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GENERALIZED MELLIN CONVOLUTIONS AND THEIR ASYMPTOTIC EXPANSIONS

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1. Introduction. A large number of important integral transforms, such as Laplace, Fourier sine and cosine, Hankel, Stieltjes, and Riemann-Liouville fractional integral transforms, can be put in the form

$$(1.1) \qquad I(x) = \int_0^\infty f(t)h(xt)dt,$$

where $f(t)$ and the kernel, $h(t)$, are locally integrable functions on $(0, \infty)$, and x is a positive parameter. Recently, two important techniques have been developed to give asymptotic expansions of $I(x)$ as $x \to +\infty$ or $x \to 0^+$. One method relies heavily on the theory of Mellin transforms [**8**] and the other is based on the use of distributions [**24**]. Here, of course, the integral $I(x)$ is assumed to exist in some ordinary sense.

If the above integral does not exist in any ordinary sense, then it may be regarded as an integral transform of a distribution (generalized function). There are mainly two approaches to extend the classical integral transforms to distributions. In one approach, the kernel of a transform is embedded in a test function space, and a generalized integral transform is defined by the action on the kernel of an element of the dual space. A typical example of this approach is provided by the generalized Laplace transform; see [**28**, Section 6] or [**21**, p. 217]. In the other approach, one first finds a test function space Φ, which is mapped continuously into another test function space Ψ by the integral transform in question, and then uses the adjoint mapping to define the generalized integral transform for the elements of the dual of Ψ. This is the approach commonly used for the Fourier transform of tempered distributions; see [**20**]. For a parallel study of these two approaches for the Stieltjes transform of generalized functions, we refer to [**3**].

Although there is a vast amount of literature on the subject of generalized integral transforms, only a few papers are devoted to discussions of the behavior of these transforms in the two limits $x \to +\infty$ and $x \to 0^+$. In [**10**], Jones has given a detailed study of the case in which the kernel is an oscillatory function such as e^{it} or the Bessel function $J_n(t)$. Jones was concerned only with the limit $x \to +\infty$, and obtained infinite

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asymptotic expansions for this class of generalized transforms. In [**15**, **16**], Lavoine and Misra have discussed the asymptotic behavior of the distributional Stieltjes transform, i.e., when $h(t) = 1/(1 + t)$. They have treated both limits $x \to +\infty$ and $x \to 0^+$, but obtain only the leading terms in the asymptotic expansions. The above two studies are complementary to each other, but neither one can be extended to include the other. In this connection, we also mention a recent paper of Zayed [**27**], who establishes a general procedure to extend certain integral transforms to distributions, and then applies a technique of one of us ([**25**]) to derive asymptotic expansions (as $x \to +\infty$) for some of these integral transforms. Zayed's procedure is essentially a unification of those given by Zemanian [**29**] and belongs to the first approach mentioned above. It should be noted, however, that the asymptotic expansions given in [**27**] are only for the conventional and not for the generalized integral transforms.

The integral in (1.1) is one of the two convolutions encountered in the theory of Mellin transforms, and is easily seen to be equivalent to the other more symmetric convolution defined by

$$(1.2) \qquad (f * g)(x) = \int_0^\infty f(t)g(xt^{-1})t^{-1}dt.$$

This suggests that one may extend the integral transforms to distributions in a third way, in addition to the two previously mentioned, that is to define first a distributional Mellin convolution and then to view the various generalized integral transforms as special cases.

In this paper, we introduce a class $\mathscr{F}$ of locally integrable functions in $(0, \infty)$, characterized by their asymptotic behavior at 0 and ∞, on which we can define the Mellin convolution $*$ as a distribution in $(0, \infty)$. This class includes functions such as e^{-t} and $1/(1 + t)$, but (unfortunately) excludes oscillatory functions such as e^{it} and $J_n(t)$. Also, we find infinite asymptotic expansions, as $x \to 0^+$ and as $x \to +\infty$, of the convolution of two functions in $\mathscr{F}$. In particular, we give asymptotic expansions of the generalized Laplace, Stieltjes, and fractional integral transforms. We expect that our results can be extended to a class $\mathscr{G} \supset \mathscr{F}$, which includes oscillatory functions such as e^{it} and $J_n(t)$, and that the asymptotic expansions of the generalized Fourier and Hankel transforms can be deduced, as special cases, from our general results on Mellin convolutions. This possibility is presently under investigation.

The present paper is arranged as follows. In Section 2 we introduce the family $\mathscr{F}$ and recall the regularization method developed in the theory of distributions. This method is then used to define infinite integrals involving functions in $\mathscr{F}$ and, in particular, the Mellin transforms of these functions. Our family $\mathscr{F}$ is similar to a class of distributions introduced by Jeanquartier [**9**] in a study on Mellin transforms of distributions. However, the aims of the two studies and the methods used differ completely. In Section 3 the generalized Mellin convolution is defined, and some specific convolutions, including $t^{\alpha}(\log t)^{n} * t^{\beta}(\log t)^{m}$, are calculated. Here α and β are any complex numbers, and m and n are non-negative integers. It should be emphasized that these are Mellin convolutions, and not the convolutions associated with the Laplace transform as given in [**6**, p. 116] or those recently studied by Jones in connection with the Fourier transform [**11**, **12**]. The main results of this paper are given in Section 4, where asymptotic expansions are derived for the generalized Mellin convolution as the parameter tends to zero or infinity. Finally, in Section 5 various generalized integral transforms are defined as (generalized) Mellin convolutions, and the results in Section 4 are then used to give asymptotic expansions of these transforms.

2. Regularization of infinite integrals. Let $\mathscr{F}$ be the family of locally integrable functions $f(t)$ on $(0, \infty)$, which have asymptotic expansions of the form

$$(2.1) \qquad f(t) \sim \sum_{s=0}^{\infty} \sum_{r=0}^{N(s)} a_{rs} t^{\alpha_s} (\log t)^r \quad \text{as } t \to 0^+$$

and

$$(2.2) \qquad f(t) \sim \sum_{s=0}^{\infty} \sum_{r=0}^{Q(s)} c_{rs} t^{-\nu_s} (\log t)^r \quad \text{as } t \to +\infty,$$

where $\{\operatorname{Re} \alpha_s\}$ and $\{\operatorname{Re} \nu_s\}$ are strictly increasing sequences with limit $+\infty$, and $N(s)$ and $Q(s)$ are finite for each s; for definition of such asymptotic expansions, see [**26**]. The following facts about $\mathscr{F}$ are easily verified and will be used later: (i) $\mathscr{F}$ is a vector space; (ii) if f and g belong to $\mathscr{F}$ then fg belongs to $\mathscr{F}$; and (iii) if f belongs to $\mathscr{F}$ then so does $\check{f}$, where $\check{f}(t) = f(1/t)$.

Without restrictions on the exponents α_0 and ν_0 in (2.1) and (2.2), the integral

$$(2.3) \qquad \int_0^{\infty} f(t) dt$$

will normally diverge. In this section we shall summarize some of the important concepts in the regularization method [**6**] which will allow us to give a meaningful definition to the infinite integral (2.3).

Let U be an open interval in $\mathbf{R} = (-\infty, \infty)$. As usual, we shall denote by $\mathscr{D}(U)$ the space of all C^{∞}-functions with compact support in U, and by $\mathscr{D}'(U)$ the space of distributions in U. The action of a distribution f on a test function η will be denoted by $\langle f, \eta \rangle$.

Let $f(t)$ be a locally integrable function in $\mathbf{R} - \{t_0\}$, but not integrable in any interval containing t_0. The integral

$$\int_{-\infty}^{\infty} f(t)\eta(t)dt,$$

where $\eta(t)$ belongs to $\mathscr{D}(\mathbf{R})$, will in general diverge. However, it will converge if $\eta(t)$ vanishes in a neighborhood of t_0.

Definition 1 ([**6**, p. 10]). A *regularization of the function* $f(t)$ is any distribution $f \in \mathscr{D}'(\mathbf{R})$ satisfying

$$\langle f, \eta \rangle = \int_{-\infty}^{\infty} f(t)\eta(t)dt$$

for all $\eta \in \mathscr{D}(\mathbf{R})$ whose support does not contain t_0.

It can be shown that every almost everywhere locally integrable function $f(t)$ with at most a finite number of algebraic singularities can be regularized, and that the regularization is determined only up to an additive functional concentrated on the singularities of $f(t)$; see [**6**, Section 1.7].

For our purpose we shall restrict our attention to those locally integrable functions $f(t)$ which vanish in $(-\infty, 0)$ but may have singularities only at 0 and ∞. Thus we need consider only integrals of the form

$$\int_0^{\infty} f(t)\eta(t)dt,$$

where the test function $\eta(t)$ is, however, still assumed to be only in $\mathscr{D}(\mathbf{R})$ and not necessarily in $\mathscr{D}(\mathbf{R}^+)$, $\mathbf{R}^+$ being the open half-line $(0, \infty)$.

For $\operatorname{Re} \lambda > -1$, the integral

$$\int_0^b t^{\lambda}\eta(t)dt, \quad 0 < b < \infty,$$

converges and is a holomorphic function of λ. Furthermore, it can be analytically continued to the entire λ-plane via

$$\begin{aligned} \int_0^b t^{\lambda}\eta(t)dt = \int_0^b t^{\lambda}\Big[\eta(t) - \eta(0) - \eta'(0)t - \\ \ldots - \eta^{(n-1)}(0)\frac{t^{n-1}}{(n-1)!}\Big]dt \\ + \eta(0)\frac{b^{\lambda+1}}{\lambda+1} + \eta'(0)\frac{b^{\lambda+2}}{\lambda+2} + \\ \ldots + \eta^{(n-1)}(0)\frac{b^{\lambda+n}}{(n-1)!(\lambda+n)}, \end{aligned} \tag{2.4}$$

the right-hand side of which makes sense for all λ with $\operatorname{Re} \lambda > -n - 1$ except for $\lambda = -1, -2, \ldots, -n$. We shall use (2.4) to define a distribution which regularizes the function

$$(2.5)\qquad \overset{\lambda}{t}_{0<t\le b} = \begin{cases} t^\lambda \text{ for } 0 < t \le b \\ 0 \text{ for all other } t, \end{cases}$$

and we shall denote this distribution also by $\overset{\lambda}{t}_{0<t\le b}$. Linearity of this distribution is obvious, and continuity follows from the Lebesgue dominated convergence theorem. Thus we define

$$(2.6)\qquad \langle \overset{\lambda}{t}_{0<t\le b}, \eta\rangle = \int_0^b t^\lambda \eta(t)dt,$$

where the integral is understood in the sense of (2.4). Note that if $\eta(t)$ vanishes in a neighborhood of $t = 0$, then this integral exists in the ordinary sense and (2.4) holds trivially.

Thus, the distribution defined in (2.6) is indeed a regularization of the function given in (2.5). If we take $\eta^*(t)$ to be a test function in $\mathscr{D}(\mathbf{R})$ which equals "1" on the interval $[0, b]$, then we have from (2.6)

$$(2.7)\qquad \langle \overset{\lambda}{t}_{0<t\le b}, \eta^*\rangle = \frac{b^{\lambda+1}}{\lambda + 1}.$$

Interpreting the integral $\int_0^b t^\lambda dt$ as the result of applying the distribution $\overset{\lambda}{t}_{0<t\le b}$ to the test function $\eta^*(t)$ gives

$$(2.8)\qquad \int_0^b t^\lambda dt = \frac{b^{\lambda+1}}{\lambda + 1},$$

valid for all $\lambda \neq -1, -2, -3, \ldots$; see two related papers by Jones [**11**, **12**]. Since the right hand side has only one singularity at $\lambda = -1$, equation (2.8) will be considered to hold also at $\lambda = -2, -3, \ldots$.

In view of the fact that

$$\frac{\partial^r}{\partial\lambda^r} t^\lambda = t^\lambda(\log t)^r \quad \text{for } t > 0,$$

we also obtain, by a similar argument, the formula

$$(2.9)\qquad \int_0^b t^\lambda(\log t)^r dt = \sum_{k=0}^{r} \frac{(-1)^k r!}{(r-k)!} \frac{b^{\lambda+1}}{(\lambda + 1)^{k+1}}(\log b)^{r-k}.$$

The case $\lambda = -1$ needs a separate treatment. The identity

$$\int_0^b t^{-1}\,\eta(t)dt = \eta(b)\log b - \int_0^b \eta'(t)\log t\,dt,$$

valid for all test functions η whose supports do not contain 0, suggests that we may define the distribution $\overset{-1}{t}_{0<t\le b}$ by

$$(2.10)\qquad \langle \overset{-1}{t}_{0<t\le b}, \eta\rangle = \eta(b)\log b - \int_0^b \eta'(t)\log t\,dt,$$

that is,

(2.11) $t^{-1}_{0<t\le b} = (\log b)\delta(t - b) + (\log t \cdot \chi_{[0,b]})'$,

where $\chi_{[0,b]}$ is the characteristic function which is equal to 1 on $[0, b]$, and where the derivative on the right-hand side is taken in the distribution sense. With $\eta^*(t)$ again being a test function in $\mathscr{D}(\mathbf{R})$ which equals "1" on $[0, b]$, we have from (2.10):

(2.12) $\langle t^{-1}_{0<t\le b}, \eta^*\rangle = \log b$.

We shall regard the integral $\int_0^b t^{-1}dt$ as the action of the distribution $t^{-1}_{0<t\le b}$ on η^*. Thus (2.12) gives

(2.13) $$\int_0^b t^{-1}dt = \log b.$$

A similar argument yields

(2.14) $$\int_0^b t^{-1}(\log t)^r dt = \frac{1}{r+1}(\log b)^{r+1}$$

for any non-negative integer r.

Having given a meaning to the integral $\int_0^b t^\lambda dt$, we now wish to do the same for the integral $\int_b^\infty t^\lambda dt$. Unfortunately there is no test function in $\mathscr{D}(\mathbf{R})$ which is equal to "1" on the infinite interval $[b, \infty)$. Thus we must proceed in a slightly different manner. Let $\hat{\mathscr{D}}(b, \infty)$ denote the space of all C^∞-functions $\phi(t)$ on $[b, \infty)$ such that for $0 < t < 1/b$, we have

$$\phi(1/t) = \eta(t) \quad \text{for some } \eta \in \mathscr{D}(\mathbf{R}).$$

The linear functional $t^\lambda_{b\le t<\infty}$ is then defined by

(2.15) $$\begin{aligned}\langle t^\lambda_{b\le t<\infty}, \phi\rangle &= \int_0^{1/b} \tau^{-\lambda-2}\eta(\tau)d\tau \\ &= \langle \tau^{-\lambda-2}_{o<\tau\le 1/b}, \eta\rangle,\end{aligned}$$

where $\eta(\tau) = \phi(1/\tau)$. We shall interpret the integral $\int_b^\infty t^\lambda\, dt$ as the result of the action of $t^\lambda_{b\le t<\infty}$ on a function $\phi^* \in \hat{\mathscr{D}}(b, \infty)$ such that

$$\phi^*(1/t) = \eta^*(t) \equiv 1 \quad \text{for } 0 < t < 1/b.$$

Thus

(2.16) $$\int_b^\infty t^\lambda dt = -\frac{b^{\lambda+1}}{\lambda+1}$$

for all $\lambda \neq -1$. The argument for (2.9) then leads to

(2.17) $$\int_b^\infty t^\lambda(\log t)^r dt = -\sum_{k=0}^{r} \frac{(-1)^k\, r!}{(r-k)!}\,\frac{b^{\lambda+1}}{(\lambda+1)^{k+1}}(\log b)^{r-k}.$$

The case $\lambda = -1$ can be treated similarly, and the corresponding formula is

$$(2.18)\quad \int_b^\infty t^{-1}(\log t)^r dt = -\frac{1}{(r+1)}(\log b)^{r+1}.$$

Formulas (2.9), (2.14), (2.17) and (2.18) together suggest that we give the meaning

$$(2.19)\quad \int_0^\infty t^\lambda(\log t)^r dt = \left(\int_0^b + \int_b^\infty\right)t^\lambda(\log t)^r\, dt = 0$$

for all complex λ and all non-negative integers r.

For $\lambda \neq -1$ there is an easier way to motivate the result (2.19). For simplicity, let us consider only the case when $r = 0$. Clearly we have

$$(2.20)\quad \int_0^b t^\lambda dt = \frac{b^{\lambda+1}}{\lambda+1} \quad \text{for } \operatorname{Re}\lambda > -1$$

and

$$(2.21)\quad \int_b^\infty t^\lambda dt = -\frac{b^{\lambda+1}}{\lambda+1} \quad \text{for } \operatorname{Re}\lambda < -1.$$

The right-hand sides of these two equations are analytic functions of λ for all λ except for $\lambda = -1$, and are hence meromorphic continuations of the integrals on the left-hand sides. If we use the integrals on the left-hand sides to denote not only the functions which they represent when they converge but also their meromorphic continuations, then it follows at once from (2.20) and (2.21) that for $\lambda \neq -1$

$$(2.22)\quad \int_0^\infty t^\lambda dt \equiv \int_0^b t^\lambda dt + \int_b^\infty t^\lambda dt = 0.$$

For another different approach to this problem, we refer to two recent articles by Jones [**11**, **12**].

The above concept of regularization will now be used to give a meaningful definition to the infinite integral in (2.3). Suggested by (2.19), we split the interval of integration at $t = b$ and consider first the integral $\int_0^b f(t)dt$. Now choose n such that $\operatorname{Re}\alpha_n > -1$ and write (2.1) as

$$(2.23)\quad f(t) = \sum_{s=0}^{n-1}\sum_{r=0}^{N(s)} a_{rs}t^{\alpha_s}(\log t)^r + f_{0,n}(t)$$

with

$$f_{0,n}(t) = O(t^{\alpha_n}(\log t)^{N(n)}) \quad \text{as } t \to 0^+.$$

We define

$$(2.24)\quad \int_0^b f(t)dt \equiv \sum_{s=0}^{n-1}\sum_{r=0}^{N(s)} a_{rs}\int_0^b t^{\alpha_s}(\log t)^r dt + \int_0^b f_{0,n}(t)dt,$$

where the integrals in the double sum are understood in the sense of either (2.9) or (2.14), depending upon whether α_s is not or is equal to -1, and where the last integral on the right exists and is hence taken in the ordinary sense.

Similarly, we choose q such that $\operatorname{Re} \nu_q > 1$ and write (2.2) in the form

$$(2.25) \quad f(t) = \sum_{s=0}^{q-1} \sum_{r=0}^{Q(s)} c_{rs} t^{-\nu_s} (\log t)^r + f_{\infty,q}(t)$$

with

$$f_{\infty,q}(t) = O(t^{-\nu_q} (\log t)^{Q(q)}) \quad \text{as } t \to \infty.$$

The integral $\int_b^\infty f(t)dt$ is then defined by

$$(2.26) \quad \int_b^\infty f(t)dt \equiv \sum_{s=0}^{q-1} \sum_{r=0}^{Q(s)} c_{rs} \int_b^\infty t^{-\nu_s} (\log t)^r dt + \int_b^\infty f_{\infty,q}(t)dt,$$

where the integrals in the double sum are understood in the sense of either (2.17) or (2.18), depending upon whether ν_s is not or is equal to 1, and where the last integral on the right-hand side exists and is taken in the ordinary sense.

Definition 2. If f is in $\mathscr{F}$ then the infinite integral $\int_0^\infty f(t)dt$ is given the meaning

$$(2.27) \quad \int_0^\infty f(t)dt \equiv \int_0^b f(t)dt + \int_b^\infty f(t)dt,$$

where the two integrals on the right are defined by the formulas (2.24) and (2.26). The value of the right-hand side is called *the regularization of the (formal) integral* on the left.

We shall also call the right-hand sides of equations (2.9), (2.14), (2.17), (2.18) and (2.19), respectively, the regularizations of the integrals on the left-hand sides of these equations.

A few points are now in order. Firstly, it is easily shown that the definitions of $\int_0^b f(t)dt$ and $\int_b^\infty f(t)dt$ given in (2.24) and (2.26) are independent of the choices of n and q as long as $\operatorname{Re} \alpha_n > -1$ and $\operatorname{Re} \nu_q > 1$. Secondly, it can also be shown that the definition of $\int_0^\infty f(t)dt$ given in (2.27) is independent of the choice of b. Thirdly, the new definition of $\int_0^\infty f(t)dt$ agrees with the usual one when the latter makes sense. Finally, for the functions in $\mathscr{F}$, we have the linearity property

$$(2.28) \quad \int_0^\infty [c_1 f_1(t) + c_2 f_2(t)]dt = c_1 \int_0^\infty f_1(t)dt + c_2 \int_0^\infty f_2(t)dt,$$

where all three integrals are understood in the sense of Definition 2.

Example 1. Let m be a non-negative integer, and consider the formal integral

$$\int_0^\infty \frac{t^{-m}}{t+x}\,dt.$$

Since this integral may be viewed as the Stieltjes transform of the distribution t_+^{-m}, it is of interest to know what appropriate value should be assigned to it; see Section 5. We first consider the case $m = 0$, and write

$$\frac{1}{t+x} = \frac{1}{t} - \frac{x}{t(t+x)}.$$

Regarding the second term on the right as the remainder in the asymptotic expansion $(t + x)^{-1} \sim t^{-1}$, we have from (2.18) and the definition in (2.26)

$$\int_b^\infty \frac{1}{t+x}dt = -\log b - \int_b^\infty \frac{x}{t(t+x)}dt.$$

The last integral exists as an ordinary integral and can be evaluated to be $\log (b + x) - \log b$. Thus

$$\int_b^\infty \frac{1}{t+x}dt = -\log (b+x).$$

Since

$$\int_0^b \frac{1}{t+x}dt = \log (b+x) - \log x,$$

Definition 2 gives

$$\int_0^\infty \frac{1}{t+x}dt = -\log x. \tag{2.29}$$

Note that for any $m \geq 1$

$$\frac{t^{-m}}{t+x} = \frac{1}{x}\left(\frac{1}{t^m} - \frac{1}{t^{m-1}(t+x)}\right).$$

From (2.22), it follows that

$$\int_0^\infty \frac{t^{-m}}{t+x}dt = -\frac{1}{x}\int_0^\infty \frac{t^{-m+1}}{t+x}dt.$$

Equation (2.29) then gives

$$\int_0^\infty \frac{t^{-1}}{t+x}dt = \frac{\log x}{x}.$$

By induction we have

$$(2.30)\quad \int_0^\infty \frac{t^{-m}}{t+x}dt = (-1)^{m+1}\frac{\log x}{x^m}.$$

Definition 3. If $f \in \mathscr{F}$, then we define the (*generalized*) *Mellin transform of* f by

$$(2.31)\quad M[f;\, z] = \int_0^\infty t^{z-1}f(t)dt,$$

where the integral on the right is understood in the sense of Definition 2.

Since $t^{z-1}f(t)$ is in $\mathscr{F}$ whenever f is in $\mathscr{F}$ and z is a complex number, $M[f;\, z]$ is a well-defined, complex-valued function of the complex variable z, for each f in $\mathscr{F}$. If the integral in (2.31) converges, then $M[f;\, z]$ is simply the conventional Mellin transform of f, evaluated at z, $M[f;\, z]$ is analytic near z and can be continued to a meromorphic function in the z-plane; see [**8**]. The following lemma shows that the generalized Mellin transform of f has similar properties, that is, it has a meromorphic extension with poles at $-\alpha_s$ and ν_s, where α_s and ν_s are the exponents in the asymptotic expansions of f, and moreover that the values $M[f;\, -\alpha_i]$ and $M[f;\, \nu_i]$ are closely related to that meromorphic function.

LEMMA 1. *Let f be in $\mathscr{F}$, with asymptotic expansions* (2.1) *and* (2.2).

(i) *The Mellin transform $M[f;\, z]$, defined in Definition* 3, *is a meromorphic function in the z-plane with poles at $z = -\alpha_s$ and $z = \nu_s$, $s = 0, 1, 2, \ldots$. The principal part of this function at $-\alpha_i$ is*

$$\sum_{r=0}^{N(i)} a_{ri}\frac{(-1)^r r!}{(\alpha_i + z)^{r+1}},$$

if $-\alpha_i \neq \nu_s$ for all s, while its principal part at ν_j is

$$-\sum_{r=0}^{Q(j)} c_{rj}\frac{(-1)^r r!}{(z-\nu_j)^{r+1}}$$

if $\nu_j \neq -\alpha_s$ for all s. If $-\alpha_i = \nu_j$ for some i and j, then the principal part of the meromorphic function at this point is

$$\sum_{r=0}^{N(i)} a_{ri}\frac{(-1)^r r!}{(\alpha_i + z)^{r+1}} - \sum_{r=0}^{Q(j)} c_{rj}\frac{(-1)^r r!}{(z-\nu_j)^{r+1}}.$$

(ii) *If $-\alpha_i \neq \nu_s$ for all s, then*

$$M[f; -\alpha_i] = \lim_{z\to-\alpha_i} \{M[f; z] - \sum_{r=0}^{N(i)} a_{ri} \frac{(-1)^r r!}{(\alpha_i + z)^{r+1}}\}.$$

(iii) *If* $\nu_j \neq -\alpha_s$ *for all s, then*

$$M[f, \nu_j] = \lim_{z\to\nu_j} \{M[f; z] + \sum_{r=0}^{Q(j)} c_{rj} \frac{(-1)^r r!}{(z - \nu_j)^{r+1}}\}.$$

(iv) *If* $-\alpha_i = \nu_j$ *for some i and j, then*

$$\begin{aligned} M[f; -\alpha_i] &= M[f; \nu_j] \\ &= \lim_{z\to-\alpha_i} \{M[f; z] - \sum_{r=0}^{N(i)} a_{ri} \frac{(-1)^r r!}{(\alpha_i + z)^{r+1}} \\ &\quad + \sum_{r=0}^{Q(j)} c_{rj} \frac{(-1)^r r!}{(z - \nu_j)^{r+1}}\}. \end{aligned}$$

Proof. (i) By Definitions 2 and 3,

$$M[f; z] = \int_0^1 t^{z-1} f(t)dt + \int_1^\infty t^{z-1} f(t)dt. \tag{2.32}$$

For convenience, we have taken $b = 1$ in (2.27). From (2.9), (2.17), (2.24) and (2.26), it follows that the first integral in (2.32) is given by

$$\sum_{s=0}^{n-1} \sum_{r=0}^{N(s)} a_{rs} \frac{(-1)^r r!}{(\alpha_s + z)^{r+1}} + \int_0^1 t^{z-1} f_{0,n}(t)dt$$

and the second integral by

$$-\sum_{s=0}^{q-1} \sum_{r=0}^{Q(s)} c_{rs} \frac{(-1)^r r!}{(z - \nu_s)^{r+1}} + \int_1^\infty t^{z-1} f_{\infty,q}(t)dt.$$

Each of the two remainder integrals above converges, and defines an analytic function of z, in the strip $-\text{Re }\alpha_n < \text{Re } z < \text{Re }\nu_q$. The two double sums give the principal parts as indicated. Since $\text{Re }\alpha_n \to \infty$ as $n \to \infty$ and $\text{Re }\nu_q \to \infty$ as $q \to \infty$, part (i) follows.

(ii), (iii), (iv). Proceed as in the proof of part (i), using (2.14) and (2.17) where necessary.

Results similar to the above have been given by Handelsman and Lew [**8**, Lemmas 1c and 2c] and Jeanquartier [**9**, Proposition 4.3].

We conclude this section with some notations which will be used later. We have found it useful to think of the generalized Mellin transform of f in $\mathscr{F}$ as the meromorphic function of Lemma 1 (i), together with certain values assigned at the poles of that function, as in Lemma 1 (ii), (iii) and (iv). These values are obtained as the limits of the meromorphic function minus its corresponding principal parts. With f as in Lemma 1, and λ one of the values $-\alpha_s$, ν_s, we write $\mathrm{Pr}_\lambda[f; z]$ for the principal part at λ given in Lemma 1 (i), and we write, for $z \neq \lambda$,

$$(2.33) \quad A_\lambda[f; z] = M[f; z] - \mathrm{Pr}_\lambda[f; z].$$

Then $A_\lambda[f; z]$ is a meromorphic function of z with a removable singularity at λ, and we have from Lemma 1 (ii), (iii), (iv), that

$$M[f; \lambda] = A_\lambda[f; \lambda].$$

Using this notation, we can easily state the following extended versions of the results of Lemma 1; in each of these formulas, the derivatives on the right are with respect to z, and the integrals on the left are in the sense of Definition 2. With f as in Lemma 1, k a non-negative integer, and z different from $-\alpha_s$ and ν_s, for all s,

$$(2.34) \quad \int_0^\infty t^{z-1}(\log t)^k f(t)dt = M^{(k)}[f; z];$$

on the other hand, if λ is one of the values $-\alpha_s$, ν_s, then

$$(2.35) \quad \int_0^\infty t^{\lambda-1}(\log t)^k f(t)dt = A_\lambda^{(k)}[f; \lambda].$$

3. Generalized Mellin convolution. In the remainder of the paper, we shall work with distributions on $(0, \infty)$, and it will be convenient to define the action of a locally integrable function f on a test function $\phi \in \mathscr{D}(\mathbf{R}^+)$ by

$$\langle f, \phi \rangle = \int_0^\infty f(t)\phi(t)t^{-1}dt.$$

Now let $f(t)$ and $g(t)$ be two locally integrable functions on $(0, \infty)$, and recall the ordinary Mellin convolution defined by (1.2):

$$(3.1) \quad (f * g)(x) = \int_0^\infty f(t)g(xt^{-1})t^{-1}dt.$$

Assume that this integral is absolutely convergent. Then the distribution on $(0, \infty)$ defined by the (locally integrable) function $f * g$ can be written in the form

$$\begin{aligned}\langle f * g, \phi \rangle &= \int_0^\infty (f * g)(t)\phi(t)t^{-1}dt \\ &= \int_0^\infty \left\{ \int_0^\infty f(x)g(tx^{-1})x^{-1}dx \right\} \phi(t)t^{-1}dt \\ &= \int_0^\infty f(x) \left\{ \int_0^\infty \phi(t)g(tx^{-1})t^{-1}dt \right\} x^{-1}dx.\end{aligned}$$

This naturally suggests that the convolution of f and g in $\mathscr{F}$ can be defined by

$$(3.2)\qquad \langle f * g, \phi\rangle = \int_0^\infty f(x)\Phi_g(x)x^{-1}dx$$

for all $\phi \in \mathscr{D}(\mathbf{R}^+)$, where

$$(3.3)\qquad \Phi_g(x) = \int_0^\infty \phi(t)g(x^{-1}t)t^{-1}dt = \int_0^\infty \phi(xt)g(t)t^{-1}dt,$$

provided that we can make sense out of the integral in (3.2).

Note that for f and g in $\mathscr{F}$, the integral in (3.1) may not exist without further restrictions on the exponents in the asymptotic expansions of these functions.

Let the asymptotic expansions of g be given by

$$(3.4)\qquad g(t) \sim \sum_{s=0}^{\infty}\sum_{r=0}^{M(s)} b_{rs}t^{\delta_s}(\log t)^r, \quad \text{as } t \to 0^+,$$

and

$$(3.5)\qquad g(t) \sim \sum_{s=0}^{\infty}\sum_{r=0}^{P(s)} d_{rs}t^{-\beta_s}(\log t)^r, \quad \text{as } t \to +\infty,$$

where $M(s)$ and $P(s)$ are finite for each s, and the sequences $\{\mathrm{Re}\ \delta_s\}$ and $\{\mathrm{Re}\ \beta_s\}$ are strictly increasing to $+\infty$.

For $\phi \in \mathscr{D}(\mathbf{R}^+)$, we define $\Phi_g(x)$ as in (3.3). Clearly, $\Phi_g(x)$ is a C^∞-function on $(0, \infty)$ and hence locally integrable there. The following result shows that $\Phi_g(x)$ belongs to $\mathscr{F}$.

LEMMA 2. *For any $\phi \in \mathscr{D}(\mathbf{R}^+)$, there exist constants b_{ks}^* and d_{ks}^* such that*

$$(3.6)\qquad \Phi_g(x) \sim \sum_{s=0}^{\infty}\sum_{k=0}^{M(s)} b_{ks}^* x^{-\delta_s}(\log x)^k \quad \text{as } x \to \infty,$$

and

$$(3.7)\qquad \Phi_g(x) \sim \sum_{s=0}^{\infty}\sum_{k=0}^{P(s)} d_{ks}^* x^{\beta_s}(\log x)^k \quad \text{as } x \to 0^+.$$

Proof. Put

$$(3.8)\qquad g(t) = \sum_{s=0}^{m-1}\sum_{r=0}^{M(s)} b_{rs}t^{\delta_s}(\log t)^r + g_{0,m}(t).$$

From (3.4) it follows that

$$g_{0,m}(t) = O(t^{\delta_m}(\log t)^{M(m)}) \quad \text{as } t \to 0^+.$$

It is easy to show that

$$(3.9) \qquad \int_0^\infty \phi(t) g_{0,m}(x^{-1}t)t^{-1}dt = O(x^{-\delta_m}(\log x)^{M(m)}) \quad \text{as } x \to \infty.$$

Coupling (3.8) and (3.9) and using the binomial theorem gives

$$\Phi_g(x) = \sum_{s=0}^{m-1} \sum_{k=0}^{M(s)} b_{ks}^* x^{-\delta_s}(\log x)^k + O(x^{-\delta_m}(\log x)^{M(m)}),$$

where

$$(3.10) \qquad (-1)^k b_{ks}^* = \sum_{r=k}^{M(s)} b_{rs}\binom{r}{k} M^{(r-k)}[\phi; \delta_s].$$

This proves the large-x expansion (3.6). A similar argument will lead to the small-x expansion given in (3.7).

Definition 4. For f and $g \in \mathscr{F}$, we define the *generalized Mellin convolution* $f * g$ by (3.2):

$$(3.11) \qquad \langle f * g, \phi \rangle = \int_0^\infty f(x)\Phi_g(x)x^{-1}dx$$

for all $\phi \in \mathscr{D}(\mathbf{R}^+)$, where the integral on the right is understood in the sense of Definition 2.

Since $\Phi_g \in \mathscr{F}$ by Lemma 2, $f\Phi_g \in \mathscr{F}$ by an earlier remark. Thus the integral in (3.11) is indeed meaningful. Furthermore, it is clear that this definition of convolution satisfies the linearity condition

$$(3.12) \qquad \langle f * g, \alpha_1\phi_1 + \alpha_2\phi_2 \rangle = \alpha_1\langle f * g, \phi_1\rangle + \alpha_2\langle f * g, \phi_2\rangle,$$

where α_1 and α_2 are any two constants and ϕ_1 and ϕ_2 are any two functions in $\mathscr{D}(\mathbf{R}^+)$. To show that this linear functional is also continuous, we let $\{\phi_\nu\}$ be a sequence of functions in $\mathscr{D}(\mathbf{R}^+)$ such that all these functions vanish outside a fixed compact set $\subset \mathbf{R}^+$ and converge uniformly to zero together with their derivatives of any order, and let $\{\Phi_{g,\nu}(x)\}$ denote the corresponding sequence of integrals defined by (3.3). It can easily be shown that the coefficients in both the large-x and the small-x expansions of $\Phi_{g,\nu}(x)$ tend to zero as $\nu \to \infty$, and moreover that the remainders in these expansions also tend to zero as $\nu \to \infty$; see equation (3.10) and the expression for the remainder given in (3.9). The same is of course true for the product $f(x)\Phi_{g,\nu}(x)$. Therefore we have

$$\int_0^\infty f(x)\Phi_{g,\nu}(x)x^{-1}dx \to 0 \quad \text{as } \nu \to \infty,$$

where the integrals are all taken in the sense of Definition 2, thus establishing that $f * g$ is indeed a distribution on $\mathscr{D}(\mathbf{R}^+)$.

It should be noted that although the ordinary convolution $*$ in (3.1) is commutative, this is no longer true for its generalization given in (3.11). We shall come across some specific examples of non-commutativity later in our discussion.

Let us now calculate convolutions of the form $t^\alpha * t^\beta$. Throughout the remaining sections, we shall use $\chi_0(t)$ to denote the characteristic function $\chi_{[0,1]}(t)$, which is 1 on [0, 1] and 0 elsewhere. Similarly we write $\chi_\infty(t)$ for the characteristic function $\chi_{[1,\infty]}(t)$.

For any α and β with $\alpha \neq \beta$, it is easy to see that $t^\alpha\chi_0 * t^\beta\chi_0$ and $t^\alpha\chi_\infty * t^\beta\chi_\infty$ exist in the ordinary sense of (3.1) and are given by

$$(3.13)\quad t^\alpha\chi_0 * t^\beta\chi_0 = -\frac{x^\alpha - x^\beta}{\alpha - \beta}\chi_0$$

$$(3.14)\quad t^\alpha\chi_\infty * t^\beta\chi_\infty = \frac{x^\alpha - x^\beta}{\alpha - \beta}\chi_\infty.$$

However, the convolutions $t^\alpha\chi_0 * t^\beta\chi_\infty$ and $t^\alpha\chi_\infty * t^\beta\chi_0$ may not exist in this sense. Nevertheless, if we replace f and g by $t^\alpha\chi_0$ and $t^\beta\chi_\infty$, respectively, in the integral (3.1), and apply the results in (2.8) and (2.16), we obtain formally

$$(3.15)\quad t^\alpha\chi_0 * t^\beta\chi_\infty = \frac{x^\alpha}{\alpha - \beta}\chi_0 + \frac{x^\beta}{\alpha - \beta}\chi_\infty$$

$$(3.16)\quad t^\alpha\chi_\infty * t^\beta\chi_0 = -\frac{x^\alpha}{\alpha - \beta}\chi_\infty - \frac{x^\beta}{\alpha - \beta}\chi_0.$$

These indeed hold, and are proved in the following lemma.

LEMMA. 3. *For all $\phi \in \mathscr{D}(\mathbf{R}^+)$ and $\alpha \neq \beta$, we have*

$$(3.17)\quad \langle t^\alpha\chi_0 * t^\beta\chi_\infty, \phi\rangle = \frac{1}{\alpha - \beta}\langle x^\alpha\chi_0, \phi\rangle + \frac{1}{\alpha - \beta}\langle x^\beta\chi_\infty, \phi\rangle$$

$$(3.18)\quad \langle t^\alpha\chi_\infty * t^\beta\chi_0, \phi\rangle = -\frac{1}{\alpha - \beta}\langle x^\alpha\chi_\infty, \phi\rangle - \frac{1}{\alpha - \beta}\langle x^\beta\chi_0, \phi\rangle.$$

Proof. By definition,

$$(3.19)\quad \langle t^\alpha\chi_0 * t^\beta\chi_\infty, \phi\rangle = \int_0^1 x^{\alpha-\beta-1}\int_x^\infty u^{\beta-1}\,\phi(u)du dx$$

with the definite integral on the right being understood in the sense of Section 2. The integrand in (3.19) is clearly equal to

$$(3.20)\quad M[\phi, \beta]x^{\alpha-\beta-1} - x^{\alpha-\beta-1}\int_0^x u^{\beta-1}\phi(u)du,$$

and the last term is actually zero for small values of x. Thus we may regard (3.20) as an asymptotic expansion of the integrand in (3.19). By (2.24), we have

$$(3.21)\quad \langle t^\alpha\chi_0 * t^\beta\chi_\infty, \phi\rangle = \frac{1}{\alpha - \beta}M[\phi; \beta]$$

$$- \int_0^1 x^{\alpha-\beta-1} \int_0^x u^{\beta-1}\phi(u)dudx.$$

The last integral is absolutely convergent and, upon reversing the order of integration, is equal to

$$\frac{1}{\alpha - \beta} \int_0^1 u^{\beta-1}\phi(u)du - \frac{1}{\alpha - \beta} \int_0^1 u^{\alpha-1}\phi(u)du.$$

Therefore

$$\begin{aligned}\langle t^\alpha\chi_0 * t^\beta\chi_\infty, \phi\rangle &= \frac{1}{\alpha - \beta} \int_0^1 u^{\alpha-1}\phi(u)du \\ &\quad + \frac{1}{\alpha - \beta} \int_1^\infty u^{\beta-1}\phi(u)du,\end{aligned}$$

which is of course equivalent to (3.17).

A similar argument applies to (3.18).

LEMMA 4. *For any complex number α, the following identities hold as functionals*:

(3.22) $\quad t^\alpha\chi_0 * t^\alpha\chi_0 = -x^\alpha(\log x)\chi_0,$

(3.23) $\quad t^\alpha\chi_\infty * t^\alpha\chi_\infty = x^\alpha(\log x)\chi_\infty,$

(3.24) $\quad t^\alpha\chi_0 * t^\alpha\chi_\infty = x^\alpha(\log x)\chi_0,$

(3.25) $\quad t^\alpha\chi_\infty * t^\alpha\chi_0 = -x^\alpha(\log x)\chi_\infty.$

Proof. The first two convolutions exist as ordinary integrals, and hence (3.22) and (3.23) can be proved in a straightforward manner.

Now consider the identity in (3.24). By definition

(3.26) $$\langle t^\alpha\chi_0 * t^\alpha\chi_\infty, \phi\rangle = \int_0^1 \left\{x^{-1} \int_x^\infty t^{\alpha-1}\phi(t)dt\right\}dx,$$

where the definite integral on the right hand side is understood in the sense of Definition 2. The function inside the bracket can be written as

$$M[\phi;\, \alpha]\, x^{-1} - x^{-1} \int_0^x t^{\alpha-1}\phi(t)dt.$$

As in (3.20), the second term here is zero for small values of x. Applying the definition (2.24) to the integral in (3.26), we have

$$\langle t^\alpha\chi_0 * t^\alpha\chi_\infty, \phi\rangle = -\int_0^1 \left\{x^{-1} \int_0^x t^{\alpha-1}\phi(t)dt\right\}dx.$$

(Note that, by (2.13), $\int_0^1 x^{-1}\, dx = 0$.) Reversing the order of integration gives

$$\langle t^\alpha\chi_0 * t^\alpha\chi_\infty, \phi\rangle = \int_0^1 t^{\alpha-1}(\log t)\phi(t)dt,$$

which is exactly the statement in (3.24). A similar reasoning applies to (3.25).

Note that coupling equations (3.24) and (3.25), we have an explicit example showing the non-commutativity of the convolution $*$ defined in (3.11).

The results in Lemmas 3 and 4 can all be extended to allow logarithmic terms of the form $(\log t)^n$, n being a positive integer. Since these formulas are rather complicated to state and their proofs are essentially the same as those previously given, we mention only the following two identities. For any $\alpha \neq \beta$, we have

$$t^\alpha(\log t)^n\chi_0 * t^\beta(\log t)^m\chi_\infty$$

$$(3.27)\quad = \sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l(m+l)!}{(\alpha-\beta)^{m+l+1}}\, x^\alpha(\log x)^{n-l}\chi_0 + (-1)^n\sum_{k=0}^{m}\binom{m}{k}\frac{(n+k)!}{(\alpha-\beta)^{n+k+1}}\, x^\beta(\log x)^{m-k}\chi_\infty;$$

and for all α, we have

$$(3.28)\quad t^\alpha(\log t)^n\chi_0 * t^\alpha(\log t)^m\chi_\infty = C_{nm}x^\alpha(\log x)^{n+m+1}\chi_0,$$

where

$$(3.29)\quad C_{nm} = \sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{n+k+1}.$$

These identities of course hold only in the sense of functionals.

Equation (3.27) can be formally obtained from (3.15) by differentiating both sides with respect to α and β. A rigorous proof can be based on the fact that from (3.21), we have

$$\langle t^\alpha(\log t)^n\chi_0 * t^\beta(\log t)^m\chi_\infty, \phi\rangle = D_\alpha^n D_\beta^m\langle t^\alpha\chi_0 * t^\beta\chi_\infty, \phi\rangle,$$

where D_α and D_β denote the partial derivatives with respect to α and β, respectively. Equation (3.27) then follows at once from (3.17). Formula (3.28) can be derived directly from the definition of generalized convolutions. In a similar manner, one can state and prove the corresponding results for $t^\alpha(\log t)^n\chi_\infty * t^\beta(\log t)^m\chi_0$ and $t^\alpha(\log t)^n\chi_\infty * t^\alpha(\log t)^m\chi_0$.

It is important to observe that the generalized convolution satisfies the distributive laws, that is, for any constants c_1, c_2, d_1 and d_2, and any f_1, f_2, f, g_1, g_2 and g in $\mathscr{F}$, we have

$$(3.30)\quad (c_1f_1 + c_2f_2) * g = c_1f_1 * g + c_2f_2 * g$$

$$(3.31)\quad f * (d_1g_1 + d_2g_2) = d_1f * g_1 + d_2f * g_2.$$

Thus, by writing $t^\alpha = t^\alpha\chi_0 + t^\alpha\chi_\infty$ and $t^\beta = t^\beta\chi_0 + t^\beta\chi_\infty$, we immediately obtain

$$(3.32)\quad t^\alpha * t^\beta = 0$$

for all α and β. The following generalization is proved in the same manner.

LEMMA 5. *For any complex numbers α and β, and for any non-negative integers n and m, we have*

$$(3.33)\quad t^\alpha(\log t)^n * t^\beta(\log t)^m = 0.$$

The following identities are also worth noting. For all $\alpha \neq \beta$ and all non-negative integers m and n, we have

$$(3.34)\quad t^\alpha(\log t)^n\chi_0 * t^\beta(\log t)^m = t^\beta(\log t)^m * t^\alpha(\log t)^n\chi_0$$

and

$$(3.35)\quad t^\alpha(\log t)^n\chi_\infty * t^\beta(\log t)^m = t^\beta(\log t)^m * t^\alpha(\log t)^n\chi_\infty.$$

That is, if $\alpha \neq \beta$ then "powers commute with truncated powers". However, these are not true in the case when $\alpha = \beta$; for instance, it is easily shown that

$$(3.36)\quad t^\alpha(\log t)^m\chi_0 * t^\alpha(\log t)^n = 0,$$

whereas

$$(3.37)\quad t^\alpha(\log t)^n * t^\alpha(\log t)^m\chi_0 = -C_{nm}x^\alpha(\log x)^{n+m+1},$$

C_{nm} being the constant given in (3.29).

The following results are needed in the next section.

LEMMA 6. *Let $f \in \mathscr{F}$ and let its asymptotic expansions be as given in (2.1)-(2.2). If $\lambda \neq \alpha_s$ and $\lambda \neq -\nu_s$ for all $s = 0, 1, 2, \ldots$, then for any non-negative integer k, we have*

$$(3.38)\quad \begin{aligned} t^\lambda(\log t)^k * f &= f * t^\lambda(\log t)^k \\ &= \sum_{j=0}^{k} \binom{k}{j}(-1)^j M^{(j)}[f; -\lambda]x^\lambda(\log x)^{k-j} \\ &= \left(\frac{d}{d\lambda}\right)^k\{M[f; -\lambda]x^\lambda\}. \end{aligned}$$

Proof. We write $f = f\chi_0 + f\chi_\infty$. From (2.23) it follows that

$$(3.39)\quad f\chi_0 = \sum_{s=0}^{n-1}\sum_{r=0}^{N(s)} a_{rs}t^{\alpha_s}(\log t)^r\chi_0 + f_{0,n}\chi_0.$$

If n is sufficiently large so that $\operatorname{Re} \alpha_n > \operatorname{Re} \lambda$, then it is easy to see that the convolution $t^\lambda(\log t)^k * f_{0,n}\chi_0$ exists as an ordinary integral and hence that the two factors commute. This, together with (3.34) and (3.39), immediately gives

(3.40) $\quad t^\lambda(\log t)^k * f\chi_0 = f\chi_0 * t^\lambda(\log t)^k.$

A similar argument leads to

(3.41) $\quad t^\lambda(\log t)^k * f\chi_\infty = f\chi_\infty * t^\lambda(\log t)^k.$

The first equality in (3.38) now follows from (3.40) and (3.41).

To prove the second equality in (3.38), we note that with $g(t) = t^\lambda(\log t)^k$ and ϕ a test function on $(0, \infty)$,

$$\Phi_g(x) = \sum_{j=0}^{k} \binom{k}{j} M^{(k-j)}[\phi; \lambda] x^{-\lambda}(-\log x)^j.$$

In view of (2.34), we also have by Definition 4

(3.42) $$\langle f * t^\lambda(\log t)^k, \phi\rangle = \sum_{j=0}^{k} \binom{k}{j} (-1)^j M^{(k-j)}[\phi; \lambda] M^{(j)}[f; -\lambda].$$

The right side of (3.42) can clearly be written as

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^j M^{(j)}[f; -\lambda] \langle x^\lambda(\log x)^{k-j}, \phi\rangle.$$

Thus, as functionals,

$$f * t^\lambda(\log t)^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^j M^{(j)}[f; -\lambda] x^\lambda(\log x)^{k-j}.$$

This completes the proof.

The corresponding result when λ is one of the values α_s, $-\nu_s$, is as follows; see the end of Section 2 for the definition of A_λ.

LEMMA 7. *If* $\lambda = \alpha_i$ *for some* i *and* $\lambda \neq -\nu_s$ *for all* s *then*

(3.43) $$f * t^\lambda(\log t)^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^j A_{-\lambda}^{(j)}[f; -\lambda] x^\lambda(\log x)^{k-j},$$

and

(3.44) $$t^\lambda(\log t)^k * f = f * t^\lambda(\log t)^k - \sum_{r=0}^{N(i)} a_{ri} C_{kr} x^\lambda(\log x)^{k+r+1},$$

where C_{kr} *is the constant given in* (3.29).

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LEMMA 8. *If* $\lambda = -\nu_i$ *for some* i *and* $\lambda \neq \alpha_s$ *for all* s *then*

$$(3.45)\quad f * t^{\lambda}(\log t)^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^j A_{-\lambda}^{(j)}[f; -\lambda] x^{\lambda}(\log x)^{k-j},$$

and

$$(3.46)\quad t^{\lambda}(\log t)^k * f = f * t^{\lambda}(\log t)^k + \sum_{r=0}^{Q(i)} c_{ri} C_{kr} x^{\lambda}(\log x)^{k+r+1}.$$

The proofs of (3.43) and (3.45) are similar to the last part of the proof of Lemma 6, using (2.35) in place of (2.34). Equations (3.44) and (3.46) follow from (3.43) and (3.45), respectively, using (3.36) and (3.37) or the corresponding results for convolutions of $t^{\alpha}(\log t)^n$ and $t^{\alpha}(\log t)^m \chi_{\infty}$. For instance, to obtain (3.44) from (3.43), we put

$$f_1(t) = f(t) - \sum_{r=0}^{N(i)} a_{ri} t^{\alpha_i}(\log t)^r \chi_0(t),$$

note that with $\lambda = \alpha_i$, Lemma 6 applies to f_1, and use (3.36) and (3.37).

There is, of course, a similar result for cases where $\alpha_i = -\nu_j$ for some i and j.

Example 2. As a simple illustration, let us calculate the convolutions $e^{-t} * t^n$ and $e^{-t} * t^n(\log t)$ for any non-negative integer n. With $f = e^{-t}$ and $\alpha_n = n$, the function $A_{-n}[f; z]$ in (2.33) becomes

$$A_{-n}[e^{-t}; z] = \Gamma(z) - \frac{(-1)^n}{n!} \frac{1}{z+n}.$$

The result [**4**, Section 1.17]

$$(3.47)\quad \Gamma(z) = \frac{(-1)^n}{n!} \left\{ \frac{1}{z+n} + \psi(n+1) + \frac{1}{2}(z+n)\left[\frac{\pi^2}{3} + \psi^2(n+1) - \psi'(n+1)\right] + O((z+n)^2) \right\}$$

then gives

$$A_{-n}[e^{-t}; -n] = \frac{(-1)^n}{n!}\psi(n+1)$$

and

$$A_{-n}^{(1)}[e^{-t}; -n] = \frac{(-1)^n}{2 \cdot n!} \left[\frac{\pi^2}{3} + \psi^2(n+1) - \psi'(n+1)\right],$$

where ψ is the logarithmic derivative of the Γ-function. From (3.43), it follows that

$$e^{-t} * t^n = \frac{(-1)^n}{n!}\psi(n + 1)x^n$$

and

$$e^{-t} * t^n \log t = -\frac{(-1)^n}{2 \cdot n!}\left[\frac{\pi^2}{3} + \psi^2(n + 1) - \psi'(n + 1)\right]x^n + \frac{(-1)^n}{n!}\psi(n + 1)x^n \log x.$$

From (3.44), we also have

$$t^n * e^{-t} = \frac{(-1)^n}{n!}[\psi(n + 1) - \log x]x^n$$

and

$$t^n \log t * e^{-t} = \frac{(-1)^n}{n!}\left\{-\frac{1}{2}\log^2 x + \psi(n + 1)\log x - \frac{1}{2}\left[\frac{\pi^2}{3} + \psi^2(n + 1) - \psi'(n + 1)\right]\right\}x^n.$$

Example 3. A slightly more complicated example is provided by $K_1 * t^s$, where s is any non-negative integer and K_1 is the modified Bessel function given by

$$K_1(t) = \frac{1}{t} + \frac{1}{2}\sum_{k=0}^{\infty}\frac{(t/2)^{2k+1}}{k!(k + 1)!} \times \left[2\log\frac{t}{2} - \psi(k + 1) - \psi(k + 2)\right]. \tag{3.48}$$

It is well known [**22**, p. 202] that

$$K_1(t) \sim \left(\frac{\pi}{2t}\right)^{1/2} e^{-t}\left[1 + \frac{3}{8}t^{-1} + \dots\right], \tag{3.49}$$

as $t \to +\infty$, and [**22**, p. 388] that

$$M[K_1; z] = 2^{z-2}\Gamma\left(\frac{z}{2} - \frac{1}{2}\right)\Gamma\left(\frac{z}{2} + \frac{1}{2}\right).$$

Thus, from (3.38), we have

$$K_1 * t^{2n} = 2^{-2n-2}\Gamma\left(-n - \frac{1}{2}\right)\Gamma\left(-n + \frac{1}{2}\right)x^{2n}.$$

For the odd exponents, we use (3.43). With $\lambda = 2n + 1$, we obtain

$$K_1 * t^{2n+1} = A_{-2n-1}[K_1; -2n - 1]x^{2n+1},$$

where the constant can be calculated as follows. In view of (3.48), we have from (2.33)

$$A_{-2n-1}[K_1; z] = M[K_1; z] + \frac{1}{2^{2n+1}n!(n+1)!}\frac{1}{(z+2n+1)^2}$$
$$+ \frac{[2\log 2 + \psi(n+1) + \psi(n+2)]}{2^{2n+2}n!(n+1)!}\frac{1}{(z+2n+1)}.$$

The Laurent expansion (3.47) then leads to

$$A_{-2n-1}[K_1; -2n - 1]$$
$$= -\frac{2^{-2n-2}}{n!(n+1)!}\left\{\log^2 2 + \log 2\,[\psi(n+1) + \psi(n+2)]\right.$$
$$+ \frac{1}{2^2}\left[2\psi(n+1)\psi(n+2) + \left(\frac{\pi^2}{3} + \psi^2(n+1) - \psi'(n+1)\right)\right.$$
$$\left.\left. + \left(\frac{\pi^2}{3} + \psi^2(n+2) - \psi'(n+2)\right)\right]\right\}.$$

4. Asymptotic expansions of convolutions. Let f and g be members of $\mathscr{F}$. We shall now study the behavior of $(f * g)(x)$ for both small and large values of x. In particular, we shall show that $f * g$ has asymptotic expansions similar to those of f and g.

THEOREM 1. *Let f and g belong to $\mathscr{F}$, and let their asymptotic expansions be given by* (2.1) - (2.2) *and* (3.4) - (3.5). *If for some m and n the exponents in these expansions satisfy the relations*

(4.1) $\quad \operatorname{Re}\delta_m > \operatorname{Re}\alpha_{n-1} > -\operatorname{Re}\nu_0$

and

(4.2) $\quad \operatorname{Re}\alpha_n > \operatorname{Re}\delta_{m-1} > -\operatorname{Re}\beta_0,$

then

$$f * g = \sum_{s=0}^{n-1}\sum_{r=0}^{N(s)} a_{rs}t^{\alpha_s}(\log t)^r * g$$
$$+ \sum_{s=0}^{m-1}\sum_{r=0}^{M(s)} b_{rs}f * t^{\delta_s}(\log t)^r + f_{0,n} * g_{0,m}, \tag{4.3}$$

where the convolution $f_{0,n} * g_{0,m}$ exists in the ordinary sense, and, with

$$\rho_{mn} = \min\{\operatorname{Re}\alpha_n, \operatorname{Re}\delta_m\},$$

satisfies

$$(f_{0,n} * g_{0,m})(x) = O(x^{\rho_{mn}}(\log x)^{K_{mn}})$$

as $x \to 0+$, *for some non-negative integer* K_{mn}.

Note that Lemmas 6 through 8 imply that each of the terms under the double sums in (4.3) is a regular distribution. Thus, Theorem 1 shows that for f and g in $\mathscr{F}$ such that (4.1) and (4.2) holds for some m and n, the convolution $f * g$ is a regular distribution.

Proof. In view of (3.33), it follows immediately from (2.23) and (3.8) that

$$\begin{aligned} f * g = &\sum_{s=0}^{n-1} \sum_{r=0}^{N(s)} a_{rs} t^{\alpha_s} (\log t)^r * g_{0,m} \\ &+ \sum_{s=0}^{m-1} \sum_{r=0}^{M(s)} b_{rs} f_{0,n} * t^{\delta_s} (\log t)^r + f_{0,n} * g_{0,m}. \end{aligned} \tag{4.4}$$

A further application of (3.33) simplifies (4.4) to

$$\begin{aligned} f * g = &\sum_{s=0}^{n-1} \sum_{r=0}^{N(s)} a_{rs} t^{\alpha_s} (\log t)^r * g \\ &+ \sum_{s=0}^{m-1} \sum_{r=0}^{M(s)} b_{rs} f * t^{\delta_s} (\log t)^r + f_{0,n} * g_{0,m}. \end{aligned} \tag{4.5}$$

All the convolutions under the double sums have been calculated in Lemmas 6-8, and it remains only to consider the remainder $f_{0,n} * g_{0,m}$. We first note that under the conditions (4.1) and (4.2), $f_{0,n} * g_{0,m}$ actually exists as an ordinary integral. Next we write

$$f_{0,n} = f_{0,n}\chi_0 + f_{0,n}\chi_\infty$$

and

$$g_{0,m} = g_{0,m}\chi_0 + g_{0,m}\chi_\infty.$$

For $x < 1$, it is clear that

$$f_{0,n}\chi_\infty * g_{0,m}\chi_\infty = 0. \tag{4.6}$$

Now recall from (2.23) that

$$f_{0,n}(t) = O(t^{\alpha_n}(\log t)^{N(n)}) \quad \text{as } t \to 0^+,$$

and that

$$f_{0,n}(t) = O(t^{\alpha_{n-1}}(\log t)^{N(n-1)}) \quad \text{as } t \to +\infty,$$

if Re $\alpha_{n-1} > -$Re ν_0. Similarly,

$$g_{0,m}(t) = O(t^{\delta_m}(\log t)^{M(m)}) \quad \text{as } t \to 0^+,$$

and

$$g_{0,m}(t) = O(t^{\delta_{m-1}}(\log t)^{M(m-1)}) \quad \text{as } t \to +\infty,$$

if Re $\delta_{m-1} > -$Re β_0. Simple estimates then show that as $x \to 0^+$,

$$(4.7) \quad f_{0,n}\chi_\infty * g_{0,m}\chi_0 = O(x^{\delta_m}(\log x)^{M(m)})$$

$$(4.8) \quad f_{0,n}\chi_0 * g_{0,m}\chi_\infty = O(x^{\alpha_n}(\log x)^{M_{mn}})$$

and

$$(4.9) \quad f_{0,n}\chi_0 * g_{0,m}\chi_0 = O(x^{\rho_{mn}}(\log x)^{N_{mn}}),$$

where $M_{mn} = M(m - 1) + N(n)$ and $N_{mn} = M(m) + N(n)$ or $M(m) + N(n) + 1$ depending upon whether α_n is not or is equal to δ_m. Adding up the results in (4.6) - (4.9) gives

$$(4.10) \quad f_{0,n} * g_{0,m} = O(x^{\rho_{mn}}(\log x)^{K_{mn}})$$

with $K_{mn} = \max \{M_{mn}, N_{mn}\}$. The final expansion (4.3) now follows from (4.5) and (4.10).

Note that under the conditions of (4.1) and (4.2), ρ_{mn} tends to infinity as $n, m \to \infty$. Furthermore, $\{\rho_{nn}\}$ is a strictly increasing sequence. The next result is proved in exactly the same manner, or can be deduced from Theorem 1 by using $f(1/t)$ and $g(1/t)$.

THEOREM 2. *Let f and g be given as in Theorem* 1. *If for some m and n the exponents in the expansions of f and g satisfy the relations*

$$(4.11) \quad \text{Re } \beta_m > \text{Re } \nu_{n-1} > -\text{Re } \alpha_0$$

and

$$(4.12) \quad \text{Re } \nu_n > \text{Re } \beta_{m-1} > -\text{Re } \delta_0,$$

then

$$(4.13) \quad \begin{aligned} f * g &= \sum_{s=0}^{n-1} \sum_{r=0}^{Q(s)} c_{rs} t^{-\nu_s}(\log t)^r * g \\ &\quad + \sum_{s=0}^{m-1} \sum_{r=0}^{P(s)} d_{rs} f * t^{-\beta_s}(\log t)^r + f_{\infty,n} * g_{\infty,m} \end{aligned}$$

where $f_{\infty,n} * g_{\infty,m}$ exists in the ordinary sense, and, with

$$\lambda_{mn} = \min \{\text{Re } \nu_n, \text{Re } \beta_m\},$$

satisfies

$$(f_{\infty,n} * g_{\infty,m})(x) = O(x^{-\lambda_{mn}}(\log x)^{L_{mn}}) \quad as\ x \to \infty,$$

for some non-negative integer L_{mn}.

If we let $P_{mn} = P(m-1) + Q(n)$ and $Q_{mn} = P(m) + Q(n)$ or $P(m) + Q(n) + 1$ depending upon whether ν_n is not or is equal to β_m, then L_{mn} in (4.13) is explicitly given by

$$L_{mn} = \max\{P_{mn}, Q_{mn}\}.$$

As with Theorem 1, Theorem 2 shows that for certain pairs f, g in $\mathscr{F}$, the convolution $f * g$ is a regular distribution. Actually, combining Theorems 1 and 2 with an elementary fact about sequences gives the following result.

THEOREM 3. *If f and g belong to $\mathscr{F}$, then so does $f * g$.*

Proof. Because of Theorems 1 and 2, it is enough to prove the following fact: if $\{\alpha_i\}$ and $\{\delta_j\}$ are real sequences, both strictly increasing to infinity, and if ν and β are real numbers, then there are integers n and m such that $\delta_m > \alpha_{n-1} > \nu$ and $\alpha_n > \delta_{m-1} > \beta$. This implies that integers n, m satisfying the hypotheses of either Theorem 1 or Theorem 2 can be chosen arbitrarily large, so that $f * g$ is regular and has the required asymptotic expansions; see the remark following the proof of Theorem 1.

To prove the fact mentioned in the last paragraph, we proceed as follows. For each $i > 0$, let $j = j(i)$ be the least integer such that $\delta_{j-1} \geq \alpha_{i-1}$; note that $j(i)$ is well-defined, since $\{\delta_j\} \uparrow \infty$. Since both sequences increase to infinity, we can ensure both $\alpha_{i-1} > \nu$ and $\delta_{j(i)-1} > \beta$ by taking i sufficiently large. Since $\{\alpha_i\} \uparrow \infty$, there is a least integer $k \geq 0$ such that $\alpha_{i+k} > \delta_{j(i)-1}$. We claim that $n = i + k$ and $m = j(i)$ satisfy all requirements. For, by choice of k, i, and $j(i)$, we have

$$\nu < \alpha_{i-1} \leq \alpha_{i+k-1} \leq \delta_{j(i)-1} < \delta_{j(i)}, \quad \text{and}$$

$$\beta < \delta_{j(i)-1} < \alpha_{i+k}.$$

This completes the proof of the theorem.

The results in Theorems 1 and 2 can be considerably simplified, if the logarithmic terms in the asymptotic expansions of f and g are all absent and, furthermore, if the exponents in these expansions are of a particular form. To be more specific, we assume that the asymptotic expansions of f are given by

$$(4.14) \quad f(t) \sim \sum_{s=0}^{\infty} a_s t^{s+\alpha}, \quad \text{as } t \to 0^+,$$

$$(4.15)\quad f(t) \sim \sum_{s=0}^{\infty} c_s t^{-s-\nu}, \quad \text{as } t \to +\infty,$$

and the corresponding expansion of g by

$$(4.16)\quad g(t) \sim \sum_{s=0}^{\infty} b_s t^{s+\delta}, \quad \text{as } t \to 0^+,$$

$$(4.17)\quad g(t) \sim \sum_{s=0}^{\infty} d_s t^{-s-\beta}, \quad \text{as } t \to +\infty,$$

where the exponents α, β, ν and δ are arbitrary complex numbers.

THEOREM 4. *Let f and g be given as in* (4.14) - (4.17), *and suppose for all non-negative integers r and s,*

$$(4.18)\quad \alpha + s \neq \delta + r, \alpha + s \neq -\beta - r, \quad \textit{and}$$
$$\delta + s \neq -\nu - r.$$

Then for sufficiently large n and m satisfying

$$(4.19)\quad m + \operatorname{Re}(\delta - \alpha) - 1 < n < m + \operatorname{Re}(\delta - \alpha) + 1,$$

we have

$$(4.20)\quad \begin{aligned} f * g &= \sum_{s=0}^{n-1} a_s M[g; -\alpha - s] x^{s+\alpha} \\ &\quad + \sum_{s=0}^{m-1} b_s M[f; -\delta - s] x^{s+\delta} + O(x^{\rho_{mn}}) \end{aligned}$$

as $x \to 0^+$, where ρ_{mn} $= \min \{n + \operatorname{Re}\alpha, m + \operatorname{Re}\delta\}$.

Proof. Choose $n > 1 - \operatorname{Re}(\alpha + \nu)$ and $m > 1 - \operatorname{Re}(\beta + \delta)$. The inequalities in (4.19) then guarantee that conditions (4.1) and (4.2) are met. Since the logarithmic terms in the asymptotic expansions are all absent, and since $\alpha + s - \delta - r$ and $\alpha + s + \beta + r$ are never zero, the non-negative integer K_{mn} is actually zero. Thus by Theorem 1

$$f * g = \sum_{s=0}^{n-1} a_s t^{s+\alpha} * g + \sum_{s=0}^{m-1} b_s f * t^{s+\delta} + O(x^{\rho_{mn}}),$$

as $x \to 0^+$. The desired result now follows from Lemma 6.

Note that the last two conditions in (4.18) are automatically satisfied, if $\operatorname{Re}(\alpha + \beta) > 0$ and $\operatorname{Re}(\delta + \nu) > 0$, in which case the convolution $f * g$ actually exists as an ordinary integral.

THEOREM 5. *Let f and g be given as in Theorem* 4. *If $\alpha + i = \delta + j$ for some i and j, and if $\alpha + s \neq -\beta - r$ and $\delta + s \neq -\nu - r$ for all s and r, then for sufficiently large n and m satisfying* (4.19) *we have*

$$
\begin{aligned}
f * g = & \sum_{0 \leq s < \delta - \alpha} a_s M[g; -s - \alpha] x^{s+\alpha} \\
& + \sum_{s \geq \delta - \alpha}^{n-1} a_s [b_s^* - b_{\alpha - \delta + s} \log x] x^{s+\alpha} \\
& + \sum_{0 \leq s < \alpha - \delta} b_s M[f; -s - \delta] x^{s+\delta} \\
& + \sum_{s \geq \alpha - \delta}^{m-1} a_s^* b_s x^{s+\delta} + O(x^{\rho_{mn}} \log x),
\end{aligned}
\tag{4.21}
$$

as $x \to 0^+$, where $\rho_{mn} = \min \{n + \operatorname{Re} \alpha, m + \operatorname{Re} \delta\}$,

$$a_s^* = \lim_{z \to -s-\delta} \left\{ M[f; z] - \frac{a_{\delta - \alpha + s}}{z + s + \delta} \right\} \tag{4.22}$$

and

$$b_s^* = \lim_{z \to -s-\alpha} \left\{ M[g; z] - \frac{b_{\alpha - \delta + s}}{z + s + \alpha} \right\}. \tag{4.23}$$

The coefficients a_s and b_s with negative subscripts are understood to be zero.

Proof. The proof proceeds exactly the same as in Theorem 4, except that here one uses both Lemmas 6 and 7. Since $n + \alpha = m + \delta$, the non-negative integer K_{mn} in (4.10) is equal to 1.

Note that if $\alpha = \delta$ then (4.21) simplifies to

$$f * g \sim \sum_{s=0}^{\infty} c_s^* x^{s+\alpha} - \sum_{s=0}^{\infty} a_s b_s x^{s+\alpha} (\log x), \tag{4.24}$$

as $x \to 0^+$, where $c_s^* = a_s b_s^* + a_s^* b_s$; the meaning of (4.24) being that whenever we terminate the two series, say after N terms in the first and M terms in the second, the error committed is of the order $O(x^{N+\alpha})$ plus $O(x^{M+\alpha}(\log x))$. The condition that m and n in (4.21) had to be sufficiently large can be dropped by a standard argument in asymptotics; see [**22**, pp. 197-198].

Theorems 4 and 5 are extensions of Theorems 1 and 2 given in [**23**] for the ordinary Mellin convolutions. Results similar to these theorems can of course be also obtained in the cases when $\alpha + i = -\beta - j$ or $\delta + i = -\nu - j$ for some i and j. We shall, however, omit the details; see Example 4 in Section 5.

To conclude this section, we include the following analogues for the large-x behavior. These results are deducible from Theorems 4 and 5, and will be used in later examples.

THEOREM 6. *Let f and g be given as in Theorem 4, and suppose that for all non-negative integers r and s,*

$$\nu + s \neq \beta + r, \nu + s \neq -\delta - r, \quad \text{and } \beta + s \neq -\alpha - r. \tag{4.25}$$

Then for sufficiently large n and m satisfying

$$m + \operatorname{Re}(\beta - \nu) - 1 < n < m + \operatorname{Re}(\beta - \nu) + 1, \tag{4.26}$$

we have

$$\begin{aligned} f * g = &\sum_{s=0}^{n-1} c_s M[g; s + \nu] x^{-s-\nu} \\ &+ \sum_{s=0}^{m-1} d_s M[f; s + \beta] x^{-s-\beta} + O(x^{-\lambda_{mn}}), \end{aligned}$$

as $x \to +\infty$, where $\lambda_{mn} = \min \{n + \operatorname{Re} \nu, m + \operatorname{Re} \beta\}$.

THEOREM 7. *Let f and g be given as in Theorem 4. If $\nu + i = \beta + j$ for some i and j, and if $\nu + s \neq -\delta - r$ and $\beta + s \neq -\alpha - r$ for all s and r, then for sufficiently large m and n satisfying* (4.26) *we have*

$$\begin{aligned} f * g = &\sum_{0 \leq s < \beta - \nu} c_s M[g; s + \nu] x^{-s-\nu} \\ &+ \sum_{s \geq \beta - \nu}^{n-1} c_s [d_s^* + d_{\nu-\beta+s} \log x] x^{-s-\nu} \\ &+ \sum_{0 \leq s < \nu - \beta} d_s M[f; s + \beta] x^{-s-\beta} \\ &+ \sum_{s \geq \nu - \beta}^{m-1} c_s^* d_s x^{-s-\beta} + O(x^{-\lambda_{mn}} \log x), \end{aligned} \tag{4.28}$$

where $\lambda_{mn} = \min \{n + \operatorname{Re} \nu, m + \operatorname{Re} \beta\}$,

$$c_s^* = \lim_{z \to \beta + s} \left\{ M[f; z] + \frac{c_{\beta-\nu+s}}{z - \beta - s} \right\} \tag{4.29}$$

$$d_s^* = \lim_{z \to \nu + s} \left\{ M[g; z] + \frac{d_{\nu-\beta+s}}{z - \nu - s} \right\}, \tag{4.30}$$

and the coefficients c_s and d_s with negative subscripts are understood to be zero.

As before, we note that if Re $(\nu + \delta) > 0$ and Re $(\beta + \alpha) > 0$ then $\nu + s \neq -\delta - r$ and $\beta + s \neq -\alpha - r$ for any pair (s, r), in which case we have $f * g$ existing as an ordinary convergent integral. Also note that if $\beta = \nu$ then (4.28) reduces to

$$(4.31)\quad f * g \sim \sum_{s=0}^{\infty} c_s d_s x^{-s-\nu} \log x + \sum_{s=0}^{\infty} (c_s d_s^* + c_s^* d_s) x^{-s-\nu},$$

as $x \to +\infty$; see the remark following (4.24).

5. Generalized integral transforms. As remarked in the introduction, a large number of classical integral transforms can be put in the form of an ordinary Mellin convolution. This suggests that extensions of these transforms to domains larger than their classical ones can be obtained by using the generalized Mellin convolution. Thus, for f and h in $\mathscr{F}$, we shall define the distribution H_f in $\mathbf{R}^+$ by

$$(5.1)\quad \langle H_f, \phi \rangle = \langle f_1 * h, \phi \rangle \quad \text{for all } \phi \in \mathscr{D}(\mathbf{R}^+),$$

where $f_1(t) = t^{-1} f(t^{-1})$, and denote it by

$$(5.2)\quad H_f(x) = \int_0^{\infty} f(t) h(xt) dt,$$

the integral being used purely in a formal sense. If the integral in (5.2) turns out to be absolutely convergent, then it is easily shown that the function defined by the integral indeed generates the distribution $H_f(x)$ defined by (5.1). We shall therefore call $H_f(x)$ the *generalized integral transform of f with kernel h*. Asymptotic behaviors of some of these generalized integral transforms will now be considered.

Generalized Laplace transform. This is the transform whose kernel is the exponential function, i.e., $h(t) = e^{-t}$. In view of its Maclaurin series and its behavior at infinity, it is clear that this function belongs to $\mathscr{F}$. Indeed, we have

$$e^{-t} \sim \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} t^s, \quad \text{as } t \to 0^+,$$

and

$$(5.3)\quad e^{-t} \sim \sum_{s=0}^{\infty} d_s t^{-\beta_s}, \quad \text{as } t \to \infty,$$

where the coefficients d_s are all zero and the exponents β_s can be chosen as desired. Now let f be a member of $\mathscr{F}$, and let $L_f(x)$ denote its generalized Laplace transform. By (5.1), $L_f(x)$ is defined by

(5.4) $\quad L_f(x) = f_1 * e^{-t}.$

Here, as before, $f_1(t) = t^{-1}f(t^{-1})$ and the convolution is understood in the generalized sense.

Let the asymptotic expansions of f be again given by (2.1) and (2.2), and choose the β_s's in (5.3) such that $\beta_s \neq \alpha_r + 1$ for all r and s, and (4.11) and (4.12) hold. If none of the exponents α_s in (2.1) is a negative integer then, in view of (5.4), we have by Theorem 2 and Lemma 6

$$L_f(x) \sim \sum_{s=0}^{\infty} \sum_{r=0}^{N(s)} a_{rs} D_s^r[\Gamma(\alpha_s + 1)x^{-\alpha_s - 1}] \tag{5.5}$$

as $x \to +\infty$, where $D_s = d/d\alpha_s$. If some of the exponents α_s are negative integers then, since Re $\alpha_s \to +\infty$, there are at most a finite number of them, say $\alpha_{s_1}, \ldots, \alpha_{s_n}$. On account of Lemma 7, we obtain

$$\begin{aligned} L_f(x) \sim \sum_{s=0}^{\infty}{}' \sum_{r=0}^{N(s)} (-1)^r a_{rs}\Bigg[&\sum_{j=0}^{r} \binom{r}{j} A_s^{(j)} x^{-\alpha_s - 1} (\log x)^{r-j} \\ &- \frac{1}{r+1} \frac{(-1)^{-\alpha_s - 1}}{(-\alpha_s - 1)!} x^{-\alpha_s - 1} (\log x)^{r+1} \Bigg] \\ + \sum_{s=0}^{\infty}{}'' \sum_{r=0}^{N(s)} a_{rs} D_s^r [\Gamma(\alpha_s + 1)&x^{-\alpha_s - 1}], \end{aligned} \tag{5.6}$$

as $x \to +\infty$, where

$$A_s^{(j)} = (-1)^j A_{\alpha_s + 1}^{(j)}[e^{-t}; \alpha_s + 1]. \tag{5.7}$$

In (5.6), $\sum'$ sums over only those s for which α_s is a negative integer and is hence finite, whereas $\sum''$ excludes exactly these s. The values of $A_s^{(j)}$ in (5.6) can be obtained as in Example 2. In particular,

$$A_s^{(0)} = \frac{(-1)^{-\alpha_s - 1}}{(-\alpha_s - 1)!} \psi(-\alpha_s)$$

and

$$A_s^{(1)} = \frac{1}{2} \frac{(-1)^{-\alpha_s - 1}}{(-\alpha_s - 1)!} \left[\frac{\pi^2}{3} + \psi^2(-\alpha_s) - \psi'(-\alpha_s) \right].$$

The results in (5.5) and (5.6) are extensions of Watson's lemma [**26**, Theorem 4.1] to generalized Laplace transforms. The leading terms in these expansions, as expected, agree with the Abelian theorems given earlier by Lavoine [**14**, Theorem 1], but our results could give much more detailed information when required.

To obtain the small-x behavior, let us suppose for simplicity that the asymptotic expansions of f are given by (4.14) and (4.15). If α is not a negative integer and ν is not an integer, then applying Theorem 4 to (5.4) gives

$$(5.8)\qquad L_f(x) \sim \sum_{s=0}^{\infty} c_s\Gamma(1-\nu-s)x^{s+\nu-1} + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!}M[f;\,1+s]x^s,$$

as $x \to 0^+$. Here we have used the fact that

$$M[f_1;\,-s] = M[f;\,1+s].$$

If ν is a non-positive integer, say $\nu = -n$, but α is not a negative integer, we have from Theorem 5

$$(5.9)\qquad \begin{aligned} L_f(x) \sim {} & \sum_{s=0}^{n} c_s\Gamma(n+1-s)x^{s-n-1} \\ & + \sum_{s=n+1}^{\infty} \frac{(-1)^{s-n-1}}{(s-n-1)!}c_s[\psi(s-n) - \log x]x^{s-n-1} \\ & + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!}c_s^* x^s \end{aligned}$$

as $x \to 0^+$, where ψ is, as before, the logarithmic derivative of the Γ-function and

$$(5.10)\qquad c_s^* = \lim_{z\to -s}\left\{M[f;\,1-z] - \frac{c_{s+n+1}}{z+s}\right\}.$$

A corresponding result follows if ν is a positive integer. In this case, the result coincides with the one given in [**18**, Theorem 4]. A similar expansion can also be derived when α is a negative integer; see Example 4.

All these results are distributional extensions of those given in [**7**] for the conventional Laplace transforms. The possibility of such an extension was also suggested in that paper, but no details were given; see [**7**, p. 129]. We also note that the Abelian theorem (for small x) given in [**14**, Theorem 2] seems to be incorrect. This fact can be verified, even in the case of ordinary Laplace transforms, by comparing it with Theorem 1 in [**1**] or Theorem 4 in [**18**].

Example 4. Consider the generalized Laplace transform of $K_1(t)$, where $K_1(t)$ is the modified Bessel function given in (3.48). As suggested by (5.2), we may denote this transform by the formal integral

$$(5.11)\qquad I(x) = \int_0^{\infty} K_1(t)e^{-xt}dt.$$

With $\alpha_0 = -1$ and $\alpha_s = 2s - 1$ for $s \geq 1$, we have from (5.6)

$$
\begin{aligned}
I(x) \sim & -\gamma - \log x \\
& + 2 \sum_{k=0}^{\infty} \frac{\Gamma(2k+2)}{k!(k+1)!}[\psi(2k+2) - \log x](2x)^{-2k-2} \\
& - \sum_{k=0}^{\infty} \frac{\Gamma(2k+2)}{k!(k+1)!}[2 \log 2 \\
& + \psi(k+1) + \psi(k+2)](2x)^{-2k-2},
\end{aligned} \tag{5.12}
$$

as $x \to +\infty$. Note that $\psi(1) = -\gamma$.

For the small-x behavior, we note that neither (5.8) nor (5.9) holds, since here we have $\alpha = -1$. Nevertheless, we may appeal directly to Theorem 1. Recall that $I(x)$ is defined by (5.4) to be

$$I(x) = t^{-1}K_1(t^{-1}) * e^{-t}.$$

Since

$$t^{-1}K_1(t^{-1}) \sim \left(\frac{\pi}{2t}\right)^{1/2} e^{-1/t} \quad \text{as } t \to 0^+,$$

the first sum in (4.3) is absent. Furthermore, we have

$$t^{-1}K_1(t^{-1}) * 1 = \log 2 - \gamma$$

by Lemma 8 and

$$t^{-1}K_1(t^{-1}) * t^s = 2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(1 + \frac{s}{2}\right)x^s, \quad s = 1, 2, \ldots,$$

by Lemma 6; cf. Example 3. Therefore, from (4.3), it follows that

$$I(x) \sim (\log 2 - \gamma) + \sum_{s=1}^{\infty} \frac{(-1)^s}{s!} 2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(1 + \frac{s}{2}\right)x^s, \tag{5.13}$$

as $x \to 0^+$.

To see that the above results are indeed what one would expect to have, let us rederive them in an alternative manner. Put

$$\mathscr{K}(t) = K_1(t) - t^{-1}. \tag{5.14}$$

In view of (3.48) and (3.49), the Laplace transform of $\mathscr{K}(t)$ exists as an ordinary integral. Hence, by Watson's lemma [**26**],

$$
\begin{aligned}
L_{\mathscr{K}}(x) \sim & \; 2 \sum_{k=0}^{\infty} \frac{\Gamma(2k+2)}{k!(k+1)!}[\psi(2k+2) - \log x](2x)^{-2k-2} \\
& - \sum_{k=0}^{\infty} \frac{\Gamma(2k+2)}{k!(k+1)!}[\log 4 + \psi(k+1) \\
& + \psi(k+2)](2x)^{-2k-2},
\end{aligned} \tag{5.15}
$$

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as $x \to +\infty$. Furthermore, it follows from [**1**, Theorem 1; 18, Theorem 4] that

$$(5.16)\quad L_{\mathscr{K}}(x) \sim (\log x + \log 2) + \sum_{s=1}^{\infty} \frac{(-1)^s}{s!} 2^{s-1} \Gamma\left(\frac{s}{2}\right) \Gamma\left(1 + \frac{s}{2}\right) x^s,$$

as $x \to 0^+$. We now recall the formula for the Laplace transform of the distribution t_+^{-1} as given in the earlier theory on this subject:

$$(5.17)\quad L_{t^{-1}}(x) = -\gamma - \log x;$$

see [**13**, p. 533; **18**, Eq. (5.5)]. Since this integral transform should be linear, the transform of K_1 must be the sum of the transforms of $\mathscr{K}$ and t^{-1}. By adding (5.17) to (5.15) and (5.16), we again obtain the expansions in (5.12) and (5.13).

It should be also noted that since $K_1(t) = -K_0'(t)$, it is tempting to define the Laplace transform of K_1 as

$$L_{K_1}(x) = \langle K_1, e^{-xt} \rangle = -\langle K_0', e^{-xt} \rangle$$
$$= -x\langle K_0, e^{-xt} \rangle = -x \int_0^{\infty} K_0(t) e^{-xt}\, dt.$$

Here $K_0(t)$ is the modified Bessel function of order zero. This definition, however, will lead to asymptotic expansions which differ from (5.12) and (5.13) by the constant $\log 2 - \gamma$. An explanation of this discrepancy is that $-K_0(t)$ is not the only antiderivative of $K_1(t)$.

Generalized Stieltjes transform. The conventional Stieltjes transform of a locally integrable function $f(t)$ on $(0, \infty)$ is defined by

$$(5.18)\quad S_f(x) = \int_0^{\infty} \frac{f(t)}{(t + x)^{\rho}} dt,$$

where ρ is a fixed real number, provided that the integral exists. Here we shall again assume that f belongs to the family $\mathscr{F}$.

If the above integral does not exist, then we shall view $S_f(x)$ as the convolution of $t^{1-\rho} f(t)$ and $(1 + t)^{-\rho}$. That is, for $\phi \in \mathscr{D}(\mathbf{R}^+)$, we define

$$(5.19)\quad \langle S_f, \phi \rangle = \langle f_{\rho} * (1 + t)^{-\rho}, \phi \rangle,$$

where $f_{\rho}(t) = t^{1-\rho} f(t)$ and the right hand side is understood in the sense of Definition 4.

To obtain the behavior of $S_f(x)$, we assume that f has the asymptotic expansions (4.14) and (4.15), and note that

$$(1 + t)^{-\rho} \sim \sum_{s=0}^{\infty} \frac{(-1)^s (\rho)_s}{s!} t^s, \quad \text{as } t \to 0^+,$$

$$(1 + t)^{-\rho} \sim \sum_{s=0}^{\infty} \frac{(-1)^s(\rho)_s}{s!} t^{-s-\rho}, \quad \text{as } t \to +\infty,$$

and

$$M[(1 + t)^{-\rho}; z] = \Gamma(z)\Gamma(\rho - z) / \Gamma(\rho),$$

where $(\rho)_s = \Gamma(\rho + s) / \Gamma(\rho)$.

If $\alpha - \rho$ is not an integer, and if $\nu + \rho - 1$ and $\alpha + 1$ are not negative integers or zero, then Theorem 4 gives

$$(5.20) \quad \begin{aligned} S_f(x) \sim & \sum_{s=0}^{\infty} a_s \frac{\Gamma(\rho - \alpha - 1 - s)\Gamma(s + \alpha + 1)}{\Gamma(\rho)} x^{s+\alpha+1-\rho} \\ & + \sum_{s=0}^{\infty} \frac{(-1)^s(\rho)_s}{s!} M[f; 1 - \rho - s]x^s, \end{aligned}$$

as $x \to 0^+$. If $\alpha - \rho$ is an integer, and if $\nu + \rho - 1$ and $\alpha + 1$ are not negative integers or zero, then by Theorem 5 we have

$$(5.21) \quad \begin{aligned} S_f(x) \sim & \sum_{s=0}^{\rho-\alpha-2} a_s \frac{\Gamma(\rho - \alpha - 1 - s)\Gamma(s + \alpha + 1)}{\Gamma(\rho)} x^{s+\alpha+1-\rho} \\ & - \sum_{s=\rho-\alpha-1}^{\infty} a_s \left[\frac{(-1)^{s+\alpha+1-\rho}}{(s + \alpha + 1 - \rho)!} (\rho)_{\alpha+1+s-\rho} \log x - b_s^* \right] \\ & \qquad \times x^{s+\alpha+1-\rho} \\ & + \sum_{s=0}^{\alpha-\rho} \frac{(-1)^s(\rho)_s}{s!} M[f; 1 - \rho - s]x^s \\ & + \sum_{s=\alpha+1-\rho}^{\infty} \frac{(-1)^s(\rho)_s}{s!} a_s^* x^s, \end{aligned}$$

as $x \to 0^+$, where

$$a_s^* = \lim_{z \to -s} \left\{ M[f; z - \rho + 1] - \frac{a_{s-\alpha-1+\rho}}{s + z} \right\}$$

and

$$b_s^* = \frac{(-1)^{s+\alpha+1-\rho}}{(s + \alpha + 1 - \rho)!} \frac{\Gamma(s + \alpha + 1)}{\Gamma(\rho)} [\psi(s + \alpha - \rho + 2) - \psi(s + \alpha + 1)].$$

In (5.21), the coefficients a_s and $(\rho)_s$ are taken to be zero when the subscripts s are negative.

Similarly, if $\nu - 1$ is not an integer, and if $\nu + \rho - 1$ and $\alpha + 1$ are not negative integers or zero, then Theorem 6 gives

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$$
\begin{aligned}
S_f(x) \sim \sum_{s=0}^{\infty} a_s \frac{\Gamma(s+\nu+\rho-1)\Gamma(1-s-\nu)}{\Gamma(\rho)} x^{1-s-\nu-\rho} \\
+ \sum_{s=0}^{\infty} \frac{(-1)^s(\rho)_s}{s!} M[f;\, s+1] x^{-s-\rho}
\end{aligned}
\tag{5.22}
$$

as $x \to +\infty$. If $\nu - 1$ is an integer, but $\nu + \rho - 1$ and $\alpha + 1$ are not negative integers or zero, then we have from Theorem 7

$$
\begin{aligned}
S_f(x) &\sim \sum_{s=0}^{-\nu} c_s \frac{\Gamma(s+\nu+\rho-1)\Gamma(1-s-\nu)}{\Gamma(\rho)} x^{1-s-\nu-\rho} \\
&+ \sum_{s=1-\nu}^{\infty} c_s \left[\frac{(-1)^{s+\nu-1}}{(s+\nu-1)!} (\rho)_{s+\nu-1} \log x + d_s^* \right] x^{1-s-\nu-\rho} \\
&+ \sum_{s=0}^{\nu-2} \frac{(-1)^s}{s!} (\rho)_s M[f;\, s+1] x^{-s-\rho} \\
&+ \sum_{s=\nu-1}^{\infty} c_s^* \frac{(-1)^s}{s!} (\rho)_s\, x^{-s-\rho},
\end{aligned}
\tag{5.23}
$$

as $x \to +\infty$ where

$$c_s^* = \lim_{z \to s+\rho} \left\{ M[f;\, z+1-\rho] + \frac{c_{1-\nu+s}}{z-\rho-s} \right\}$$

and

$$
\begin{aligned}
d_s^* = \frac{(-1)^{\nu-1+s}}{(\nu-1+s)!} \frac{\Gamma(s+\nu+\rho-1)}{\Gamma(\rho)} \\
\times \{\psi(\nu+s) - \psi(s+\nu+\rho-1)\},
\end{aligned}
$$

the coefficients c_s and $(\rho)_s$ in (5.23) again being taken to be zero when s is a negative integer.

The above results should be compared with some of those in [**15**, **16**], where only the leading terms in the expansions (5.20) - (5.23) are given; see also Example 1 above. For asymptotic expansions of the ordinary Stieltjes transform, we refer to [**8**], [**18**] and [**23**].

Example 5. Consider the formal integral

$$H(x) = \int_0^{\infty} \frac{|1-t|^{3/2}}{(x+t)^2} dt,$$

which is used only symbolically to represent the Stieltjes transform of $|1 - t|^{3/2}$ defined by (5.19) with $\rho = 2$. The first three terms in ascending powers of x in (5.21) give

$$H(x) \sim \frac{1}{x} + \frac{3}{2}\log x + \frac{3}{2}\left[\psi\left(\frac{3}{2}\right) - \pi + \gamma\right],$$

as $x \to 0^+$. Similarly, from (5.22) we have

$$H(x) \sim -\frac{3}{2}\pi x^{1/2} - \frac{3}{4}\pi x^{-1/2} + \frac{3}{16}\pi x^{-3/2}$$

as $x \to +\infty$. The behavior of a similar but conventional integral has been obtained by Handelsman and Lew [**8**, p. 430].

Generalized fractional integral transform. The above technique can of course be repeated for the generalized fractional integral transform

$$(5.24)\quad I^\mu g(x) = \frac{1}{\Gamma(\mu)}\int_0^x (x - t)^{\mu-1} g(t)dt, \quad g \in \mathscr{F},$$

with the right-hand side being understood in the sense of the generalized Mellin convolution

$$(5.25)\quad \frac{x^\mu}{\Gamma(\mu)}(f * g)(x),$$

where

$$(5.26)\quad f(t) = \begin{cases} 0 & 0 < t \leq 1, \\ t^{-1}(1 - t^{-1})^{\mu-1} & t > 1. \end{cases}$$

Fractional integrals of generalized functions have been studied via a different approach by Erdélyi-McBride [**5**] and Erdélyi [**2**]. Their results have been further extended in [**17**]. For the asymptotic expansions of the conventional fractional integrals, see [**19**] and [**23**]. The results in Section 4 will enable us to write down the asymptotic expansions of the generalized fractional integrals at once.

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