have conducted extensive research in this area. A class of matrix functions that have recently aroused considerable interest is the generalized matrix functions in the sense of I. Schur **(7)**. These are defined as follows: let S_m be the full symmetric group of degree m and let λ be a function on S_m with values in F. The matrix function associated with λ is defined by

$$d_\lambda(X) = \sum_{\sigma \in S_m} \lambda(\sigma) \prod_{i=1}^{m} x_{i\sigma(i)}.$$

These functions clearly include the classical determinant, permanent **(5)**, and immanent functions **(8)**.

Let G be a subgroup of S_m and λ a non-trivial homomorphism of G into the multiplicative group of F (i.e., λ is a character of degree one on G). If we extend λ to all of S_m by defining $\lambda(\sigma) = 0$ if $\sigma \notin G$, then the matrix function associated with λ will be denoted by G_λ. Our main result is a characterization of all linear maps $T: M_m(F) \rightarrow M_m(F)$ that satisfy:

$$G_\lambda(T(X)) = G_\lambda(X) \quad \text{for all } X, \tag{1}$$

where G is a doubly transitive or regular proper subgroup of S_m.

If $G = S_m$ and λ is a character of degree one, then the function G_λ is either the determinant or permanent. The structure of all linear maps satisfying (1) in these cases has been obtained by Marcus and May **(5)** and Marcus and Moyls **(6)**. If G is transitive and cyclic and λ is any function such that $\lambda(\sigma) = 0$ if $\sigma \notin G$, then I **(1)** have characterized the linear maps that preserve the values of the matrix function associated with λ.

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2. Definitions and main results. Throughout the remainder of this paper we suppose that the field F contains more than m elements, where m is the size of the matrices under consideration, and that m is greater than two.

We shall also assume that G is either a proper doubly transitive or regular subgroup of S_m. We denote by $G(i, j)$ the set of $\sigma \in G$ such that $\sigma(i) = j$. If G is doubly transitive, then clearly if $i \neq p, j \neq q$, there exists $\sigma \in G(i, j)$ such that $\sigma(p) = q$. If G is regular, then $G(i, j)$ consists of only one permutation for each i and j; hence G is transitive and is of order m.

Definition. A subspace A of $M_m(F)$ is a 0-subspace for G_λ if $\dim A = m^2 - m$ and if $X \in A$ implies $G_\lambda(X) = 0$.

The following characterizations of 0-subspaces turn out to be very useful in the determination of all linear maps of $M_m(F)$ into itself satisfying (1).

PROPOSITION 1. *Let $G = \{\sigma_1, \ldots, \sigma_m\}$ be a regular subgroup of S_m. A subspace A is a 0-subspace for G_λ if and only if there exist m distinct pairs of integers $(i_1, j_1), \ldots, (i_m, j_m)$, $1 \leqslant i_t, j_t \leqslant m$, such that $\sigma_k(i_k) = j_k$ and if $X \in A$, $X_{i_t j_t} = 0$, $t = 1, \ldots, m$.*

PROPOSITION 2. *Let G be a doubly transitive proper subgroup of S_m. A subspace A is a 0-subspace for G_λ if and only if there exists an integer i, $1 \leqslant i \leqslant m$, such that A consists either of all matrices with row i zero or of all matrices with column i zero.*

If $\sigma \in S_m$, then the permutation matrix corresponding to σ, $P(\sigma)$, is defined by $P(\sigma)_{ij} = \delta_{i\sigma(j)}$ where $\delta_{st} = 1$ if $s = t$ and 0 otherwise. If $G = \{\sigma_1, \ldots, \sigma_m\}$ is regular and $X \in M_m(F)$, then it is clear that we may uniquely write

$$X = \sum_{i=1}^{m} X_i P(\sigma_i)$$

where the X_i are diagonal matrices. We use this representation to define the following type of maps of $M_m(F)$ into itself: If $\mu_1, \ldots, \mu_m, \alpha \in S_m$, then

$$S(\mu_1, \ldots, \mu_m, \alpha)(X) = \sum_{i=1}^{m} \lambda(\sigma_{\alpha(i)}) YP(\sigma_{\alpha(i)})$$

where if $X_i = \operatorname{diag}(x_{i1}, \ldots, x_{im})$, then

$$Y_i = \operatorname{diag}(x_{i\mu(1)}, \ldots, x_{i\mu(m)}), \qquad \mu = \mu_i.$$

If B and C belong to $M_m(F)$, then the Hadamard product $D = B * C$ is defined by $d_{ij} = b_{ij} c_{ij}$. If we denote by X' the transpose of the matrix X, we can now state our main results.

THEOREM 1. *Let G be a regular subgroup of S_m and λ a character of degree one on G. A linear map T of $M_m(F)$ into itself satisfies*

$$G_\lambda(T(X)) = G_\lambda(X) \quad \textit{for all } X$$

if and only if there exists a matrix C belonging to $M_m(F)$ and a map

$$K = S(\mu_1, \ldots, \mu_m, \alpha)$$

such that for each σ_t in G

$$\prod_{i=1}^{m} c_{i\sigma_t(i)} = \lambda(\sigma_t \sigma_{\alpha(t)}^{-1}) \quad \text{and} \quad T(X) = C * K(X).$$

THEOREM 2. *Let G be a doubly transitive proper subgroup of S_m and λ a character of degree one on G. A linear transformation T of $M_m(F)$ into itself satisfies*

$$G_\lambda(T(X)) = G_\lambda(X) \quad \text{for all } X$$

if and only if there exist permutations μ, τ in S_m and a matrix C in $M_m(F)$ such that $\mu\tau$ belongs to G; and either

$$\text{(a)} \qquad T(X) = C * P(\mu)XP(\tau) \quad \text{with} \prod_{i=1}^{m} c_{i\sigma(i)} = \lambda(\sigma\tau^{-1}\mu)$$

for all σ in G, or

$$\text{(b)} \qquad T(X) = C * P(\tau)X'P(\mu) \quad \text{with} \prod_{i=1}^{m} c_{i\sigma(i)} = \lambda(\sigma^{-2}\mu^{-1}\tau)$$

for all $\sigma \in G$.

When $G = S_m$ and λ is identically equal to one, the matrix function G_λ is the permanent and it is known **(5)** that the same result holds and that C is of rank one, so $C * X = DXL$ for suitable diagonal matrices D and L. This is not true in general, as the following example shows. Let G be the alternating group of degree four and suppose λ is identically equal to one. Let

$$C = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

Clearly the rank of C is greater than one, so we cannot have $C * X = DXL$ for any fixed diagonal matrices D and L. A direct computation shows that $G_\lambda(C * X) = G_\lambda(X)$ for all X.

When $G = S_m$ and $\lambda(\sigma) = 1$ or -1 according as σ is an even or odd permutation, then the matrix function G_λ is the determinant. Marcus and Moyls **(6)** have shown that in this case $\det T(X) = \det X$ for all X if and only if $T(X) = UXV$ or $UX'V$ for fixed non-singular U and V satisfying $\det UV = 1$.

If the group G is not transitive, then the transformations T satisfying (1) may be singular. If G is singly transitive but not regular or doubly transitive, then our techniques fail. In particular, the analogues of Propositions 1 and 2 do not hold. A counterexample may be found by examining the dihedral group of degree four.

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3. Proofs. Suppose A is a subspace of $M_m(F)$, $\dim A = m^2 - m$. By using the reduction of a basis for A to Hermite normal form we can assume that there exist m distinct pairs of integers $\{(i_1, j_1), \ldots, (i_m, j_m)\} = M$ such that the matrices

$$A_{ij} = E_{ij} + \sum_{t=1}^{m} c_t^{ij} E_{i_t j_t}, \qquad c_t^{ij} \in F,\ (i, j) \notin M,$$

form a basis for A. Here E_{ij} is the matrix with a one in the (i, j) position and zeros elsewhere.

If S is any finite set, $|S|$ will denote the number of elements in S.

The group G is transitive so $|G| = nm$ and $|G(i, j)| = n$ for some integer $n \geqslant 1$ **(9)**. If $\sigma \in S_m$, let $D(\sigma) = \{(i, \sigma(i)) : i = 1, \ldots, m\}$.

We now establish some lemmas that will be used to prove Propositions 1 and 2.

LEMMA 1. *If for some $\sigma \in G$, $|D(\sigma) \cap M| > 1$, then there exists $\tau \in G$ such that $|D(\tau) \cap M| = 0$.*

Proof. If $|D(\sigma) \cap M| > 1$, then for some $t \neq s$, $|G(i_s, j_s) \cap G(i_t, j_t)| \geqslant 1$. We know that $|G| = nm$ and $|G(i, j)| = n$; hence

$$\left| \bigcup_{k=1}^{m} G(i_k, j_k) \right| \leqslant \sum_{k=1}^{m} |G(i_k, j_k)| - |G(i_s, j_s) \cap G(i_t, j_t)|$$
$$\leqslant mn - 1 = |G| - 1.$$

Therefore there exist $\tau \in G$ such that

$$\sigma \notin \bigcup_{k=1}^{m} G(i_t, j_t)$$

and clearly τ has the desired properties.

We may assume that the pairs (i_t, j_t) of M are arranged so that if $c_r^{ij} = 0$ for i, j, then for all $s \geqslant r$, $c_s^{ij} = 0$ for all i, j. Let $n(A) = \max\{0, t : c_t^{ij} \neq 0$ for some $i, j\}$.

LEMMA 2. *If for some $\sigma \in G$, $|D(\sigma) \cap M| = 0$, then there exists a matrix $B \in A$ such that:*

(a) $B = P(\sigma) + cE_{i_1 j_1}$,

(b) $G_\lambda(B) \neq 0$.

Proof. If $n(A) = 0$, let

$$B = \sum_{i=1}^{m} A_{i\sigma(i)} = \sum_{i=1}^{m} E_{i\sigma(i)} = P(\sigma).$$

Clearly (a) is satisfied and since $G_\lambda(P(\sigma)) = \lambda(\sigma) \neq 0$, (b) is satisfied.

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If $n(A) = 1$, let

$$B = \sum_{i=1}^{m} A_{i\sigma(i)} = \sum_{i=1}^{m} E_{i\sigma(i)} + \sum_{i=1}^{m} c_1^{i\sigma(i)} E_{i_1 j_1}$$
$$= P(\sigma) + cE_{i_1 j_1}, \qquad c = \sum_{i=1}^{m} c_1^{i\sigma(i)}.$$

Clearly B satisfies (a). Now notice that if $\tau \neq \sigma$, then there exist p, q ($p \neq q$) such that $\sigma(p) \neq \tau(p)$ and $\sigma(q) \neq \tau(q)$. Therefore

$$G_\lambda(B) = \sum_{\tau \in G} \lambda(\tau) \prod_{i=1}^{m} b_{i\tau(i)}$$
$$= \lambda(\sigma) \prod_{i=1}^{m} b_{i\sigma(i)} + \sum_{\tau \neq \sigma} \lambda(\tau) \prod_{i=1}^{m} b_{i\tau(i)}.$$

For $\tau \neq \sigma$, let p and q be as above. We may assume that one of p and q, say p, is different from i_1. Then $(p, \tau(p)) \notin \{(i_1, j_1)\} \cup D(\sigma)$; hence

$$b_{p\tau(p)} = 0 \quad \text{and} \quad \prod_{i=1}^{m} b_{i\tau(i)} = 0.$$

Therefore $G_\lambda(B) = \lambda(\sigma) \neq 0$.

Suppose that the result holds for all subspaces L with $n(L) < k$ and that $n(A) = k > 1$. Let B be the subspace generated by the set

$$\{E_{ij}\colon (i, j) \neq (i_k, j_k)\}.$$

Then $\dim B = m^2 - 1$. Let $C = A \cap B$; then

$$\dim C \geqslant \dim A + \dim B - m^2 = m^2 - m - 1.$$

Now note that since $C \subset A$, $E_{i_1 j_1} \notin C$, and let $\bar{C}$ be the subspace generated by adjoining $E_{i_1 j_1}$ to C. Clearly $\dim \bar{C} = m^2 - m$ and $n(\bar{C}) = k - 1$. Hence, by the induction hypothesis, there exists $\bar{B} \in \bar{C}$ such that $\bar{B} = P(\sigma) + cE_{i_1 j_1}$ and $G_\lambda(\bar{B}) \neq 0$. Now $\bar{B} = B - aE_{i_1 j_1}$ for some $B \in C$, $a \in F$, so

$$B = P(\sigma) + (a + c)E_{i_1 j_1},$$

and since $C \subset A$, we know that $B \in A$. Clearly B satisfies (a) and the same computation as in the case of $n(A) = 1$ shows that $G_\lambda(B) = \lambda(\sigma) \neq 0$.

Using Lemmas 1 and 2, we see that if A is a 0-subspace for G_λ, then for all $\sigma \in G$, $|D(\sigma) \cap M| = 1$. Let x_{ij}, $(i, j) \notin M$, and x be commuting indeterminates over F and

$$L_t = \sum_{(i,j) \notin M} c_t^{ij} x_{ij}.$$

If $B \in A$, then since $\{A_{ij}\colon (i, j) \notin M\}$ is a basis for A, it follows that

$$B = \sum_{(i,j) \notin M} a_{ij} A_{ij}$$

and

$$b_{ij} = a_{ij} \quad \text{if } (i, j) \notin M, \qquad b_{i_t j_t} = \sum_{(i,j) \notin M} c_t^{ij} a_{ij}.$$

LEMMA 3. *If for each* $\sigma \in G$, $|D(\sigma) \cap M| = 1$ *and for some* t, $L_t \neq 0$, *then there exists a matrix* B *in* A *such that* $G_\lambda(B) \neq 0$.

Proof. Since $L_t \neq 0$, $c_t^{ij} \neq 0$ for some pair $(i, j) \notin M$. Choose $\sigma \in G(i_t, j_t)$ and let

$$B(x) = \sum_{k \neq i_t} A_{i\sigma(i)} + xA_{ij}.$$

The element in the (i_t, j_t) position of $B(x)$ is a non-zero polynomial of degree one in x so we may choose $c \in F$ such that this position is non-zero. Let $B(c) = b_{rs}$ and note that $b_{rs} = 0$ if $(r, s) \notin M \cup D(\sigma) \cup \{(i, j)\}$, and $b_{i\sigma(i)} \neq 0$, $i = 1, \ldots, m$. Then

$$\begin{aligned} G_\lambda(B(c)) &= \sum_{\tau \in G} \lambda(\tau) \prod_{k=1}^{m} b_{k\tau(k)} \\ &= \lambda(\sigma) \prod_{k=1}^{m} b_{k\sigma(k)} + \sum_{\tau \neq \sigma} \lambda(\tau) \prod_{k=1}^{m} b_{k\tau(k)}. \end{aligned}$$

If $\tau \neq \sigma$, then there exist $p \neq q$ such that $\tau(p) \neq \sigma(p)$, $\tau(q) \neq \sigma(q)$. If one of either $(p, \tau(p))$ or $(q, \tau(q))$ does not belong to $M \cup D(\sigma) \cup \{(i, j)\}$, then

$$\prod_{k=1}^{m} b_{k\tau(k)} = 0.$$

If this case does not occur, then we may assume that $(p, \tau(p)) = (i, j)$ and $(q, \tau(q)) = (i_s, j_s)$ for some $s \neq t$, because neither $(p, \tau(p))$ nor $(q, \tau(q))$ belongs to $D(\sigma)$ and $|D(\tau) \cap M| = 1$. Further, notice that if $k \neq p, q$, then $(k, \tau(k)) = (k, \sigma(k))$, for otherwise $(k, \tau(k)) \notin M \cup D(\sigma) \cup \{(i, j)\}$.

If G is regular, this clearly implies $\sigma = \tau$, a contradiction.

If G is doubly transitive, then notice that $\tau^{-1}\sigma$ is the transposition (pq). If $r \neq s$, choose $\mu \in G$ such that $\mu(r) = p$, $\mu(s) = q$. Then $\mu^{-1}\tau^{-1}\sigma\mu$ is the transposition (rs). Hence G contains all transpositions and is equal to S_m, contradicting the fact that G is a proper subgroup of S_m.

It now follows from Lemma 3 and the preceding remark that if A is a 0-subspace for G_λ, then A consists of all matrices with m fixed positions $(i_1, j_1), \ldots, (i_m, j_m)$ equal to zero.

To prove Proposition 1 we simply note that since $|D(\sigma) \cap M| = 1$, we have $(i_t, j_t) = (i_t, \sigma(i_t))$ for some $\sigma \in G$.

To prove Proposition 2 suppose that $i_t \neq i_s, j_t \neq j_s$. Choose $\sigma \in G$ such that $\sigma(i_t) = j_t$, $\sigma(i_s) = j_s$. Then $|D(\sigma) \cap M| > 1$, so by Lemma 1 there exists $B \in A$ such that $G_\lambda(B) \neq 0$, a contradiction.

The following proposition will be needed in the remaining portions of this paper and may be of some use in handling the simply transitive case.

PROPOSITION 3. *Suppose* G *is a transitive subgroup of* S_m *and* λ *a character of degree one on* G. *Let* T: $M_m(F) \to M_m(F)$ *be a linear transformation satisfying*

$$G_\lambda(T(X)) = G_\lambda(X) \quad \textit{for all } X.$$

Then T *is non-singular.*

Proof. If T were singular, then for some $A \neq 0$, $T(A) = 0$. Then

$$G(X + A) = G_\lambda(T(X + A)) = G_\lambda(T(X) + T(A)) = G(T(X)) = G_\lambda(X)$$

for all X. If we recall that G is transitive and use the techniques in **(1)**, it is easy to construct a matrix B such that $G_\lambda(B) \neq 0$ but $G_\lambda(B + A) = 0$, a contradiction.

Suppose now that T is a linear map of $M_m(F)$ into itself satisfying

$$G_\lambda(T(X)) = G_\lambda(X)$$

for all X. It is convenient to consider a matrix X of m^2 indeterminates x_{ij} and to consider the entries of $T(X)$ as linear forms in the x_{ij}. Write

$$T(X)_{ij} = L_{ij} = \sum_{r=1}^{m} \sum_{s=1}^{m} c(i, j, r, s) x_{rs}$$

where $c(i, j, r, s) \in F$. Let R_i (R^j) be the subspace of $M_m(F)$ consisting of all matrices with row i (column j) zero. Clearly R_i and R^j are 0-subspaces for G_λ. The map T is non-singular; hence by Propositions 1 and 2 $T(R_i)$ and $T(R^j)$ consist of all matrices with m fixed positions zero. Let $\{(r(i, t),\ s(i, t)): t = 1, \ldots, m\}$ be the positions that are zero in $T(R_i)$ and $\{(\alpha(j, t), \beta(j, t)): t = 1, \ldots, m\}$ the positions that are zero in $T(R^j)$.

LEMMA 4. *If $i \neq k$, then for all $p, q = 1, \ldots, m$:*
(a) $(r(i, p), s(i, p)) \neq (r(k, q), s(k, q))$,
(b) $(\alpha(i, p), \beta(i, p)) \neq (\alpha(k, q), \beta(k, q))$.

Proof. Suppose that for some $i \neq k$ there exist integers p and q such that $(r(i, p), s(i, p)) = (r(k, q), s(k, q)) = (u, v)$. Then for all $X \in R_i + R_k$, $(T(X))_{uv} = 0$. However, $M_m(F) = R_i + R_k$ since $i \neq k$. Therefore T is singular, a contradiction. The other case is identical.

LEMMA 5. *For each $i, j = 1, \ldots, m$ there exist integers $p(i, j)$ and $q(i, j)$ and a non-zero constant c_{ij} such that $L_{ij} = c_{ij} x_{pq}$ $(p = p(i, j),\ q = q(i, j))$. Further, if $(i, j) \neq (k, n)$, then $(p(i, j), q(i, j)) \neq (p(k, n), q(k, n))$.*

Proof. By Lemma 4 there are m^2 pairs $(r(i, t),\ s(i, t))$ and these are all distinct. Since $1 \leqslant r(i, t), s(i, t) \leqslant m$, the set of these pairs must be

$$\{(u, v): u, v = 1, \ldots, m\}.$$

Hence, given $1 \leqslant u,\ v \leqslant m$, there exist unique integers i and j such that $(u, v) = (r(i, j), s(i, j))$. We know that if $X \in R_i$, then $x_{ik} = 0, k = 1, \ldots, m$. We also know that the zeros in $T(R_i)$ appear in the $(r(i, t), s(i, t))$ positions. Hence if $x_{ik} = 0, k = 1, \ldots, m; L_{uv} = 0; u = r(i, j), v = s(i, j), t = 1, \ldots, m$. It follows that $c(r(i, t), s(i, t), p, q) = 0$ unless $p = i$.

Similarly, there exist unique integers a and b such that

$$(u, v) = (\alpha(a, b), \beta(a, b)).$$

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If we consider R^b and proceed as above, we may show that $x_{kb} = 0$ $k = 1, \ldots, m$, implies that $L_{uv} = 0$. Hence it follows that

$$c(\alpha(a, b), \beta(a, b), p, q) = 0$$

unless $q = b$. Therefore $L_{uv} = c(u, v, i, b)x_{ib}$, and since T is non-singular, $c(u, v, i, b) = c_{uv} \neq 0$.

It is clear from Lemma 5 that the matrix representation of T with respect to the natural (i.e., E_{ij}: $i, j = 1, \ldots, m$) basis for $M_m(F)$ is a generalized permutation matrix. Hence we have $T(X) = C * P(X)$, where $P(X)$ permutes the elements of X and $c_{ij} \neq 0$. Further, if $\sigma \in G$, then there exists $\tau \in G$ such that $T(P(\sigma)) = C * P(\tau)$. Therefore, since

$$G_\lambda(P(\sigma)) = \lambda(\sigma) \quad \text{and} \quad G_\lambda(C * P(\tau)) = \prod_{i=1}^{m} c_{i\tau(i)} \, \lambda(\tau),$$

we must have

$$\prod_{i=1}^{m} c_{i\tau(i)} = \lambda(\sigma)/\lambda(\tau) = \lambda(\sigma\tau^{-1}).$$

We now prove Theorem 1. Let $G = \{\sigma_1, \ldots, \sigma_m\}$; then since G is regular, if $X \in M_m(F)$ we may uniquely write

$$X = \sum_{i=1}^{m} X_i P(\sigma_i),$$

where the X_i are diagonal matrices. Let $T(P(\sigma_i)) = C * P(\tau_j)$. It follows from the facts that G is regular and T non-singular that if $i \neq j$, then $\tau_i \neq \tau_j$; hence since $|G| = m$ we know that $\tau_j = \sigma_{\alpha(i)}$ for some $\alpha \in S_m$. If $X = \operatorname{diag}(x_{i1}, \ldots, x_{im})P(\sigma_j)$, then

$$T(X) = C * \operatorname{diag}(x_{i\mu(1)}, \ldots, x_{i\mu(m)})P(\sigma_{\alpha(i)})$$

for some $\mu = \mu_i \in S_m$, because the polynomials $G_\lambda(X)$ and $G_\lambda(T(X))$ in $x_{i1}, \ldots, x_{im}$ must be equal.

A straightforward computation using the linearity of T shows that

$$T(X) = C * S(\mu_1, \ldots, \mu_m, \alpha)(X),$$

and that

$$\prod_{i=1}^{m} c_{i\sigma_t(i)} = \lambda(\sigma_t \sigma_{\alpha(t)}^{-1})$$

for each $\sigma_t \in G$. Clearly, if K and C satisfy the conditions of the theorem, then $G_\lambda(C * K(X)) = G_\lambda(X)$ for all X. This completes the proof of Theorem 1.

Suppose now that the group G is doubly transitive. We first show that $T(R_i) = R_j$ or R^j for some integer j. If this were not the case, then for some i there would exist integers j, k such that $T(E_{ij}) = aE_{pq}$, $T(E_{ik}) = bE_{st}$, $a, b \in F$ and $p \neq s$, $q \neq t$. Choose $\sigma \in G$ such that $\sigma(p) = q$, $\sigma(s) = t$. Note that $G_\lambda(T^{-1}(X)) = G_\lambda(X)$ for all X. However, $T^{-1}(P(\sigma))$ has two non-zero entries in row i and only m non-zero entries in all, so some row of $T^{-1}(P(\sigma))$ must be zero. Therefore $G_\lambda(T^{-1}(P(\sigma)) = 0$, a contradiction. Similarly we may show that $T(R^i) = R^j$ or R_j for some integer j.

We now show that if $T(R_i) = R_j$ for some i, j, then $T(R_k) = R_{\mu(k)}$ and $T(R^k) = R^{\tau(k)}$ for all k and some μ, τ in S_m. If this were not the case, then for some $k \neq i$ we would have $T(R_k) = R^n$ for some n. Then for all $X \in R_i + R_k$, $T(X)_{jn} = 0$. This contradicts the fact that T is non-singular, since

$$M_m(F) = R_i + R_k.$$

Similar arguments establish that if $i \neq j$, then $T(R_i) \neq T(R_j)$, and that if $T(R_i) = R^j$ for some i, j, then $T(R_k) = R^{\mu(k)}$ and $T(R^k) = R_{\tau(k)}$ for all k and some μ, τ in S_m.

Clearly, the above argument shows that either $T(X) = C * P(\mu)XP(\tau)$ or $C * P(\mu)X'P(\tau)$. If the first case occurs, we take X to be the identity matrix and it follows that $\mu\tau$ must belong to G; for otherwise

$$G_\lambda(C * P(\mu)P(\tau)) = G_\lambda(C * P(\mu\tau)) = 0.$$

An easy computation shows that if $\mu\tau$ belongs to G, then

$$G_\lambda(P(\mu)XP(\tau)) = \lambda(\tau^{-1}\mu)G_\lambda(X),$$

and by taking X to be appropriate permutation matrices we find that for each $\sigma \in G$

$$\prod_{i=1}^{m} c_{i\sigma(i)} = \lambda(\sigma\tau^{-1}\mu).$$

If the second case occurs, then, as above, we must have $\mu\tau$ belonging to G. Then if σ belongs to G,

$$\begin{aligned} T(P(\sigma)) &= C * P(\mu)P(\sigma)'P(\tau) = C * P(\mu)P(\sigma^{-1})P(\tau) = G_\lambda(T(P(\sigma)) \\ &= \lambda(\tau^{-1}\mu)G_\lambda(C * P(\sigma^{-1})) = \lambda(\tau^{-1}\mu)\lambda(\sigma^{-1})\prod_{i=1}^{m} c_{i\sigma^{-1}(i)} \\ &= G_\lambda(P(\sigma)) = \lambda(\sigma). \end{aligned}$$

Hence

$$\prod_{i=1}^{m} c_{i\sigma^{-1}(i)} = \lambda(\sigma^2)\lambda(\mu^{-1}\tau) = \lambda(\sigma^2\mu^{-1}\tau).$$

Conversely, if T satisfies the conditions of Theorem 2, an easy computation shows that $G_\lambda(T(X)) = G_\lambda(X)$ for all X. This completes the proof of Theorem 2.

4. Related results. If the group G is regular, it is possible to remove the restriction that λ be a character on G. By using the techniques in **(1)**, in particular Lemma 7, it is possible to characterize the linear maps of $M_m(F)$ into itself satisfying

$$G_\lambda(T(X)) = G_\lambda(X) \quad \text{for all } X,$$

where G is a regular subgroup of S_m and λ is any function on G to the field F.

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