

# 1

## Preliminaries

### 1.1 Notation

We denote by  $\mathcal{L}^n$  the Lebesgue measure in the Euclidean  $n$ -space  $\mathbb{R}^n$ . In a metric space  $X$ ,  $d(A)$  stands for the diameter of  $A$ ,  $d(A, B)$  the minimal distance between the sets  $A$  and  $B$ , and  $d(x, A)$  the distance from a point  $x$  to a set  $A$ . The closed ball with centre  $x \in X$  and radius  $r > 0$  is denoted by  $B(x, r)$  and the open ball by  $U(x, r)$ . In  $\mathbb{R}^n$  we sometimes denote  $B^n(x, r)$ . The unit sphere in  $\mathbb{R}^n$  is  $S^{n-1}$ . The Grassmannian manifold of linear  $m$ -dimensional subspaces of  $\mathbb{R}^n$  is  $G(n, m)$ . It is equipped with an orthogonally invariant Borel probability measure  $\gamma_{n,m}$ . For  $V \in G(n, m)$ , we denote by  $P_V$  the orthogonal projection onto  $V$ .

For  $A \subset X$ , we denote by  $\mathcal{M}(A)$  the set of non-zero finite Borel measures  $\mu$  on  $X$  with support  $\text{spt } \mu \subset A$ . We shall denote by  $f_{\#}\mu$  the push-forward of a measure  $\mu$  under a map  $f$ :  $f_{\#}\mu(A) = \mu(f^{-1}(A))$ . The restriction of  $\mu$  to a set  $A$  is defined by  $\mu \lfloor A(B) = \mu(A \cap B)$ . The notation  $\ll$  stands for absolute continuity.

The characteristic function of a set  $A$  is  $\chi_A$ . By the notation  $M \lesssim N$ , we mean that  $M \leq CN$  for some constant  $C$ . The dependence of  $C$  should be clear from the context. The notation  $M \sim N$  means that  $M \lesssim N$  and  $N \lesssim M$ . By  $c$  and  $C$ , we mean positive constants with obvious dependence on the related parameters.

### 1.2 Hausdorff Measures

For  $m \geq 0$ , the  $m$ -dimensional Hausdorff measure  $\mathcal{H}^m = \mathcal{H}_d^m$  in a metric space  $(X, d)$  is defined by

$$\mathcal{H}^m(A) = \liminf_{\delta \rightarrow 0} \left\{ \sum_{i=1}^{\infty} \alpha(m) 2^{-m} d(E_i)^m : A \subset \bigcup_{i=1}^{\infty} E_i, d(E_i) < \delta \right\}.$$

Then  $\mathcal{H}^0$  is the counting measure. Usually  $m$  will be a positive integer and then  $\alpha(m) = \mathcal{L}^m(B^m(0, 1))$ , from which it follows by the isodiametric inequality that  $\mathcal{H}^m = \mathcal{L}^m$  in  $\mathbb{R}^m$ . The isodiametric inequality says that among the subsets of  $\mathbb{R}^m$  with a given diameter, the ball has the largest volume; see, for example, [203, 2.10.33]. For non-integral values of  $m$  the choice of  $\alpha(m)$  does not really matter. We denote by  $\dim$  the Hausdorff dimension. The *spherical Hausdorff measure*  $S^m$  is defined in the same way but using only balls as covering sets.

The lower and upper  $m$ -densities of  $A \subset X$  are defined by

$$\Theta_*^m(A, x) = \liminf_{r \rightarrow 0} \alpha(m)^{-1} r^{-m} \mathcal{H}^m(A \cap B(x, r)),$$

$$\Theta^{*m}(A, x) = \limsup_{r \rightarrow 0} \alpha(m)^{-1} r^{-m} \mathcal{H}^m(A \cap B(x, r)).$$

The density  $\Theta^m(A, x)$  is defined as their common value if they are equal.

We have

**Theorem 1.1** *If  $A$  is  $\mathcal{H}^m$  measurable and  $\mathcal{H}^m(A) < \infty$ , then*

$$2^{-m} \leq \Theta^{*m}(A, x) \leq 1 \text{ for } \mathcal{H}^m \text{ almost all } x \in A,$$

$$\Theta^{*m}(A, x) = 0 \text{ for } \mathcal{H}^m \text{ almost all } x \in X \setminus A.$$

When  $m \leq 1$  the constant  $2^{-m}$  is sharp; for  $m > 1$  the best constant is not known.

We also have

**Theorem 1.2** *If  $A \subset X$  is  $\mathcal{H}^m$  measurable and  $\mathcal{H}^m(A) < \infty$ , then*

$$\limsup_{\delta \rightarrow 0} \{d(B)^{-m} \mathcal{H}^m(A \cap B) : x \in B, d(B) < \delta\} = 1 \text{ for } \mathcal{H}^m \text{ almost all } x \in A.$$

For general measures, we have

**Theorem 1.3** *Let  $\mu \in \mathcal{M}(X)$ ,  $A \subset X$ , and  $0 < \lambda < \infty$ .*

- (1) *If  $\Theta^{*m}(A, x) \leq \lambda$  for  $x \in A$ , then  $\mu(A) \leq 2^m \lambda \mathcal{H}^m(A)$ .*
- (2) *If  $\Theta^{*m}(A, x) \geq \lambda$  for  $x \in A$ , then  $\mu(A) \geq \lambda \mathcal{H}^m(A)$ .*

For the above results, see [203, 2.10.17–19], [190, Section 2.2] or [321, Chapter 6].

We say that a closed set  $E$  is AD- $m$ -regular (AD for Ahlfors and David) if there is a positive number  $C$  such that

$$r^m/C \leq \mathcal{H}^m(E \cap B(x, r)) \leq Cr^m \text{ for } x \in E, 0 < r < d(E).$$

A measure  $\mu$  is said to be AD- $m$ -regular if

$$r^m/C \leq \mu(B(x, r)) \leq Cr^m \text{ for } x \in \text{spt } \mu, 0 < r < d(\text{spt } \mu),$$

which means that  $\text{spt } \mu$  is an AD- $m$ -regular set.

### 1.3 Lipschitz Maps

Since Lipschitz maps are at the heart of rectifiability, we state here some basic well-known facts about them. We say that a map  $f: X \rightarrow Y$  between metric spaces  $X$  and  $Y$  is *Lipschitz* if there is a positive number  $L$  such that

$$d(f(x), f(y)) \leq Ld(x, y) \text{ for } x, y \in X.$$

The smallest such  $L$  is the Lipschitz constant of  $f$ , which is denoted by  $\text{Lip}(f)$ .

Euclidean valued Lipschitz maps  $f: A \rightarrow \mathbb{R}^k, A \subset X$ , can be extended: there is a Lipschitz map  $g: X \rightarrow \mathbb{R}^k$  such that  $g|_A = f$ , see [203, 2.10.43–44] or [321, Chapter 7].

Any Lipschitz map  $g: \mathbb{R}^m \rightarrow \mathbb{R}^k$  is almost everywhere differentiable by Rademacher's theorem, see [203, 3.1.6] or [321, 7.3].

There is the Lusin type property: if  $f: A \rightarrow \mathbb{R}^k, A \subset \mathbb{R}^m$  is Lipschitz, then for every  $\varepsilon > 0$  there is a  $C^1$  map  $g: \mathbb{R}^m \rightarrow \mathbb{R}^k$  such that

$$\mathcal{L}^m(\{x \in A: g(x) \neq f(x)\}) < \varepsilon, \quad (1.1)$$

see [203, 3.1.16].