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On the Singular Behavior of the Inverse Laplace Transforms of the Functions $\frac{I_n(s)}{sI_n'(s)}$

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Abstract. Exact analytical expressions for the inverse Laplace transforms of the functions $\frac{I_n(s)}{sI_n'(s)}$ are obtained in the form of trigonometric series. The convergence of the series is analyzed theoretically, and it is proven that those diverge on an infinite denumerable set of points. Therefore it is shown that the inverse transforms have an infinite number of singular points. This result, to the best of the author's knowledge, is new, as the inverse transforms of $\frac{I_n(s)}{sI_n'(s)}$ have previously been considered to be piecewise smooth and continuous. It is also found that the inverse transforms have an infinite number of points of finite discontinuity with different left- and right-side limits. The points of singularity and points of finite discontinuity alternate, and the sign of the infinity at the singular points also alternates depending on the order n. The behavior of the inverse transforms in the proximity of the singular points and the points of finite discontinuity is addressed as well.

1 Introduction

The inverse Laplace transforms of the functions

$$\psi_n^L(s) = \frac{I_n(s)}{sI_n'(s)}, \tag{1}$$

where $I_n(s)$ is the modified Bessel function of the first kind of integer order n and a complex argument s, and the prime denotes the first derivative with respect to the argument, play a very important role in the analytical treatment of the problem of the non-stationary loading on a fluid-filled circular cylindrical shell structure. These inverse transforms are called the *response functions* (referred sometimes as the transfer functions), and the notation $\psi_n(t)$ (or $\xi_n(t)$) is adopted for them. The response functions allow one to express hydrodynamic pressure in terms of shell deformations in the integral form. The most important and practically useful feature of the response functions is that they do not depend on either structure or fluid properties. An approach based on use of the response functions seems to be very computationally attractive because the time-consuming numerical inversion of the Laplace transform is avoided. Once calculated and tabulated, the response functions can be used for any other problem of the same geometry; therefore the exact values of the inverse transforms of (1) are of both practical and theoretical interest.

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The idea of dealing with property-independent functions (applying to the considered class of problems) was initially introduced by Geers [4]: he considered the 'exterior' problem of the non-stationary interaction between a hydrodynamic shock wave and a hollow submerged shell. The 'interior' functions (1) were first introduced in the research monograph of Pertsev and Platonov [6]. Some results based on numerical inversion were presented in that work. In particular, as a result of numerical computations, it was found that the inverse transforms of (1) are piecewise smooth and continuous. However, the authors did not provide any theoretical basis to support their numerical observations, and did not address the question of the total error given by the used numerical procedure. The very similar functions were addressed in the monograph [5], but no analytical treatment was discussed, and the authors introduced only numerical results. Again, no singular behavior was observed.

Thus, by the late 1980s, the response functions $\psi_n(t)$ were considered to be known and tabulated, and believed to be continuous. However, working on the study of the behavior of a fluid-filled cylindrical structure under shock loading, the author had to calculate the response functions with higher precision, and for a wider range of n than it was available in the published literature. While carrying out the numerical inversion, some obvious problems with the convergence of Mellin's integral were observed at certain points t. After a very careful numerical analysis it was realized that those problems were not just numerical effects, and the response functions seemed to have at least a finite number of singular points. Therefore, the careful verification of the previously published results was needed to prove or contradict the assumption of the potential existence of singular points.

Thus, the main goal of the present paper is to obtain an analytical expression for the response functions, and to prove that they do have singular points. The behavior in the proximity of the singular points is of interest as well.

2 Mellin's Integral and Singular Points

In order to obtain the inverse transform of (1), let us consider Mellin's integral for the function $\psi_n^L(s)$

$$\psi_n(t) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \psi_n^L(s) \mathrm{e}^{st}\, ds, \tag{2}$$

where all the singular points of the integrand lie in the half-plane $\operatorname{Re} s < \epsilon$.

First, let us determine the singular points of ψ_n^L. It is obvious that all the zeros of $I_n'(s)$ will be the singular points as well as possibly the point $s = 0$. We know [1]

$$I_n(s) = \mathrm{e}^{-\frac{n\pi i}{2}} J_n(is), \tag{3}$$

$$I_n'(s) = i\mathrm{e}^{-\frac{n\pi i}{2}} J_n'(is), \tag{4}$$

where $J_n(s)$ is the Bessel function of order n and an argument s. It is also known that all the zeros of $J_n'(s)$ are real. Therefore, all the zeros of $I_n'(s)$ are pure imaginary. Noting that $J_n'(-\omega) = (-1)^n J_n'(\omega)$, those can be written as

$$s_{\pm k}^n = \pm i\omega_k^n, \quad k = 1, 2, \dots, \tag{5}$$

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where ω_k^n are the positive zeros of $J_n'(\omega)$. All the zeros (5) are of first order.

Thus, ψ_n^L has an infinite number of simple poles defined by (5). It is easy to show (Appendix A) that $s = 0$ is a pole of second order for $n = 0$, and a removable singular point for $n \geq 1$. In the next section we will introduce an approach based on the theory of residues to treat Mellin's integral.

3 Asymptotic Behavior of $\psi_n^L(s)$ on a Circle of Large Radius

Let us consider a circle of a sufficiently large radius $R \gg 1$. In this case

$$s = Re^{i\phi}, \quad \phi \in [0, 2\pi], \quad \text{and} \quad |s| \gg 1. \tag{6}$$

The asymptotic expansions for I_n and I_n' at large $|s|$ are [3]

$$I_n(s) = \frac{1}{\sqrt{2\pi s}}\left(\mathrm{e}^s - (-1)^n i\mathrm{e}^{-s}\right)\left(1 + O(s^{-1})\right), \tag{7}$$

$$I_n'(s) = \frac{1}{\sqrt{2\pi s}}\left(\mathrm{e}^s + (-1)^n i\mathrm{e}^{-s}\right)\left(1 + O(s^{-1})\right), \tag{8}$$

and hence

$$\psi_n^L(s) \sim \frac{\mathrm{e}^s - (-1)^n i\mathrm{e}^{-s}}{s\left(\mathrm{e}^s + (-1)^n i\mathrm{e}^{-s}\right)}, \quad |s| \gg 1. \tag{9}$$

We will consider only odd n as the approach for even n is the same. Noting (6) we have

$$|\psi_n^L(s)| \sim \frac{1}{R}\chi(R, \phi), \tag{10}$$

where

$$\chi(R, \phi) = \sqrt{\frac{\mathrm{e}^{2R\cos\phi} + \mathrm{e}^{-2R\cos\phi} + 2\sin(2R\sin\phi)}{\mathrm{e}^{2R\cos\phi} + \mathrm{e}^{-2R\cos\phi} - 2\sin(2R\sin\phi)}}. \tag{11}$$

The denominator in (11) becomes 0 when

$$R = \frac{\pi}{4} + \pi k, \quad k = 0, 1, 2, \dots, \ \phi = \frac{\pi}{2}, \tag{12}$$

and when

$$R = \frac{3\pi}{4} + \pi k, \quad k = 0, 1, 2, \dots, \ \phi = \frac{3\pi}{2}. \tag{13}$$

Thus, $\chi(R, \phi)$ is unbounded on the family of circles

$$R = \frac{\pi}{4} + \frac{\pi k}{2}, \quad k = 0, 1, 2, \dots. \tag{14}$$

Now let us choose a family of circles R_k such that, first, it does not coincide with the family (14), and, second, R_k do not fall in the close proximity of the numbers

$$\left(k+\frac{n}{2}-\frac{3}{4}\right)\pi, \quad n=0,1,\dots,\ k=0,1,\dots. \tag{15}$$

As we will see later, (15) represents the leading term of the asymptotic expansion of the zeros of J_n' at large n, and thus it is guaranteed that R_k do not coincide with the poles of ψ_n^L. The family of circles

$$R_k=\pi k, \quad k=1,2,\dots \tag{16}$$

satisfies both of these conditions.

It can be shown (Appendix C) that on the family of circles (16)

$$\chi(R_k,\phi)<2, \quad \phi\in[0,2\pi], \tag{17}$$

when $k \gg 1$. Hence

$$\max_{\phi\in[0,2\pi]}|\psi_n^L(R_k e^{i\phi})|<\frac{2}{R_k}, \quad k\gg 1, \tag{18}$$

and we have shown that the functions $\psi_n^L(s)$ uniformly tend to zero with respect to ϕ ($\phi=\arg s$) on the infinite family of circles (16).

4 Residue Theory

Let us consider the contour Γ_k (Figure 1)

$$\Gamma_k=C_k+P_k, \tag{19}$$

where C_k is an arc of a circle of radius R_k (16) centered on the origin, and P_k is a segment of the line $\operatorname{Re} s=\epsilon$. Let us now address the contour integral

$$\int_{\Gamma_k}\psi_n^L(s)e^{st}\,ds. \tag{20}$$

On the one hand, this integral can be decomposed into a sum of two contour integrals as

$$\int_{\Gamma_k}\psi_n^L(s)e^{st}\,ds=\int_{C_k}\psi_n^L(s)e^{st}\,ds+\int_{P_k}\psi_n^L(s)e^{st}\,ds, \tag{21}$$

and, on the other hand, it can be expressed in terms of the residues of the integrand as

$$\int_{\Gamma_k}\psi_n^L(s)e^{st}\,ds=2\pi i\sum_{s_j^n\in D_k}R_{s_j^n}^n, \tag{22}$$

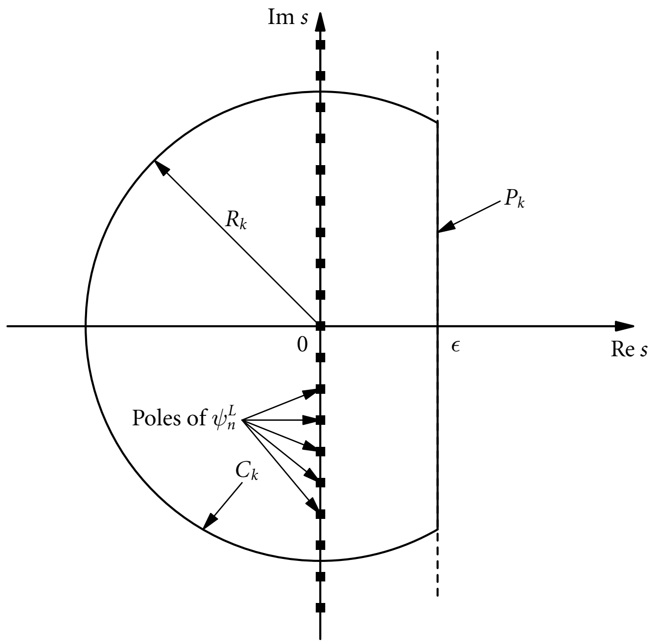

Figure 1: Contour Γ_k.

where $R^n_{s^\star}$ is the residue of the function $\psi_n^L(s)e^{st}$ at the point $s = s^\star$, and D_k is the domain bounded by the contour Γ_k.

When $k \to \infty$, the second term in the right-hand side of (21) becomes

$$\int_{\epsilon - i\infty}^{\epsilon + i\infty} \psi_n^L(s)e^{st}\, ds, \tag{23}$$

and the right-hand side of (22) is

$$2\pi i \sum_{k=\pm 1, \pm 2, \dots} R^n_{s^n_k}. \tag{24}$$

We have shown that $\psi_n^L(s)$ uniformly (with respect to ϕ) tends to zero on the family of circles R_k. Thus, by virtue of Jordan's modified lemma (Appendix D),

$$\int_{C_k} \psi_n^L(s)e^{st}\, ds \to 0 \quad \text{when } k \to \infty, \tag{25}$$

and finally we have

$$2\pi i \sum_{k=\pm 1, \pm 2, \dots} R^n_{s^n_k} = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \psi_n^L(s)e^{st}\, ds. \tag{26}$$

Looking at (2) we can write

$$\psi_n(t) = \sum_{k=\pm 1,\pm 2,\dots} R^n_{s^n_k}, \tag{27}$$

where s^n_k are all the poles defined by (5), as well as $s = 0$ for $n = 0$.

It can be shown (Appendix B) that the residues of $\psi^L_n(s)e^{st}$ at $s = 0$ and $s = s^n_k$ are

$$R^0_0 = 2t, \tag{28}$$

$$R^n_{i\omega^k_n,\, k=1,2,\dots} = \frac{i\omega^n_k}{n^2 - (\omega^n_k)^2}\{\cos(\omega^n_k t) + i\sin(\omega^n_k t)\}, \tag{29}$$

$$R^n_{-i\omega^n_k,\, k=1,2,\dots} = -\frac{i\omega^n_k}{n^2 - (\omega^n_k)^2}\{\cos(\omega^n_k t) - i\sin(\omega^n_k t)\}, \tag{30}$$

where ω^n_k, $k = 1, 2, \dots$, are the positive zeros of $J'_n(\omega)$. Taking into account these formulas, the functions $\psi_n(t)$ can be obtained in final form as

$$\psi_0(t) = 2t + 2\sum_{k=1}^{\infty} \frac{\sin(\omega^0_k t)}{\omega^0_k}, \tag{31}$$

$$\psi_n(t) = 2\sum_{k=1}^{\infty} \frac{\omega^n_k}{(\omega^n_k)^2 - n^2}\sin(\omega^n_k t), \quad n \geq 1. \tag{32}$$

Now, convergence of the series (31)–(32) is of interest.

5 Convergence of the Series For $\psi_n(t)$

We will address only convergence of the series (32) as (31) is essentially a special case of (32). It is known [1] that the asymptotic representation for the zeros of $J'_n(\omega)$ is

$$\omega^n_k = \beta^n_k - \frac{\mu + 1}{8\beta^n_k} + O\Big(\frac{1}{k^3}\Big), \quad k \gg 1, \tag{33}$$

where

$$\beta^n_k = \Big(k + \frac{n}{2} - \frac{3}{4}\Big)\pi, \quad \mu = 4n^2. \tag{34}$$

Now

$$\sin(\omega^n_k t) = \sin(\beta^n_k t) + \frac{\cos(\beta^n_k t)(1 + 4n^2)t}{8\pi k} + O\Big(\frac{1}{k^2}\Big), \quad k \gg 1, \tag{35}$$

and

$$\frac{\omega^n_k}{(\omega^n_k)^2 - n^2} = \frac{1}{\pi k} + \frac{(3 - 2n)}{4\pi}\frac{1}{k^2} + O\Big(\frac{1}{k^3}\Big), \quad k \gg 1. \tag{36}$$

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Hence, the N-th remainder ($N \gg 1$) of the series in (32) can be decomposed into a sum of two series as

$$\sum_{k=N}^{\infty} \frac{\omega_k^n}{(\omega_k^n)^2 - n^2} \sin(\omega_k^n t) = S_1 + S_2, \tag{37}$$

where

$$S_1 = \frac{1}{\pi} \sum_{k=N}^{\infty} \frac{\sin(\beta_k^n t)}{k}, \tag{38}$$

(39)
$$S_2 = \sum_{k=N}^{\infty} a_k, \quad a_k = \left\{ \frac{\sin(\beta_k^n t)(3-2n)}{4\pi} + \frac{\cos(\beta_k^n t)(1+4n^2)t}{8\pi^2} \right\} \frac{1}{k^2} + O\Big(\frac{1}{k^3}\Big).$$

The series in (39) is absolutely convergent whereas convergence of the series in (38) depends on t. Noting (34) we can write the series in (38) as

$$\sum_{k=N}^{\infty} \frac{\sin(\beta_k^n t)}{k} = \sum_{k=N}^{\infty} \frac{\sin\{(k + \frac{n}{2} - \frac{3}{4})\pi t\}}{k}. \tag{40}$$

Let us consider the behavior of the series (40) at

$$t = 2(2j+1), \quad j = 0, 1, 2, \dots. \tag{41}$$

Note that at all other t, the series in (38) converges by virtue of Dirichlet's test. When t is as defined by (41), the numerator in the right-hand side of (40) is equal to either -1 or 1 for all k. Hence, for these t the series in (38) is

$$\sum_{k=N}^{\infty} \frac{\sin(\beta_k^n t)}{k} = \pm \sum_{k=N}^{\infty} \frac{1}{k}. \tag{42}$$

The series in the right-hand side of (42) diverges, and, consequently, the series in (38), as well as the series in (32), also diverge. The sign of the infinity depends on both n and t, and for even n and $n = 0$ is

$$\psi_n(t) = \begin{cases} +\infty & \text{at } t = 2(4l+1), l = 0, 1, \dots \ (t = 2, 10, 18, \dots) \\ -\infty & \text{at } t = 2(4l+3), l = 0, 1, \dots \ (t = 6, 14, 22, \dots) \end{cases}, \tag{43}$$

while for odd n we have

$$\psi_n(t) = \begin{cases} +\infty & \text{at } t = 2(4l+3), l = 0, 1, \dots \ (t = 6, 14, 22, \dots) \\ -\infty & \text{at } t = 2(4l+1), l = 0, 1, \dots \ (t = 2, 10, 18, \dots) \end{cases}. \tag{44}$$

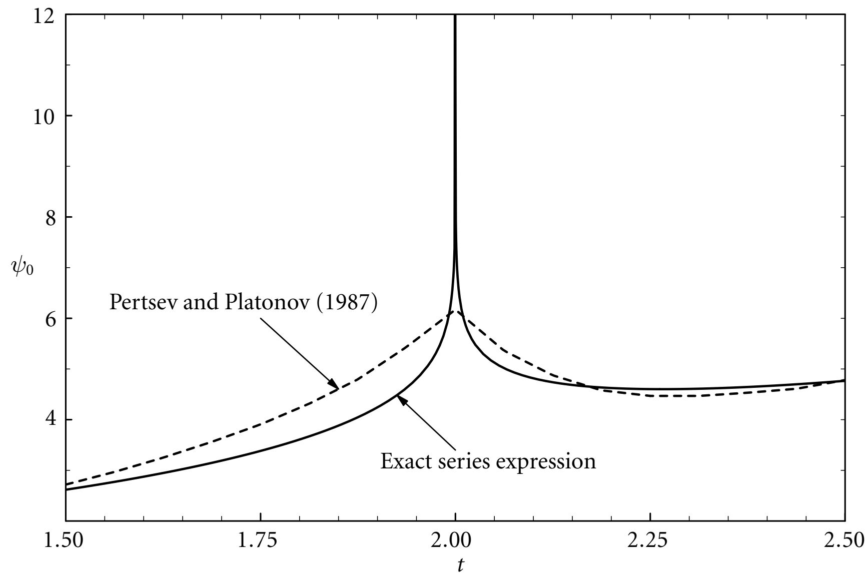

Figure 2: The function $\psi_0(t)$ in the proximity of the point $t = 2$ in comparison with the results by Pertsev and Platonov [6].

Thus, we have shown that the functions $\psi_n(t)$ have an infinite number of singular points, and that the sign of the infinity at those alternates depending on the order n and t according to (43) and (44). As an example, the behavior of the function $\psi_0(t)$ in the proximity of the singular point $t = 2$ is shown in Figure 2. The figure also shows the comparison of the present results to the previously published ones.

Let us now address

$$t = 4m, \quad m = 1, 2, \ldots. \tag{45}$$

The series in (32) converges at these t since $S_1 = 0$. However, this case needs special treatment because the numerical results for $\psi_n(t)$ in the proximity of the points (45) look very 'suspicious', and it is possible that the functions have discontinuities at these t as well. We will show that, in fact, this is truth, and $\psi_n(t)$ do have different left- and right-side limits at the points (45).

Let us consider

$$t = 4m \pm \delta, \tag{46}$$

where $0 < \delta \ll 1$. One might expect that S_1 is close to 0 for these values of t. However, this is not the case. To show it, let us first substitute $t = 4m \pm \delta$ into the expression for S_1. We have

$$S_1|_{t=4m\pm\delta,\, m=1,2,\dots} = \frac{1}{\pi}\sum_{k=N}^{\infty}\frac{\sin\bigl(\beta_k^n(4m\pm\delta)\bigr)}{k}$$

$$= \begin{cases} \mp\dfrac{1}{\pi}\displaystyle\sum_{k=N}^{\infty}\frac{\sin\bigl(\delta\pi(k+\frac{n}{2}-\frac{3}{4})\bigr)}{k}, & m=1,3,5,\dots \\ \pm\dfrac{1}{\pi}\displaystyle\sum_{k=N}^{\infty}\frac{\sin\bigl(\delta\pi(k+\frac{n}{2}-\frac{3}{4})\bigr)}{k}, & m=2,4,6,\dots. \end{cases} \tag{47}$$

Let us define $Q(\delta, N)$ as

$$Q(\delta,N) = \sum_{k=N}^{\infty}\frac{\sin\bigl(\delta\pi(k+\frac{n}{2}-\frac{3}{4})\bigr)}{k}, \tag{48}$$

then we can write

$$S_1|_{t=4m\pm\delta,\, m=1,2,\dots} = \begin{cases} \mp\dfrac{1}{\pi}Q(\delta,N), & m=1,3,5,\dots \\ \pm\dfrac{1}{\pi}Q(\delta,N), & m=2,4,6,\dots. \end{cases} \tag{49}$$

In order to analyze the function $Q(\delta, N)$, let us make use of the transcendental function $\Phi(z, s, a)$ (called sometimes the Lerch transcendental function) which is defined as

$$\Phi(z,p,a) = \sum_{k=0}^{\infty}\frac{z^k}{(a+k)^p}. \tag{50}$$

Details regarding this function can be found in [2], pages 27–31 (note that the authors do not use the term "Lerch function").

It is easy to see that $Q(\delta, N)$ can be written in terms of the Lerch functions as

$$Q(\delta,N) = \frac{\mathrm{e}^{i\delta\pi(N+\frac{n}{2}-\frac{3}{4})}}{2i}\Phi(\mathrm{e}^{i\delta\pi},1,N) - \frac{\mathrm{e}^{-i\delta\pi(N+\frac{n}{2}-\frac{3}{4})}}{2i}\Phi(\mathrm{e}^{-i\delta\pi},1,N). \tag{51}$$

It is known ([2], page 30, formula (13)) that $\Phi(z, 1, a) \sim -\log(1 - z)$ as $z \to 1$. Using this fact, for small δ we have

$$Q(\delta,N) \sim \frac{1}{2i}\log(-\mathrm{e}^{-i\delta\pi}) = \frac{1}{2}\arg(-\mathrm{e}^{-i\delta\pi}), \tag{52}$$

and now it is obvious that

$$Q(\delta,N) \to \frac{\pi}{2} \quad \text{as } \delta \to 0. \tag{53}$$

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It follows that S_1 has different one-side limits at $t = 4m$,

(54)
$$\lim_{t\to 4m^-} S_1 = \begin{cases} \frac{1}{2}, & m = 1, 3, 5, \dots \\ -\frac{1}{2}, & m = 2, 4, 6, \dots, \end{cases}$$

and

(55)
$$\lim_{t\to 4m^+} S_1 = \begin{cases} -\frac{1}{2}, & m = 1, 3, 5, \dots \\ \frac{1}{2}, & m = 2, 4, 6, \dots . \end{cases}$$

In particular, it follows from these formulas that the difference between one-side limits of S_1 at $t = 4m$ is equal to 1 for any m. Thus, we have shown that S_1 has a discontinuity with the magnitude 1, which corresponds to a discontinuity with the magnitude 2 in $\psi_n(t)$ (for any n). It should be especially noted that the one-side limits of S_1 do not depend on N: although the magnitude of S_1 does decrease with increasing N, the one-side limits at $t = 4m$ are still the same.

However, the behavior of $\psi_n(t)$ in the proximity of the points $t = 4m$ depends not only on S_1 but also on $\sum_{k=1}^{N-1} \frac{\omega_k^n}{(\omega_k^n)^2 - n^2} \sin(\omega_k^n t)$ and S_2. The first term is just a finite sum of continuous functions, and it affects only the magnitude of $\psi_n(t)$, not the discontinuity about $t = 4m$. However, comparing the expressions for S_1 and S_2, one might expect that S_2 has different one-side limits at $t = 4m$ as well. Thus, let us now address the behavior of S_2 in the proximity of these points.

First of all, it should be noted that we have already proven $\psi_n(t)$ to have a discontinuity at $t = 4m$ because of the S_1 contribution. The contribution of S_2 has a much smaller magnitude, and the 'addition' provided by S_2 is, to some extent, irrelevant: it does not change the discontinuous nature of $\psi_n(t)$ at $t = 4m$. However, it is desirable to gain more certainty in this regard.

Using the same idea as for S_1, it is possible to express S_2 in terms of the Lerch functions. In fact, if we consider all the terms in the asymptotic expansion for ω_k^n, there will be a series of the Lerch functions of integer orders with n-dependent coefficients: $\Phi(\mathrm{e}^{\pm i\delta\pi}, 2, N)$ will represent the infinite series with $\frac{1}{k^2}$, $\Phi(\mathrm{e}^{\pm i\delta\pi}, 3, N)$ will stand for the series with $\frac{1}{k^3}$, etc. Asymptotic analysis of this series of the Lerch functions would allow one to make a conclusion about the behavior of S_2. However, this approach seems too complicated. Instead, we can apply the Cauchy criterion to the convergent series S_2 and see that S_2 can be made as small as desired by choosing N large enough. Hence, for any fixed t and n, S_2 decays with increasing N, whereas S_1 has a point of discontinuity with a constant difference between left- and right-side limits. Thus, we have demonstrated that the contribution of S_2 is negligible, even if it has different one-side limits at $t = 4m$. However, as numerical computations show, S_2 does have the zero limit at $t = 4m$. Moreover, as one can observe from the graphs of $\psi_n(t)$, the magnitude of the discontinuity at $t = 4m$ is always approximately equal to 2, which demonstrates once again that S_1 is the only (noticeable) contributor to the discontinuous nature of $\psi_n(t)$ at the points $t = 4m$.

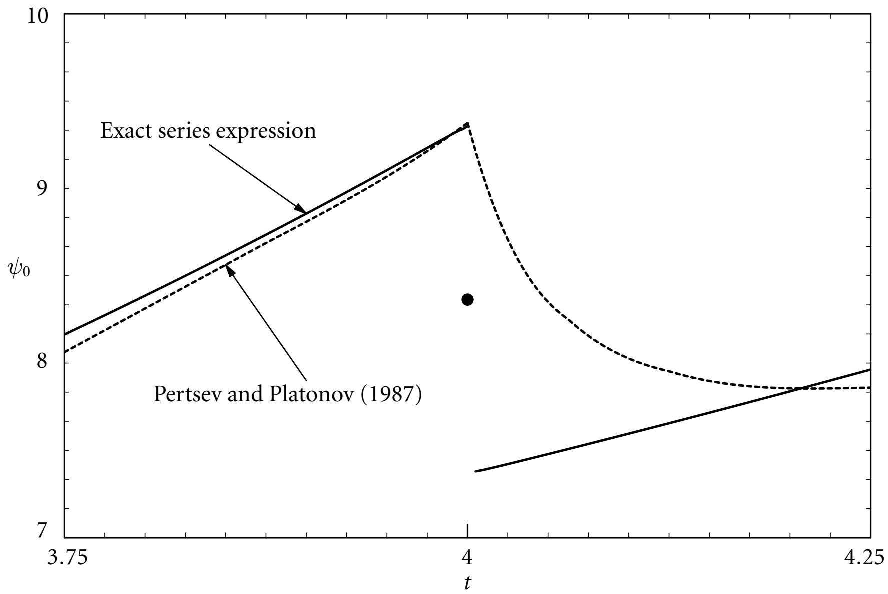

Figure 3: The function $\psi_0(t)$ in the proximity of the point $t = 4$ in comparison with the results by Pertsev and Platonov [6].

Finally, we have shown that the functions $\psi_n(t)$ have an infinite number of points of finite discontinuity at $t = 4m$, $m = 1, 2, \dots$. As an example, Figure 3 shows the behavior of the function $\psi_0(t)$ in the proximity of the point $t = 4$. Comparison with the previously published results is presented as well.

Thus, we have shown that the functions $\psi_n(t)$ have an infinite number of singular points at $t = 2(2j + 1)$, $j = 0, 1, 2, \dots$, as well as an infinite number of points of finite discontinuity at $t = 4j$, $j = 1, 2, \dots$. The points of discontinuity and the points of singularity alternate, and the sign of the infinity at the singular points also alternates for specific n.

6 Numerical Results

The functions $\psi_n(t)$ were calculated for $n = 0 \cdots 30$. The number of terms in the series (31)–(32) was chosen to ensure a final accuracy about 0.1%. Figure 4 shows the function $\psi_0(t)$ for $t = 0 \cdots 15$. One can clearly see the alternation of signs of the infinity at the singular points depending on t, as well as the alternation of a type of the singular points. The main contribution is provided by the first term in (31) whereas the singular 'fluctuations' appear due to the second term. Note that the numerical simulations based on the obtained response functions have shown that this essentially singular behavior clearly represents the complex wave nature of the interaction process in real structures. Figure 5 shows the functions $\psi_n(t)$ for $n = 1$ and 2. One can see the alternation of the sign of the infinity at $t = 2$, as well as the

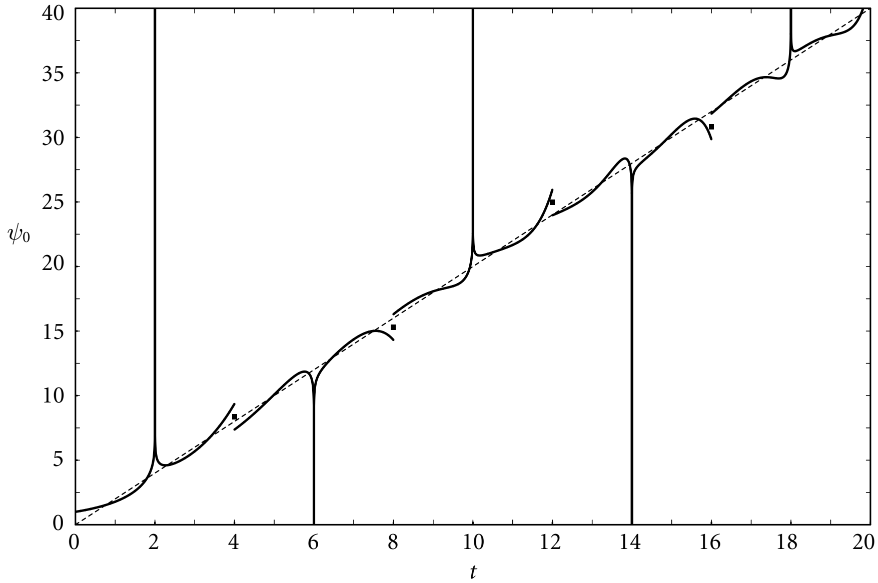

Figure 4: The function $\psi_0(t)$.

behavior at $t = 4$, depending on n. Finally, Figure 6 shows the function $\psi_n(t)$ for $n = 5$.

7 Conclusions

An analytical procedure for the inversion of the Laplace transform for the functions $\frac{I_n(s)}{sI_n'(s)}$ is established. The inverse transforms $\psi_n(t)$, being the response functions of the 'interior' problem of fluid-circular cylindrical structure interaction, are obtained in the form of not everywhere convergent trigonometric series. The convergence of the series is analyzed in detail, and it is shown that the functions $\psi_n(t)$ have an infinite number of singular points as well as an infinite number of points of finite discontinuity. The graphs of the response functions are presented for several n, and comparison with the previously published results is provided, showing the substantial difference.

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Appendix A. Type of the Point $s = 0$

We know [1]

$$I_n(s) = \frac{s^n}{2^n\Gamma(n+1)} + O(s^{n+2}), \quad |s| \ll 1. \tag{56}$$

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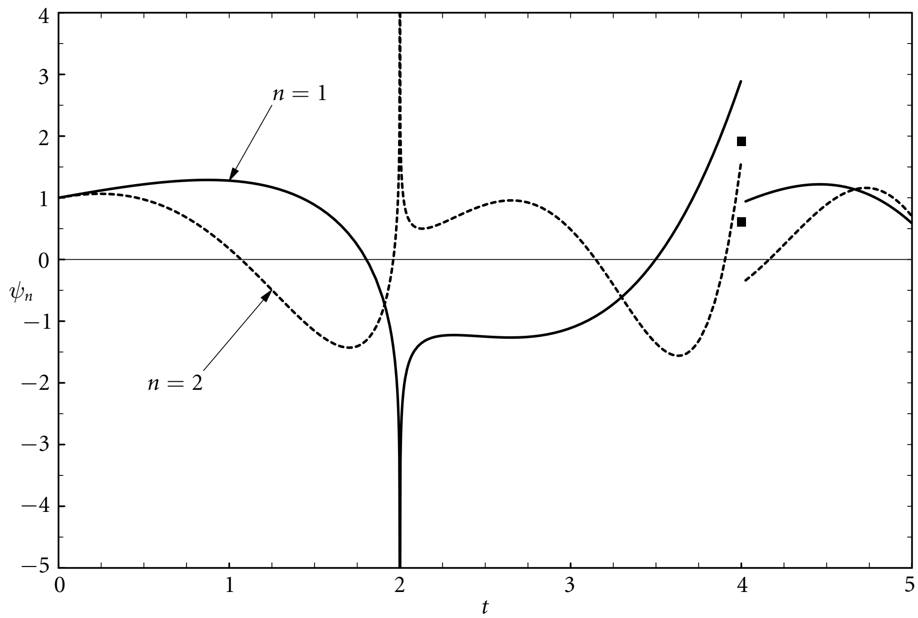

Figure 5: The functions $\psi_1(t)$ and $\psi_2(t)$.

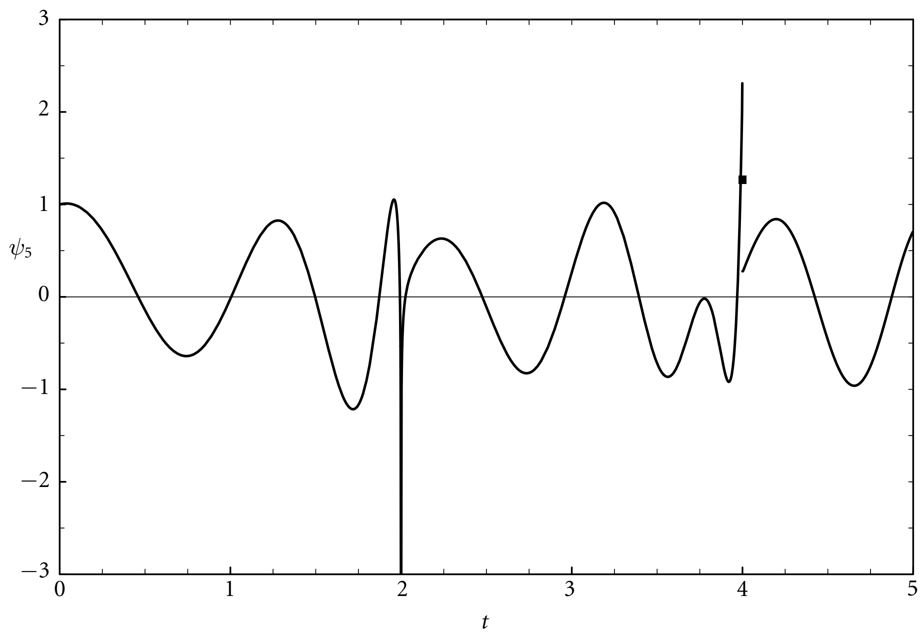

Figure 6: The function $\psi_5(t)$.

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We also know

$$I_n'(s) = I_{n-1}(s) - \frac{n}{s} I_n(s), \quad n \geq 1, \tag{57}$$

$$I_0'(s) = I_1(s). \tag{58}$$

Hence, the asymptotics for $\psi_n^L(s)$ are

$$\psi_n^L(s) = \frac{1}{n} + O(s^2), \quad n \geq 1, \tag{59}$$

$$\psi_0^L(s) = \frac{2}{s^2} + O(1), \tag{60}$$

and we have shown that the function $\psi_n^L(s)$ has a removable singularity at $s = 0$ when $n \geq 1$, and a pole of second order when $n = 0$.

Appendix B. Residues of $\psi_n^L(s)\mathrm{e}^{st}$

Residue at $s = 0$ For $n = 0$, $s = 0$ is a pole of second order, and the residue at this point is

$$\begin{aligned} R_0^0 &= \lim_{s\to 0} \frac{d}{ds}\left(\mathrm{e}^{st} \frac{I_0(s)}{I_1(s)} s \right) \\ &= \lim_{s\to 0} \left\{ \mathrm{e}^{st} \frac{I_0(s)}{I_1(s)} + t\mathrm{e}^{st} s \frac{I_0(s)}{I_1(s)} + \mathrm{e}^{st} s \left[\frac{I_0'(s) I_1(s) - I_1'(s) I_0(s)}{I_1^2(s)} \right] \right\}. \end{aligned} \tag{61}$$

Noting (58) and (57), the third term in (61) can be rewritten as

$$\frac{I_0'(s) I_1(s) - I_1'(s) I_0(s)}{I_1^2(s)} = 1 - \left\{ \frac{I_0(s)}{I_1(s)} \right\}^2 + \frac{1}{s} \frac{I_0(s)}{I_1(s)}. \tag{62}$$

At small s, $I_0(s)/I_1(s)$ is

$$\frac{I_0(s)}{I_1(s)} = \frac{2}{s} + O(s), \tag{63}$$

and

$$\mathrm{e}^{st} \frac{I_0(s)}{I_1(s)} + t\mathrm{e}^{st} s \frac{I_0(s)}{I_1(s)} + \mathrm{e}^{st} s \left\{ 1 - \left[\frac{I_0(s)}{I_1(s)} \right]^2 + \frac{1}{s} \frac{I_0(s)}{I_1(s)} \right\} = 2t + O(s). \tag{64}$$

Hence

$$R_0^0 = 2t. \tag{65}$$

Residue at $s = s_k^n$ These points are simple poles, and the equation for the residue is

$$R^n_{s^n_k} = \frac{I_n(s)}{\big(sI_n'(s)\big)'} e^{st} \Big|_{s=s^n_k}. \tag{66}$$

The denominator of (66) can be represented as

$$I_n(s)\Big(s + \frac{n^2}{s}\Big), \tag{67}$$

and we have

$$R^n_{s^n_k} = \frac{s^n_k}{(s^n_k)^2 + n^2} e^{s^n_k t}. \tag{68}$$

Thus, for $s^n_k = i\omega^n_k$ the residue is

$$R^n_{i\omega^n_k} = \frac{i\omega^n_k}{n^2 - (\omega^n_k)^2} \{\cos(\omega^n_k t) + i\sin(\omega^n_k t)\}, \tag{69}$$

and for $s^n_k = -i\omega^n_k$ we arrive at

$$R^n_{-i\omega^n_k} = -\frac{i\omega^n_k}{n^2 - (\omega^n_k)^2} \{\cos(\omega^n_k t) - i\sin(\omega^n_k t)\}. \tag{70}$$

Appendix C. $\chi(R, \phi)$

Let us consider the function

$$\chi(R,\phi) = \sqrt{\frac{e^{2R\cos\phi} + e^{-2R\cos\phi} + 2\sin(2R\sin\phi)}{e^{2R\cos\phi} + e^{-2R\cos\phi} - 2\sin(2R\sin\phi)}} \tag{71}$$

at sufficiently large $R = R_k = \pi k$, $k \gg 1$, and find its maximum when $\phi \in [0, 2\pi]$. We will address the following two cases: (first) when the first summand in the numerator and denominator of (71) is substantially large than the second one, and (second) when the first and second summands are of the same order.

First case Let

$$|\cos\phi| > \delta > 0, \tag{72}$$

where

$$\delta = \frac{1}{R}. \tag{73}$$

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Let us address the case $\cos\phi > 0$ first. Taking into account (73) we have

$$e^{2R\cos\phi} > e^2, \tag{74}$$

$$e^{-2R\cos\phi} < \frac{1}{e^2}, \tag{75}$$

and we can estimate the numerator and denominator of (71) as

$$e^{2R\cos\phi} + e^{-2R\cos\phi} + 2\sin(2R\sin\phi) < e^{2R\cos\phi}\Big(1+\frac{2e^2+1}{e^4}\Big), \tag{76}$$

$$e^{2R\cos\phi} + e^{-2R\cos\phi} - 2\sin(2R\sin\phi) > e^{2R\cos\phi}\Big(1-\frac{2}{e^2}\Big). \tag{77}$$

Now for $\chi(R,\phi)$ we have

$$\chi(R,\phi) < \frac{e^2+1}{e\sqrt{e^2-2}} = \xi < 2. \tag{78}$$

For the case $\cos\phi < 0$ and $|\cos\phi| > \delta$ the estimate (78) can be obtained in the similar manner. Thus, we have shown that

$$\chi(R,\phi) < \xi, \quad |\cos\phi| > \frac{1}{R}. \tag{79}$$

Note that the estimate (79) is valid for any ϕ except for the close proximity of the points $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

Second Case Let

$$|\cos\phi| \le \delta. \tag{80}$$

Again, let us consider $\cos\phi > 0$. Let us also assume that ϕ is close to $\frac{\pi}{2}$ rather than $\frac{3\pi}{2}$. Since $\arccos\delta = \frac{\pi}{2} - \delta + O(\delta^3)$,

$$\phi = \frac{\pi}{2} - \gamma, \quad 0 < \gamma \ll 1. \tag{81}$$

Then

$$\sin\phi = 1 - \frac{\gamma^2}{2} + O(\gamma^4), \tag{82}$$

and, taking into account that $R = \pi k$, for $\sin(2R\sin\phi)$ we have

$$\sin(2R\sin\phi) = -R\gamma^2 + O(\gamma^3). \tag{83}$$

For ϕ satisfying (81) we also have

$$e^{2R\cos\phi} + e^{-2R\cos\phi} = e^{2R\gamma} + e^{-2R\gamma} + O(\gamma^2), \tag{84}$$

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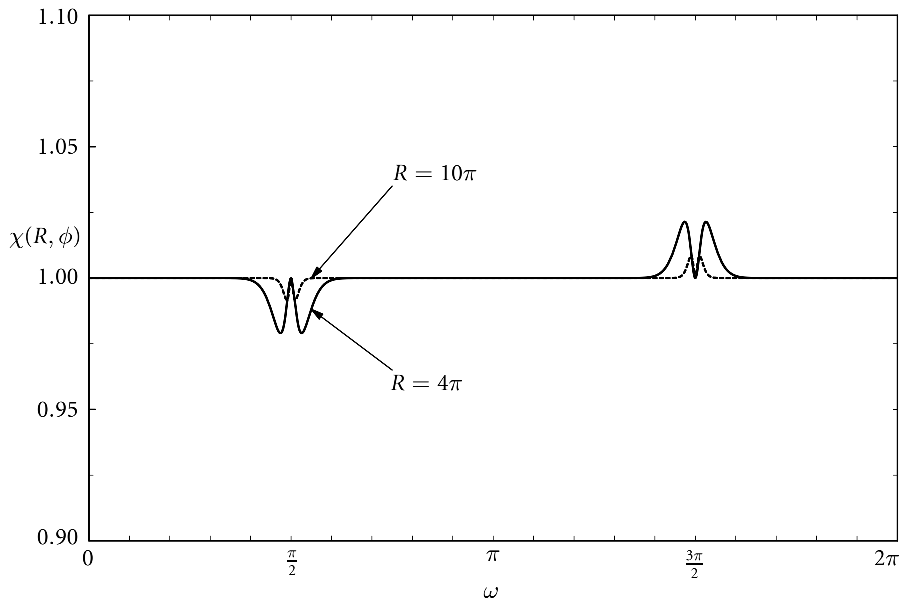

Figure 7: $\chi(R,\phi)$ versus ϕ for two various values of R.

and

$$\chi(R,\phi) = 1 + O(\gamma), \quad \phi = \frac{\pi}{2} - \gamma,\ 0 < \gamma \ll 1. \tag{85}$$

Treatment of the case $\phi = \frac{3\pi}{2} + \gamma$, $0 < \gamma \ll 1$, as well as the case $-\delta < \cos\phi < 0$, can be proceeded in the similar manner, leading to the same estimate (85).

Thus, finally we have shown that for $\phi \in [0, 2\pi]$ and sufficiently large $R = \pi k$

$$\chi(R,\phi) < 2. \tag{86}$$

Figure 7 shows $\chi(R,\phi)$ versus ϕ for two various values of R.

8 Appendix D. Jordan's Modified Lemma

Let $f(z)$ be a complex-valued function of a complex argument. Let also η_n be an infinite family of arcs defined as

$$\eta_n = \{z : |z| = \rho_n, \operatorname{Re} z < a\}, \quad n = 1, 2, \dots, \tag{87}$$

where $a > 0$, $\rho_1 < \rho_2 < \cdots < \rho_k < \rho_{k+1} < \cdots$, and $\rho_n \to \infty$ as $n \to \infty$.

If $f(z)$ tends to zero on the family of arcs (87) uniformly with respect to $\arg z$, *i.e.* if

$$|f(z)| \le \Delta_n, \quad z \in \eta_n \text{ with } \Delta_n \to 0 \text{ as } n \to 0, \tag{88}$$

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then, for any $\lambda > 0$,

$$\lim_{n\to\infty} \int_{\eta_n} f(z) e^{\lambda z}\, dz = 0. \tag{89}$$

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