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REPRESENTATIONS OF LIE GROUPS BY CONTACT TRANSFORMATIONS, I: COMPACT GROUPS

BY

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ABSTRACT. The action of Lie groups as transitive groups of restricted contact transformations of compact manifolds are classified.

RÉSUMÉ. On classifie les actions de groupe de Lie par transformations de contact, au sens restreint, de variété compacte.

1. **Statements of results.** An *exact contact manifold* (Δ, θ) is a smooth $(2n+1)$-dimensional manifold Δ endowed with a 1-form θ such that $\theta \wedge d\theta^n$ gives a volume element, i.e. it vanishes nowhere. A restricted contact transformation is a diffeomorphism S of Δ such that, for the tangent functor T, one has $\theta \circ T(S) = \theta$.

We consider the case of a connected Lie group $\mathbf{G}$ acting as a transitive group of contact transformations of (Δ, θ). Put $\mathfrak{g}$ for the Lie algebra of $\mathbf{G}$ and write Δ_X for the vector field on Δ corresponding to the infinitesimal action of $X \in \mathfrak{g}$. We shall henceforth assume that the action of $\mathbf{G}$ is *infinitesimally faithful*, i.e.

$$X \in \mathfrak{g} \text{ and } \Delta_X(z) = 0 \quad \forall z \in \Delta \qquad \text{implies } X = 0.$$

One can give a simple complete classification.

PROPOSITION 1.1. *Suppose* $\mathbf{G}$ *acts infinitesimally faithfully as a transitive group of restricted contact transformations of a compact manifold* Δ. *There are three cases:*

(i) $\mathbf{G}$ *is compact semi-simple;*

(ii) there exists a compact semi-simple subgroup $\mathbf{K}$ *of* $\mathbf{G}$ *of codimension 1 which acts as a transitive group of contact transformations of* Δ*;*

(iii) $\mathbf{G}$ *is 1-dimensional and* Δ *is a circle.*

The contact manifolds (Δ, θ) are heavily restricted by admitting a transitive group of restricted contact transformations. Replacing θ by $c\theta$ where $c > 0$ changes nothing important. Such a change will be called a *normalization*. For any exact contact manifold there is a unique vector field Θ defined on Δ by the properties

$$(1.2) \qquad \theta(\Theta) \equiv 1 \qquad d\theta(\Theta(z), \xi) = 0 \quad \text{for each } \xi \in T_z(\Delta).$$

The group invariance of θ leads to

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PROPOSITION 1.3. *Suppose* (Δ, θ) *is a compact contact manifold admitting a transitive group,* $\mathbf{G}$, *of restricted contact transformations. Then the* Θ *defined by (1.2) commutes with the action of* $\mathbf{G}$, *i.e.* $\Theta \circ S = T(S) \circ \Theta$ *for each* $S \in \mathbf{G}$. *Moreover one can normalize* θ *so that the flow of* Θ *has period* 2π *everywhere. Thus, in* $\Omega = \Delta \times \mathbb{R}_+$ *one can introduce an action of* $\mathbb{C}_*$ *where multiplication by* i *is given by* $\exp(\frac{1}{2}\pi\Theta)$. Ω *can be given a canonical complex structure so that it appears as a holomorphic* C_**-bundle over a compact Kähler manifold* $\mathcal{H}$ *on which* $\mathbf{G}$ *acts as a transitive group of holomorphic isometries, and the Kähler form on* $\mathcal{H}$ *is carried back by the projection map of the circle bundle* $\Delta \longrightarrow \mathcal{H}$ *to* $d\theta$.

In view of Proposition 1.1 we may confine our attention to $\mathbf{G}$ compact semi-simple. Let λ be the highest weight of an irreducible unitary representation of $\mathbf{G}$ in a complex Hilbert space $\mathcal{U}$ with inner product $\langle \cdot, \cdot \rangle$ which is conjugate linear in the first variable and linear in the second. (The existence of a highest weight vector implies that the representation is P-extreme in the sense of [5; p. 90]; see also [1]). We may as well suppose that the representation is faithful, and that $\mathbf{G}$ is defined as a group of unitary transformations of $\mathcal{U}$. Fix z_0 as a highest weight vector of norm 1, and put Δ for the $\mathbf{G}$-orbit of z_0. The tangent space $T_z(\Delta)$ is viewed as a subspace of $\mathcal{U}$. Define a 1-form on Δ by

$$\theta(z, u) = \operatorname{Im}\langle z, u \rangle \quad \text{for } u \in T_z(\Delta) \tag{1.4}$$

For $X \in \mathfrak{g}$ we have

$$\Delta_X(z) = Xz.$$

LEMMA 1.5. *The* θ *defined by (1.4) is a* $\mathbf{G}$*-invariant contact structure for* Δ. Δ *is invariant under scalar multiplication by* ϵ *where* $\epsilon \in \mathbb{C}$, $|\epsilon| = 1$.

Put Ω for the cone through Δ; thus $\Omega = \{zc : z \in \Delta, c > 0\}$. Then Ω admits an action of $\mathbb{C}_*$ by scalar multiplication and has a complex structure given by multiplication by i in the tangent space viewed as a subspace of $\mathcal{U}$. The $\mathcal{H}$ of Proposition 1.3 is the image of Ω in the complex projective space $\mathcal{U}/\mathbb{C}_*$.

ILLUSTRATION. Take $\mathbf{G} = \mathbf{SU}(3)$ in its standard representation π in $\mathbb{C}^3$. Then $\mathfrak{g}$ may be identified with the 3×3 complex skew-Hermitian matrices of trace 0. $\mathbb{C} \otimes \mathfrak{g}$ is viewed as the complex 3×3 matrices of trace 0. The highest weights of $\mathbb{C} \otimes \mathfrak{g}$ are equivalent to some dominant weight $\lambda_{m,n}$ where

$$\lambda_{m,n} X = mX_1^1 - nX_3^3 \quad \text{with } m, n \in \mathbb{Z},\ m, n \geqq 0,$$

and we suppose $m+n > 0$. The corresponding representation is contained in $\pi^{\otimes m} \otimes \bar{\pi}^{\otimes n}$, and, in terms of the standard basis for $\mathbb{C}^3$, the highest weight vector may be taken to be the tensor product of e_1 with itself m times tensored with the tensor product of e_3^* with itself n times. There are only two different $\mathcal{H}$ which arise. If $n = 0$ we have $\mathcal{H} = \mathrm{PROJ}\,(2, \mathbb{C})$, and Δ is the standard circle bundle corresponding to $m\omega_1$ where ω_1 is the standard generator for $H^2(\mathcal{H})$ given by the Kähler form. Thus Δ may be viewed as standard S^5 modulo identification by a cyclic group of order m. If $m = 0$ the situation is the same except that Δ corresponds to $-n\omega_1$. The contact manifolds obtained for $\lambda_{n,0}$ and $\lambda_{0,n}$ are distinct as $\mathbf{G}$-manifolds. If $m \neq 0, n \neq 0$ one has $\mathcal{H} \cong \mathbf{G}/\mathbb{T}^2$ where $\mathbb{T}^2$

is a maximal torus. To make matters clearer we examine in detail the weight $\lambda_{1,1}$. The representation is the one on the space, $\mathcal{U}$, of 3×3 complex matrices of trace 0 given by $(S, Z) \mapsto SZS^{-1}$. Observe that the equivalence class of this representation is usually taken as the action on the Hermitian 3×3 matrices of trace 0, but in order to have a highest weight vector one must use a complex vector space. One may choose z_0 here to be the 3×3 matrix with 1 in the upper right-hand corner and 0 elsewhere. Here we have

$$\Omega = \{Z \in \mathcal{U} \setminus \{0\} : Z^2 = 0\}.$$

Thus $\mathcal{H}$ appears as an algebraic variety of complex dimension 3 in $\mathrm{PROJ}\,(7, \mathbb{C})$. $\mathbf{SU}(3)$ acts faithfully on Δ except when m and n are both divisible by 3 in which case the centre acts trivially.

THEOREM 1.6. *Let* $\mathbf{G}$ *be a compact semi-simple Lie group acting as a transitive group of restricted contact transformations of* (Δ, θ). *After suitable normalization* Δ *is equivalent as a* $\mathbf{G}$*-manifold to the orbit of a highest-weight vector in an irreducible unitary representation of* $\mathbf{G}$ *with the contact form given by (1.4).*

PROPOSITION 1.7. *The* $\mathcal{H}$ *given by Proposition 1.3 is a symmetric space of* $\mathbf{G}$ *iff the representation has highest weight* λ *with the property that if* α *and* β *are distinct positive roots such that* $\alpha + \beta$ *is a root and* $\lambda - \alpha$ *and* $\lambda - \beta$ *are weights of the representation then* $\lambda - \alpha - \beta$ *is not a weight of the representation.*

Combining (1.1) and (1.6) we can assert

PROPOSITION 1.8. *In order that a connected Lie group* $\mathbf{G}$ *admit a faithful representation by restricted contact transformations of a compact manifold it is necessary and sufficient that* $\mathbf{G}$ *be compact with finite cyclic centre or* $\mathbf{G} \cong \mathbb{T}$.

The upshot is that we get no new, interesting information about Lie groups by considering restricted contact transformations. This contrasts with representations of non-compact simple Lie groups by unrestricted contact transformations acting on compact manifolds; see [3] and [4]. The contact manifolds which occur for the non-compact simple Lie groups are always among the ones described here.

ILLUSTRATION. Consider the exceptional simple compact 14-dimensional group $\mathbf{G}_2$. It acts as a faithful transitive group of transformations of (S^6, ω) where ω is the non-degenerate 2-form corresponding to the quasi-complex structure of S^6; see [2] for a detailed description. This arises when one considers the fundamental representation π_1 of $\mathbf{G}_2$ by orthogonal transformations of $\mathbb{R}^7$. When one views this representation as a unitary representation in $\mathbb{C}^7$, the orbit of a highest-weight vector is 11-dimensional. The system of positive roots in the complexified Lie algebra may be represented as

$$\gamma,\ \delta,\ \gamma+\delta,\ 2\gamma+\delta,\ 3\gamma+\delta,\ 3\gamma+2\delta.$$

The highest weight corresponding to π_1 is $\lambda = 2\gamma + \delta$. The elements $X \in \mathfrak{g}$ such that $\Delta_X(z_0) = 0$ where z_0 is the weight vector for λ are the skew-Hermitian elements of

$$\mathbb{C}\, H_\delta + \mathfrak{g}(\delta) + \mathfrak{g}(-\delta)$$

where H_δ is the co-root vector and $\mathfrak{g}(\pm\delta)$ are the root spaces in $\mathbb{C} \otimes \mathfrak{g}$. If we write $\mathbf{SU}(2)$ for the subgroup corresponding to this Lie algebra then the contact manifold for $\mathbf{G}_2$ given by Theorem 1.6 is isomorphic to $\mathbf{G}_2/\mathbf{SU}(2)$. This contrasts with case of the adjoint group of the non-compact version of the Lie algebra G_2 which acts faithfully as a transitive group of unrestricted contact transformations of a 5-dimensional manifold; see [2, Theorem (6.3)].

2. **Proofs.** We start with the general situation of $\mathbf{G}$ a connected Lie group acting transitively as restricted contact transformations of (Δ, θ). A general property of infinitesimal restricted contact transformations gives

$$d\theta(\Delta_X, \Delta_Y) = \theta(\Delta_{[X,Y]}) \quad \text{for } X, Y \in \mathfrak{g}. \tag{2.1}$$

Write $\mathfrak{g}^*$ for the dual vector space of the Lie algebra $\mathfrak{g}$. There is a canonical map

$$\Delta \longrightarrow \mathfrak{g}^*, \quad z \longmapsto \hat{z} \text{ where } \hat{z}X = \theta(\Delta_X(z)). \tag{2.2}$$

Under the action of $\mathbf{G}$ on Δ we have

$$\widehat{Sz} = \hat{z}\,\mathrm{ad}\,S^{-1}$$

where $\mathrm{ad}\,S$, $S \in \mathbf{G}$, acts on the left on $\mathfrak{g}$ and on the right in $\mathfrak{g}^*$. Put $\mathcal{H}$ for the image of this map. Write $\mathcal{H}_X$ for the infinitesimal action of $X \in \mathfrak{g}$ on $\mathcal{H}$. Obviously $\mathcal{H}$ is a co-adjoint orbit. It's Kostant-Souriau form ω_1 is, by definition

$$\omega_1(\mathcal{H}_X(q), \mathcal{H}_Y(q)) = q[X, Y] \quad \text{for } q \in \mathcal{H}. \tag{2.3}$$

Therefore, under the mapping (2.2) the Kostant-Souriau form of $\mathcal{H}$ lifts back to $d\theta$ on Δ.

LEMMA 2.4. *Suppose* $\mathbf{G}$ *acts as an infinitesimally faithful transitive group of contact transformations of a compact manifold* Δ. *Put* $\mathfrak{c}$ *for the centre of the Lie algebra* $\mathfrak{g}$. *If* $N \in \mathfrak{g}$ *is such that* $\mathrm{Ad}\,N$ *is nilpotent then* $N \in \mathfrak{c}$. *Moreover* $\dim(\mathfrak{c}) \leq 1$. *The projection* $\Delta \longrightarrow \mathcal{H}$ *given by (2.2) is a finite covering map.*

PROOF. IT is obvious that $\dim \mathcal{H} = \dim \Delta - 1$. Define

$$\mathfrak{c} = \{N \in \mathfrak{g} : \mathcal{H}_N \equiv 0\}.$$

If $N \in \mathfrak{g}$ is such that $\mathrm{ad}\,N$ is nilpotent then for fixed $q \in \mathcal{H}$ and $X \in \mathfrak{g}$, $t \mapsto q\exp(t\,\mathrm{ad}\,N)X$ is a bounded polynomial function. We conclude that $N \in \mathfrak{c}$. Conversely, if $N \in \mathfrak{c}$ then $d\theta(z; \Delta_N, \Delta_X) = 0$ for all $X \in \mathfrak{g}$. This implies that $\theta(\Delta_N)$ is a constant function on Δ. By hypothesis, if $\theta(\Delta_N) \equiv 0$ then $N = 0$. Thus $\mathfrak{c}$ has dimension at most 1, and $[\mathfrak{c}, \mathfrak{g}] = 0$ which shows that $\mathfrak{c}$ is indeed the centre of $\mathfrak{g}$.

PROOF OF PROPOSITION 1.1. Consider the Levi decomposition $\mathfrak{g} = \mathfrak{s} + \mathfrak{r}$ where $\mathfrak{s}$ is a semi-simple subalgebra and $\mathfrak{r}$ is the radical. By (2.4), $\mathfrak{r} = \mathfrak{c}$. Thus $\mathfrak{s}$ is an ideal. Again by (2.4), $\mathfrak{s}$ contains no Ad-nilpotent elements. Therefore it is of compact type. Let $\mathbf{K}$ be the connected compact subgroup of $\mathbf{G}$ corresponding to $\mathfrak{s}$. If $\mathfrak{c} = 0$ we have

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case (i) and there is nothing to prove. If $\mathfrak{s} = 0$ we have the trivial case (iii). Finally if $\mathfrak{c}$ is 1-dimensional and $\dim \mathbf{G} > 1$ it is easy to see that for each $z \in \Delta$ there must exist $X \in \mathfrak{s}$ such that $\theta(\Delta_X(z)) \neq 0$.

Henceforth we assume that $\mathfrak{g}$ is semi-simple of compact type.

The flow corresponding to Θ, defined by (1.2), must commute with the action of $\mathbf{G}$ by uniqueness, and hence it is strictly confined to the fibres of Δ over $\mathcal{H}$ which are compact and 1-dimensional. One easily sees that the flow is circular, and we can normalize θ so that the period is exactly 2π. Write $t \mapsto \exp(t\Theta)$ for the flow of Θ. In $\Omega = \Delta \times \mathbb{R}_+$ introduce an action on the right of $\mathbb{C}_*$ by

$$(2.5) \qquad (z,c)(re^{it}) = \big(\exp(t\Theta)z, cr\big).$$

The action of $\mathbf{G}$ is extended to Ω by $S(z,c) = (Sz, c)$; thus it commutes with the action of $\mathbb{C}_*$. Put E for the vector field on Ω corresponding to the infinitesimal action of scalar multiplication on $\mathbb{R}_+$. Put $V(z) = \{\xi \in T_z(\Delta) : \theta(\xi) = 0\}$. In an obvious way we have a direct sum decomposition

$$T_{(z,c)}(\Omega) = E(z,c)\mathbb{R} \oplus \Theta(z)\mathbb{R} \oplus V(z).$$

To define a complex structure, J, on $T_{(z,c)}(\Omega)$ it suffices to set $JE(z,c) = \Theta(z)$, $J\Theta(z) = -E(z,c)$, and impose a complex structure $J(z)$ on $V(z)$. To do this we note that $V(z)$ is canonically isomorphic to $T_{\hat{z}}(\mathcal{H})$. Proposition 1.3 will be proved once we can show that there is a $\mathbf{G}$-invariant complex structure J on $\mathcal{H}$ such that $\omega_1(\cdot, J\cdot)$ is a Riemannian metric where ω_1 is the Kostant-Souriau form. To do this we need the theory of semi-simple Lie algebras and their root systems.

Let $\mathbf{P}(z)$ be the subgroup of $\mathbf{G}$ leaving $z \in \Delta$ fixed. Put $\mathbf{Q}(z)$ for the subgroup of $\mathbf{G}$ leaving $\hat{z}$ fixed. Put $\mathfrak{z}$ for the 1-dimensional subspace of the Lie algebra $\mathfrak{q}(z)$ which is orthogonal to $\mathfrak{p}(z)$ under the Killing form. Observe that for $K \in \mathfrak{z}$ we have $K = 0$ iff $\hat{z}K = 0$. It follows that $\mathfrak{z}$ is a central ideal in $\mathfrak{q}(z)$.

LEMMA 2.6. $\mathbf{Q}(z)$ *contains the identity component of the centralizer of* $\mathfrak{z}$ *in* $\mathbf{G}$.

PROOF. Using the Killing form one can choose $K \in \mathfrak{z}$ so that

$$\hat{z}X = \mathrm{KILL}\,(K, X) \qquad \forall X \in \mathfrak{g}.$$

If $S \in \mathbf{G}$ is such that $\operatorname{ad} S\mathfrak{z} = \mathfrak{z}$ then $\operatorname{ad} S\, K = \pm K$, and hence $\hat{z} \operatorname{ad} S^{-1} = \pm \hat{z}$.

Take $\mathfrak{a}$ to be a maximal abelian subalgebra of $\mathfrak{g}$ which contains $\mathfrak{z}$ By Lemma 2.6, $\mathfrak{a} \subset \mathfrak{q}(z)$. Take $K \in \mathfrak{a}$ such that $H_\alpha = -iK$ is a co-root vector for a root α of $\mathbb{C} \otimes \mathfrak{a}$ in $\mathbb{C} \otimes \mathfrak{g}$, i.e.

$$\alpha(H) = 2\mathrm{KILL}\,(H, H_\alpha) / \mathrm{KILL}\,(H_\alpha, H_\alpha) \qquad \text{for all } H \in \mathbb{C} \otimes \mathfrak{a}.$$

In particular $\exp(2\pi K) = 1$. Observe that

$$\Delta_K(z) = \Theta(z)c \quad \text{where} \quad c = \theta(z; \Delta_K) \equiv \hat{z}K.$$

We must have c integral. This is to say that $\hat{z} = -i\lambda$ where λ is a *weight* for $\mathbb{C} \otimes \mathfrak{a}$. For a suitable ordering of the roots, λ is the highest weight of an irreducible unitary representation of $\mathbf{G}$.

At this stage we need

PROOF OF LEMMA 1.5. We have to prove, for $\xi \in T_z(\Delta)$

$$\theta(\xi) = 0 \quad \text{and} \quad d\theta(\xi, \eta) = 0\, \forall \eta \in T_z(\Delta) \qquad \text{imply } \xi = 0.$$

It suffices to take $z = z_0$ and ξ of the form Xz_0, $X \in \mathfrak{g}$. Observe that

$$d\theta(z; \Delta_X, \Delta_Y) = \theta(z; \Delta_{[X,Y]}) = 2\mathrm{Im}\langle Yz, Xz\rangle.$$

We have only to show that if $\langle Xz_0, Yz_0\rangle$ is real for every $Y \in \mathfrak{g}$ and $\langle z_0, Xz_0\rangle = 0$ then $Xz_0 = 0$. Let $\mathfrak{a}$ be the maximal abelian subalgebra of $\mathfrak{g}$ used to define the weights. Let $\mathfrak{g}_+$ be the subspace of $\mathbb{C} \otimes \mathfrak{g}$ spanned by the positive root vectors for $\mathbb{C} \otimes \mathfrak{a}$ for some ordering of the roots. Then every element of $\mathfrak{g}$ can be written in the form

$$X = K + (M - M*) \qquad \text{where } K \in \mathfrak{a} \text{ and } M \in \mathfrak{g}_+.$$

By the fact that z_0 is a highest weight vector we have $Mz_0 = 0$ for $M \in \mathfrak{g}_+$. Therefore

$$Xz_0 = z_0 i\lambda(K) - M^* z_0.$$

By the hypothesis on X, $\lambda(K) = 0$. Take $Y = -i(M + M^*)$; this defines an element of $\mathfrak{g}$ and $\langle Xz_0, Yz_0\rangle = i\|M^* z_0\|^2$ which is real iff $Xz_0 = 0$. Since λ is supposed to be a non-zero weight, there exists $K \in \mathfrak{a}$ such that $\lambda K = i$, and $\exp(tK)z_0 = z_0 e^{it}$ which proves the invariance of Δ under scalar multiplication by complex numbers of modulus 1.

Now let us assume that $\mathbf{G}$ acts faithfully and transitively as restricted contact transformations of (Δ, θ). For a given $z_0 \in \Delta$ we get a weight $\lambda = -i\hat{z}_0$. Consider the irreducible unitary representation π of $\mathbf{G}$ in the complex Hilbert space $\mathcal{U}$ with highest weight vector $\tilde{z}_0$; the maximal abelian subalgebra $\mathbb{C} \otimes \mathfrak{a}$ and the ordering of the roots being that constructed for z_0. Let $(\tilde{\Delta}, \tilde{\theta})$ be the $\mathbf{G}$-orbit through $\tilde{z}_0$ with the contact form given by (1.4). Since π is a representation of $\mathbf{G}$, and not merely of its universal covering group, the transitivity of $\mathbf{G}$ on Δ gives a map

$$\Delta \longrightarrow \tilde{\Delta}, \quad Sz_0 \longmapsto \pi(S)\tilde{z}_0, \quad S \in \mathbf{G}$$

such that $\tilde{\theta}$ is carried back to θ. We claim that the map is a diffeomorphism; one only need prove that it is one-to-one. It is easy to see that if $C \in \mathbf{G}$ and $\pi(C) = I$ then C must be in the centre of $\mathbf{G}$. Thus C belongs to the maximal torus with Lie algebra $\mathfrak{a}$, and we can write $C = \exp(K)$ where $K \in \mathfrak{a}$. Since $\pi(C) = 1$ we have $\theta(\Delta_K(z_0)) = 0$ as well as $d\theta(\Delta_K(z_0), \Delta_X(z_0)) = 0$ for all $X \in \mathfrak{g}$. This implies that $\Delta_K(z_0) = 0$, so $Cz_0 = z_0$, i.e. $C = 1$.

We have now completed the proof of (1.3), but see the next paragraph for more details.

PROOF OF PROPOSITION 1.7. Once again we may assume that we are in the situation of Lemma 1.4 with $\mathbf{G}$ a group of unitary transformations of $\mathcal{U}$; here $\mathbf{Q}(z_0)$ is the subgroup leaving invariant the complex line containing the highest weight vector $z_0 \in \mathcal{U}$ invariant. To say that

$$\mathcal{H} \cong \mathbf{G}/\mathbf{Q}(z_0)$$

is a symmetric space of $\mathbf{G}$ is to say that there exists an involution σ of $\mathfrak{g}$ whose fixed subspace is $\mathfrak{q}(z_0)$. To investigate this involution we take an ordering of the roots of $\mathbb{C} \otimes \mathfrak{a}$ in $\mathbb{C} \otimes \mathfrak{g}$ for which λ is a highest weight. Put $\mathfrak{g}(\alpha)$ for the root space in $\mathbb{C} \otimes \mathfrak{g}$ of the root α, and let H_α be the corresponding co-root. One has that

$$[M,N] = c(M,N)H_\alpha \qquad \text{if } M \in \mathfrak{g}(\alpha),\ N \in \mathfrak{g}(-\alpha)$$
$$\text{where } 2c(M,N)\,\mathrm{KILL}\,(H_\alpha, H_\alpha) = \mathrm{KILL}\,(M,N).$$

As in the proof of (1.5) we can identify $T_{\hat{z}_0}(\mathcal{H})$ with the subspace $V(z_0)$ of $\mathfrak{g}$ generated by the vectors of the form $(M_\alpha - M_\alpha^*)$ where $\lambda H_\alpha > 0$. If the involution σ exists we must have $\sigma = -I$ on $V(z)$, but this requires that

$$(2.7) \qquad [V(z_0), V(z_0)] \subset \mathfrak{q}(z_0).$$

One calculates

$$[M_\alpha - M_\alpha^*, N_\beta - N_\beta^*] = [M_\alpha, N_\beta] - [M_\alpha, N_\beta]^* - ([M_\alpha, N_\beta^*] - [M_\alpha, N_\beta^*]^*).$$

Now $[M_\alpha, N_\beta] \in \mathfrak{q}(z_0)$ unless $\alpha + \beta$ is a positive root and $\lambda H_{\alpha+\beta} > 0$. Similar considerations apply to the second term; so (2.7) holds unless we have positive roots α and β with $\lambda H_\alpha > 0$, $\lambda H_\beta > 0$ such that $\alpha + \beta$ is a positive root with $\lambda H_{\alpha+\beta} > 0$ or $\alpha - \beta$ is a positive root with $\lambda H_{\alpha-\beta} > 0$. The two cases need not be distinguished. To say the α is a root and $\lambda H_\alpha > 0$ is to say that $\lambda - \alpha$ is a weight of the representation π with highest weight λ.

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