GENERATION OF THE LOWER CENTRAL SERIES II

by ROBERT M. GURALNICK

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1. Introduction. In this article, we obtain results on commutators in Sylow subgroups of the lower central series, extending the work of Dark and Newell [2], Rodney [12, 13] and Aschbacher and the author [1, 6, 7].

Some notation is required for the statement of the main results. Let r be a positive integer and define

\[ [x_1] = x_1, \quad [x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2, \]

and

\[ [x_1, \ldots, x_r] = [[x_1, \ldots, x_{r-1}], x_r] \quad \text{for} \quad r \geq 3 \]

where \( x_1, \ldots, x_r \) are elements in a group \( G \). Let \( \Gamma_r G = \{ [x_1, \ldots, x_r] \mid x_i \in G \} \) be the set of \( r \)-fold commutators in \( G \). Then \( L_r G = (\Gamma_r G) \) is the \( r \)th term in the lower central series of \( G \). Set \( L_\infty G = \bigcap L_r G \).

**Theorem A.** Suppose \( L_r G \) is finite and \( P \in \text{Syl}_p(L_r G) \) is abelian of rank at most 2. If any of the following conditions hold then \( P \leq \Gamma_r G \).

(i) \( p \geq 5 \).
(ii) \( P \) is cyclic.
(iii) \( P \) has exponent \( p \).
(iv) \( P \cap L_\infty G \neq 1 \).
(v) \( P \cap L_{r+1} G = 1 \).
(vi) \( r \leq 2 \).

This result is known for \( r = 2 \). It was first proved by Rodney [13] for \( P \in \text{Syl}_p(G) \) of exponent \( p \). The complete proof of (vi) is [7, Theorem A]. The main idea of the proof is to reduce to the case where \( P = L_r G \). With this hypothesis, (iii) and (iv) are given in [6], while (ii) and (v) are proved in [2]. However, (i) is still a new result even in this more restricted situation. By examples in [1], [2], and [6], rank 2 cannot be replaced by rank 3. Moreover, (i) fails for \( p = 2 \) (and possibly for \( p = 3 \)).

The proof that when \( p \geq 5 \) and \( P = L_r G \) is an abelian rank 2 \( p \)-group then \( P = \Gamma_r G \) splits essentially into two cases. The first is when \( P = L_\infty G \) and is handled by [1, Theorem C]. The more difficult case is when \( G \) is a \( p \)-group. In fact, we consider the more general problem of when \( P \leq (\Gamma_r G)^k \). This also breaks up into the two cases described above (Theorem 2.1). An example (Section 4) is given to show that the \( p \)-group situation is the relevant obstruction to determining \( k \) in terms of the rank of \( P \).

Combining these techniques with a result of Gallagher, we obtain the following results.

THEOREM B. Suppose $G'$ is a $p$-group of order $p^k$ with $k < n(n + 1)$.
(a) If $L_nG$ is abelian, then $G' = (\Gamma_2G)^n$.
(b) $G' = (\Gamma_2G)^{2n}$.

THEOREM C. If $G' \leq Z(G)$ and $G'$ is a $p$-group of rank less than $n(n + 1)$, then $G' = (\Gamma_2G)^n$.

By examples in [5], the bounds in Theorems B and C are of the right order of magnitude. We define a function $f = f(p, r, d)$ by the following: if $P = L_rG$ is an abelian $p$-group of rank $d$, then $P = (F rG)^f$ and $f$ is the least positive integer satisfying this.

THEOREM D. (i) $\sqrt{d-1} \leq f \leq 2d$, for $r = 2$ and 3. (ii) $d(r-2)/(r-1) < f < 2d$, for $r > 3$.

Finally, we note that:

THEOREM E. If $|G| < 96$ or $|L_rG| < 8$, then $L_rG = \Gamma_rG$.

Moreover, these bounds are best possible for $r > 2$ (for $r = 2$, replace 8 by 16). See [6] and [7] for examples.

The paper is organized as follows. In Section 2, the proof of Theorem A is reduced to the case $L_rG = P$. This case is handled in Section 3. Some examples pertaining to lower bounds for $f$ are given in Section 4. Finally, Theorems B–E are proved in Section 5. We shall use notation as in [4]. I wish to thank the referee for his very careful reading of the article and many valuable comments.

2. Reduction of Theorem A. The first result describes how the condition that $L_rG$ has an abelian Sylow $p$-subgroup splits a Sylow $p$-subgroup into two nicer pieces.

THEOREM 2.1. Let $G$ be a finite group with $S \in \text{Syl}_p(G)$. Suppose $n \geq r$ and set $P = S \cap L_rG$ and $T = S \cap L_nG$.
(a) $P = (L_rS)T$.
(b) If $T$ is abelian and $N = N_G(T)$, then $P = (L_rS)[T, H] \leq L_rN$, where $H$ is a Hall $p'$-subgroup of $N$.
(c) If $S \triangleleft G$ and $V = S \cap L_nG$ is abelian, then $S = CV$ and $C \cap V = 1$, where $C = C_S(H)$ and $H$ is a Hall $p'$-subgroup of $G$. Moreover, if $G$ is solvable and $m$ is a positive integer such that $L_rC = (\Gamma_rG)^m$ and rank $V < 2^{m+1} - 1$, then $P \leq (\Gamma_rG)^m$.

Proof. Since $G/L_nG$ is nilpotent, $(L_rS)L_nG/L_nG$ is a Sylow $p$-subgroup of $L_rG/L_nG$. Thus $P \supseteq (L_rS)T \in \text{Syl}_p(L_rG)$ and (a) follows. If $T$ is abelian, then by [1, Corollary 5.2] $T = (L_nS)[T, H] \leq L_nN$, where $H$ is a Hall $p'$-subgroup of $N$. Hence (b) holds.

Now assume $S \triangleleft G$ and $V$ is abelian. Clearly $V = [S, H]$ and so by [4, Theorems 5.2.3 and 5.3.5], $S = CV$, $V = [V, H]$ and $V \cap C = 1$. If $G$ is solvable, then by [1, Theorem 4.1] there exist $h_1, \ldots, h_m \in H$ such that

$$V = \prod_{i=1}^m [V, h_i] = \prod_{i=1}^m [V, h_i, \ldots, h_i].$$

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Fix $h \in H$. Then for any $v \in V$, and $c_1, c_2 \in C$,
\[
[v c_1, c_2 h] = [v, c_2 h]^{c_1} [c_1, c_2].
\]
Since $\langle c_2 h \rangle \supseteq \langle h \rangle$, $V < G$ is abelian, and $[C, h] = 1$,
\[
\{[v, c_2 h]^{c_i} | v \in V\} \supseteq [V, h].
\]
Similarly, it follows that
\[
\{[v c_1, c_2 h, \ldots, c_r h] | v \in V\} = [V, h, \ldots, h] [c_r, \ldots, c_1]
\]
\[
= [V, h, c_1, \ldots, c_r].
\]
Hence
\[
\Delta_i = \{[v c_1, c_2 h_i, \ldots, c_r h_i] | v \in V, c_i \in C\} = [V, h_i] \Gamma_r(C).
\]
Finally, note that since $C$ normalizes $[V, h_i]$, $\Gamma_i = [\prod_{i=1}^m [V, h_i]] (\Gamma_r C)^m = V(L_i C)$.

Now (c) follows for $V(L_i C) = V(L_i S) = P$.

We now prove Theorem A (modulo results in section 3). So assume $L_i G$ is finite and $P \in \text{Syl}_p(L_i G)$ is abelian of rank at most 2. By [6, Lemma 1.4], we can assume $G$ is finite. By Theorem 2.1b, we can also assume $P < G$ and $P \cap L_i G = T = [T, H]$ for $H$ a Hall $p'$-subgroup (note that conditions (i)–(vi) remain valid under these reductions). Now $

P = T \times C_p(H)$ by [4, Theorem 5.2.3]. If $P = T$, then $P < \Gamma_i G$ by [1, Theorem C]. Otherwise $T$ is cyclic and so there exists $h \in H$ with $[T, h] = T$. Moreover, since $G = N_G(S) G' = N_G(S) C_G(T)$, we can assume $h \in N_G(S)$. Consider $M = \langle S, h \rangle$. Then $L_i M = L_i S[h, T] = P$. Thus it suffices to assume $G = M$ and $P = L_i G$. Now (ii), (iii), and (iv) follow by [6, Theorem 3.2] and (v) follows by [2, Theorem 2]. As remarked before (vi) is [7, Theorem A]. Finally, if (i) holds but (iv) fails, then $T = 1$, $P = L_i S$ and Theorem 3.6 applies.

3. The case $L_i G = P$. We need some commutator calculus. Suppose $g \in G$. Define $\gamma_0(g) = \langle g \rangle$ and $\gamma_{i+1}(g) = [\gamma_i(g), G]$. Note that $\gamma_i(g) < G$ for $i > 0$, and by the three subgroup lemma,
\[
[\gamma_i(g), L_r G] \leq \gamma_{i+r}(g),
\]
\[
[\gamma_i(g), \gamma_i(g)] \leq \gamma_{i+i+1}(g).
\]

**Proposition 3.3.**

(a) $[s, u, x_1, \ldots, x_r] = [s, t, x_1, \ldots, x_r] [s, u, x_1, \ldots, x_r]$
\[
\times [s, t, u, x_1, \ldots, x_r] \bmod \gamma_{r+2}([s, t]).
\]
(b) $[s, t', x_1, \ldots, x_r] = [s, t, x_1, \ldots, x_r]$
\[
\times [s, t, u, x_1, \ldots, x_r] \bmod \gamma_{r+2}([s, t]).
\]
Proof. (a) Set \( y = [s, t] \). Induct on \( r \). If \( r = 0 \), then
\[
[s, tu] = [s, u][s, t][s, t, u].
\]
(*)

The result holds in this case since \([s, t, G'] \leq \gamma_2(y)\). Now assume \( r > 0 \) and \( \gamma_{r+2}(y) = 1 \). By induction
\[
[s, tu, x_1, \ldots, x_{r-1}] = abcd,
\]
where \( a = [s, t, x_1, \ldots, x_{r-1}] \), \( b = [s, u, x_1, \ldots, x_{r-1}] \), \( c = [s, t, u, x_1, \ldots, x_{r-1}] \) and \( d \in \gamma_{r+1}(y) \leq Z(G) \). Thus,
\[
[s, t, u, x_1, \ldots, x_r] = [abcd, x_r] = [abc, x_r].
\]

However, by (*) (or its inverse),
\[
[abc, x_r] = [ab, x_r][ab, x_r, c][c, x_r] = [a, x_r][a, x_r, b][b, x_r][ab, x_r, c][c, x_r].
\]

Now (a) follows by noting that \([ab, x_r, c] \in [G', \gamma_r(y)] \leq \gamma_{r+2}(y)\) and \([a, x_r, b] \in [\gamma_{r-1}(y), G, G'] \leq \gamma_{r+2}(y)\).

(b) follows from (a) by a straightforward induction argument.

Lemma 3.4. Suppose \( L_rG \) is an abelian \( p \)-group of rank at most two and \( L_{r+1}G \leq U^1L_rG \).

(a) There exist \( j, 1 \leq j \leq r \), and \( u_i, 1 \leq i \neq j \leq r \), such that
\[
L_rG = [u_1, \ldots, u_{i-1}, G, u_{i+1}, \ldots, u_r].
\]

(b) Moreover, if \( p > 2 \), then
\[
L_rG = \{[u_1, \ldots, u_{i-1}, g, u_{i+1}, \ldots, u_r] \mid g \in G\} = \Sigma.
\]

Proof. (a) Without loss of generality, \( U^1L_rG = L_{r+1}G = 1 \). Choose \( J \subset I = \{1, \ldots, r\} \) maximal so that there exist \( u_j, j \in J \) with \( L_rG = [E_1, \ldots, E_r] \) where \( E_j = u_j \) if \( j \in J \) and \( E_j = G \) otherwise. Assume \( k < l \in I - J \). Hence \( L_rG = \langle [x_1, \ldots, x_r], [y_1, \ldots, y_r] \rangle \) where \( x_i = y_i = u_l \) if \( j \in J \). Suppose \( z = [x_1, \ldots, y_k, \ldots, x_r] \neq 1 \). Then either
\[
L_rG = \langle [x_1, \ldots, x_r], [x_1, \ldots, y_p, \ldots, x_r] \rangle
\]
or
\[
L_rG = \langle [x_1, \ldots, y_k, \ldots, x_r], [y_1, \ldots, y_r] \rangle.
\]

In the first case \( I - \{k\} \) satisfies the conclusion, and in the second case \( J \cup \{j\} \) satisfies the same condition as \( J \). This contradicts the maximality of \( J \). So \( z = 1 \). Similarly \([y_1, \ldots, x_0, \ldots, y_r] = 1 \); but now \( J \cup \{k\} \) satisfies the condition with \( u_k = x_ky_k \).

(b) We note that we can assume \( j \neq 1 \) (for if \( j = 1 \) then \( j = 2 \) will also satisfy the conclusion in (a)). Set \( s = [u_1, \ldots, u_{i-1}] \) and define \( \phi : G \to \Sigma \) by \( \phi(g) = [s, g, u_{i+1}, \ldots, u_r] \). First we shall show that for any \( g, v_{i+1}, \ldots, v_r \in G \) we have
\[
y = [s, g, v_{i+1}, \ldots, v_r] = [s, g, v_{i+1}, \ldots, v_r] \mod U^{i+1}L_rG.
\]

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Induct on \( i \). If \( i = 0 \), this follows from Proposition 3.3b and the fact that \( L_{r+1} \leq U^1L_r \). So assume \( i > 0 \). Again by Proposition 3.3b,

\[
y = [s, g^{p_i}, u_{i+1}, \ldots, u_r] \mod B,
\]

where \( B = \gamma_{r-i+1}(s, g^{p_i}) \gamma_{r-i+2}(s, g^{p_i}) \). By induction,

\[
\gamma_{r-i}(s, g^{p_i}) \leq U^{i-1}L_r.
\]

Since \( L_{r+1} \leq U^1L_r \) it follows that \( [G, U^kL_r] \leq U^{k+1}L_r \) and so \( B \leq U^{i+1}L_r \). Thus \((**)\) holds.

Suppose now that \( U^1L_r = U^{i+1}L_r \). By (a), there exists \( g \in G \) such that \( \phi(g) \) has order \( p^{k+1} \) and so by \((**)\), \( z = \phi(g^{p_k}) = \phi(g)^{p_k} \). In particular, \( z \in U^kL_r \leq Z(G) \). By induction on \( |L_r| \), if \( x \in L_r \) then \( xz = \phi(h) \) for some positive integer \( e \) and \( h \in G \). By Proposition 3.3a,

\[
\phi(g^{-ep_k}h) = \phi(g^{-ep_k})\phi(h) \mod \gamma_{r-i+1}(s, g^{-ep_k}).
\]

However by \((**)\),

\[
\phi(g^{-ep_k}) = z^{-e} \quad \text{and} \quad \gamma_{r-i}(s, g^{-ep_k}) \leq U^kL_r \leq Z(G).
\]

Thus

\[
\phi(g^{-ep_k}h) = z^{-e}z = x,
\]

and so \( \phi \) is surjective as desired.

The reason we assume \( p > 3 \) is apparent in the next lemma. If \( p > 3 \), then

\[
\sum_{i=0}^{n-1} i^2 \equiv 0 \mod p.
\]

**Lemma 3.5.** Let \( G \) be a \( p \)-group with \( p > 3 \). If \( x, y \in G \) with \( a = [x, y], \langle a^G \rangle = \langle a, b \rangle \) abelian and \( [a, G] \leq \langle a^p, b \rangle \), then \( [x, y^p] \equiv a^p \mod B \), where \( B = \langle a^{p^2}, b^p \rangle \).

**Proof.** Set \( A = \langle a^G \rangle \). Then \( [A, G] \leq \langle a^p, b \rangle \). If \( c = [a, y] \in U^1A \), then \( [y, U^1A] \leq U^2A \), and so

\[
[x, y^p] \equiv a^i c^{i(i-1)/2} \mod U^2A
\]

by Proposition 3.3(b). Hence as \( p \neq 2 \), \( [x, y^p] \equiv a^p \mod B \). Otherwise, we can take \( b = [a, y] \).

Now \( [b, y] = a^p b \). Then a straightforward tedious computation shows that modulo \( B \)

\[
[x, y^p] \equiv a^p \prod_{i=1}^{p-1} [a, y^i] \equiv a^p \prod_{i=1}^{p-1} a^{\alpha_p(i-1)/2} b^i \equiv a^p
\]

since \( p > 3 \).

**Theorem 3.6.** If \( L_rG \) is a rank 2 abelian \( p \)-group with \( p > 3 \), then \( L_rG = \Gamma_rG \).
Proof. As in Section 2, we can assume $G$ is a $p$-group. Let $G$ be a counterexample with $|L,G|$ minimal. Choose $a \in L,G - \Gamma,G$. Set $A = \langle a^G \rangle = \langle a, b = [a, g] \rangle$ for some $g \in G$. Let $p^\alpha$ and $p^\beta$ denote the orders of $a$ and $b$ respectively. Note that $\alpha \equiv \beta$. First assume $\beta < \alpha$. Then $B = U^{-1}A = \langle a^{p^{\beta - 1}} \rangle \leq Z(G)$. If $\alpha > 1$, then by passing to $G/B$, some generator of $\langle a \rangle$ is in $\Gamma^1,G$, and thus $a \in \Gamma^1,G$ by [6, Lemma 1.3]. So $\alpha = 1$ and $\beta = 0$. If $a = c^p$ with $c \in L,G$, then as above $c \in \Gamma^1,G$. Say $c = [d, h]$ with $d \in \Gamma_{r+1}G$ and set $e = [c, h]$. Then $e^p = [c^p, h] = 1$. Moreover as $e \in \Omega \Gamma_{r+1}G$ and $a \notin \Gamma_{r+1}G$ (as $\Gamma_{r+1}G = \Gamma_{r+1}G \leq \Gamma,G$ since $|L_{r+1}G| < |L,G|$, $e \in Z(G)$. Thus, $[d, h^p] = c^p e^{p(p-1)/2} = a$ since $p \neq 2$. Thus, $a \notin \Omega^1L,G$ and so $L,G = \langle a, x \rangle$. We can assume $x \in \Gamma,G$ and so $x = [g, y]$ for some $g \in G$ and $y \in \Gamma_{r-1}G$. If $[G, y] = \langle x \rangle$, then $\langle x \rangle < G$ and so $\Gamma_{r-1}G = [G, x] \leq \Omega^1L,G$. Then $L,G = \Gamma,G$ by Lemma 3.4b. Otherwise $[G, y] = L,G$. We claim $a = [h, y]$ for some $h \in G$. Induct on the order of $x$. If $x^p = 1$, then the map $h \mapsto [h, y] \in Z(G)$ is an endomorphism from $G$ onto $L,G$. So assume $x$ has order $p^{k+1}$. By induction ($x^p \in Z(G)$), we see that $[h, y] = ax^{epk}$ for some integer $e$. By Lemma 3.5, $[g^p, y] = x^p \mod(x^p)$, and continuing we see that $[g^p, y] = x^p$. Hence $[g^{-epk}h, y] = a \in \Gamma,G$. So $\beta = \alpha$. Thus $B = \langle b^{p^{\alpha - 1}} \rangle = U^{-1}[G, A] \leq Z(G)$. Hence by passing to $G/B$, $a(b^{p^{\alpha - 1}}) \in \Gamma,G$ for some $\lambda$. By Lemma 3.5, $[a, g^p] \equiv b^p \mod \Omega^2A$, and continuing we find that $[a, g^{p^{\alpha - 1}}] = b^{p^{\alpha - 1}}$ and so $a$ is conjugate to $ab^{p^{\alpha - 1}}$. This completes the proof.

By Example 3.1 in [6], $p > 2$ is necessary. If $r = 2$, and $p > 3$, one can replace rank 2 by rank 3 [7, Theorem B]. This would seem to provide some evidence that there is a counterexample with $p = 3$ since there is such for $r = 2$ with $L,G$ of rank 3.

4. Lower bounds for $f$. Many examples have been given with $L,G \neq (\Gamma,G)^k$, particularly for $r = 2$ or for $k = 1$ (see [1], [2], [5], [6], [7], [8], [10] and [11]). We construct one which gives a good lower bound for $f = f(p, r, d)$. First note that for $r = 2$, it follows from [5] and [7] that:

**Proposition 4.1.**

(a) $f(d) < \sqrt{d - 1}$.

(b) $f(3) = 1 \Leftrightarrow p > 3$.

(c) $f(2) = 1$.

For the rest of the section, assume $r > 2$. By Theorem A, $f(1) = 1$ for all $p$ and $f(2) = 1$ for $p > 3$. Also by [1], $f(d) > \log_2(d + 1) - 1$.

Now fix a prime $p$ and $r > 2$. Let $F$ be the free group on $n$ generators. Set $H = F/(L,F)^pL_{r+1}F$. By Witt's formula, $L,H$ is a free elementary abelian $p$-group of rank $t$, where

$$t = \frac{1}{r} \sum_{k | r} \mu(k)n^{r/k},$$

and $\mu$ is the Moebius function. It follows easily that $n \geq n^{r-1}$. Now suppose $d$ is a positive integer with $(n - 1)^s < d < n^s$, $s = r - 1$. Choose a subgroup $M$ of $L,H$ of index $p^d$ in $L,H$. Set $G = H/M$. Then $G$ is nilpotent, $L_{r+1}G = 1$, and $L,G$ is elementary abelian of rank $d$. By Proposition 3.3, the $r$-fold commutator is multiplicative in each variable. Thus
|Γ, G| < p^n. Hence if f = f(p, r, d), then p^d = |L_2 G| = |(Γ, G)^d| < |Γ, G|^d < p^{nd}. So f > d/nr. Since
d^{1/s} > (n-1)r^{-1/s} > \frac{1}{2}r^{-1-(1/s)}(nr),
it follows that
f > d/nr > \frac{1}{2}r^{-1-(1/s)}d^{(s-1)/s} \geq d^{(s-1)/s}/6r.

We remark that for r = 3, one can construct a group G similar to that in [2, Proposition 3] in which L_3 G is elementary abelian of rank n^2 and L_3 G \neq (Γ_3 G)^n. Thus f(p, 3, d) > d - 1. Also as r gets large, we can replace 6 by numbers tending to 2. We conjecture that f(d) > d^{(r-1)/r - 1} (this is true for r = 2).

5. Theorems B–E. Theorem B follows as an easy consequence of Theorem 2.1 and a result of Gallagher.

**Theorem B.** Suppose G' is a p-group of order p^k with k < n(n + 1).
(a) If L_2 G is abelian then G' = (Γ_2 G)^n.
(b) G' = (Γ_2 G)^2^n.

**Proof.** Note that (b) follows from (a) and [3, Theorem 1b] by considering G/(L_2 G). As usual, we assume G is finite. For (a), note that if S ∈ Syl_p(G), and C is as in Theorem 2.1c then (Γ_2 C)^n = C' by Gallagher [3, Theorem 2]. The result now follows by Theorem 2.1c (for n(n + 1) < 2^{n+1} - 1).

Theorem C also follows from the same result of Gallagher and the next lemma.

**Lemma 5.1.** Let G be a p-group with G' ≤ Z(G) and G' \neq (Γ_2 G)^n. Moreover, assume |G| is minimal with respect to this property (for a fixed p and n). Then G' has exponent p.

**Proof.** Choose a ∈ L_2 G - (Γ_2 G)^n. Suppose d = [u, v]^p = [u, v^p] ≠ 1 for some u, v ∈ G. As |G| > |G/(d)|, ad^i ∈ (Γ_2 G)^n for some i. By replacing d by d^i, we may assume that ad ∈ (Γ_2 G)^n. Say

ad = \prod_{i=1}^{n} [s_i, t_i].

Then

a = \prod_{i=1}^{n} [s_i, t_i][u, v^{-p}].

Since a ∈ H, H = \langle s_1, t_1, \ldots, s_n, t_n, u, v^p \rangle = G. However, v^p is an element of the Frattini subgroup of G, and so G can be generated by 2n + 1 elements. However, this implies G' = (Γ_2 G)^n by [5, Theorem 5.2]. So [u, v]^p = 1 for all u, v ∈ G, proving the lemma.

Now Theorem C follows from Theorem B. Moreover, by [5, Section 5], n(n + 1) can not be replaced by (n + 1)^2 + 1 in either Theorem B or C. Recall that f = f(p, r, d) is defined as follows: if P = L_2 G is an abelian p-group of rank d then P = (Γ, G)^f and f is as small as possible. We first obtain an upper bound for f.
Proposition 5.2. Let $G$ be a finite group.

(a) If $\langle x \rangle = \langle y \rangle$ and $x \in (\Gamma, G)^k$, then $y \in (\Gamma, G)^k$.

(b) If $x \in (\Gamma, G)^k$, then $\langle x \rangle \leq (\Gamma, G)^{3k}$.

(c) If $x \in (\Gamma, G)^k$ is a $p$-element, then $\langle x \rangle \leq (\Gamma, G)^{2k}$.

(d) If $P \in \text{Syl}_p(L; G)$ is abelian of rank $d$, then $P \leq (\Gamma, G)^{2d}$.

Proof. As remarked before, (a) is [6, Lemma 1.3]. Now (b) and (c) follow by noting that if $y \in \langle x \rangle$, then $y = ab$ or $abc$, where $a$, $b$ and $c$ are some generators for $\langle x \rangle$. Moreover, if $x$ is a $p$-element, then either $\langle y \rangle = \langle x \rangle$ or $y = ab$. Now (d) follows from (c) and the observation that if $P$ has rank $d$, it can be generated by $d$ elements of $\Gamma, G$ (see Theorem 2.1).

Theorem D now follows from Proposition 5.2(d) and the results in Section 4. We note that $f(1) = f(2) = 1$ (for $p > 3$ or $r = 2$) and $f(3) \geq 2$ (for $r > 2$ and $p \leq 3$).

Finally, we shall prove Theorem E. Note if $|L, G| < 8$, then $L, G = \Gamma, G$ by Theorem A and [6] unless perhaps $L, G = S_3$. However if $r \geq 3$, then as $A = (L, G)' \lhd G$, $L, G \leq G' \leq C_G(A)$, a contradiction. So $L, G \neq S_3$. Now assume $|G| < 96$. Suppose $G$ is a counterexample. Then $r \geq 3$ by [7, Theorem D]. If $G/G'$ is cyclic, then $\Gamma_2 G = \{[g, h] \mid g \in G', h \in G\}$, and $G' = L, G$ for $r \geq 2$. Since $|G| < 96$, $G' = \Gamma_2 G$, and so by induction, we see that

$$\Gamma_{r+1}(G) = \{[g, h] \mid g \in \Gamma_r G, h \in G\} = \Gamma_2 G = G' = L, G_{r+1}.$$ 

Since $|L, G| \geq 8$ and indeed by an argument similar to the one above $|L, G|$ is divisible by at least three primes, the only possibilities remaining are that $[G : G'] = 4$ and $|L, G| = 8, 12, 18$ or 18 or that $[G : G'] = 9$ and $|G'| = |L, G| = 8$. The last possibility is easily eliminated by inspection. If $[G : G'] = 4$ and $|L, G| = 8$, then $G$ is a 2-group and $G'$ is cyclic. Thus Theorem A applies. If $[G : G'] = 4$ and $|L, G| = 12$ or 18, then either $L, G$ is the union of its Sylow subgroups (and so $L, G = \Gamma, G$ by Theorem A) or $G' = L, G$ is abelian. It then follows that $L, G \neq G'$, and this contradiction completes the proof.

References


DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTHERN CALIFORNIA
LOS ANGELES
CALIFORNIA 90089–1113
U.S.A.