SOME REMARKS ON EXTREME DERIVATES

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In 1957 Hájek [1] proved that the extreme bilateral derivates of an arbitrary finite real valued function of a real variable, are Borel measurable of class $\leqq 2$. It was later shown by Staniszewska [3] that Hájek's result is the best possible (even among the class of functions satisfying a Lipschitz condition). Staniszewska exhibited a Lipschitz function whose extreme bilateral derivates are not in Borel class 1. Staniszewska's proof makes use of a result of Zahorski's [4] concerning kernel functions. The purpose of this note is two-fold: first to provide a simpler example of a Lipschitz function whose extreme derivates are in class 2; and, second, to modify our example to indicate a certain type of behavior possible of extreme derivates.

Let E be a measurable subset of $[0,1]$ such that both E and its complement have positive Lebesgue measure in every subinterval of $[0,1]$. Such a set can be constructed by using nowhere dense perfect sets of positive measure (see the lemma below). Let f be the characteristic function of E. Define a function F by

$$F(x) = \int_0^x f(t)dt, \qquad 0 \leqq x \leqq 1.$$

Then $F'(x) = f(x)$ a.e. In particular, both the set $\{x: F'(x) = 0\}$ and the set $\{x: F'(x) = 1\}$ have positive measure in every interval. Now, on the set where F' exists, the two extreme bilateral derivates of F (as well as the four Dini derivates) all are equal to F'. It follows that each of these derivates is everywhere discontinuous. Therefore, none of these derivates is in Borel class 1.

By modifying our example, we can exhibit certain behavior possibilities of extreme derivates of absolutely continuous functions. We first state a lemma.

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LEMMA. The interval $[0, 1]$ can be expressed as a disjoint union of measurable sets, $[0, 1] = \bigcup_{k=1}^{\infty} B_k$, each of which has positive measure in every subinterval of $[0, 1]$.

Proof. We shall use the term "thick Cantor set" to mean a nowhere dense perfect set of positive measure.

Let A_1 be a thick Cantor set contained in $[0, 1]$. Let $A_2 = A_2^0 \cup A_2^1$ where, for $i = 0, 1$, A_2^i is a thick Cantor set contained in $(\frac{i}{2}, \frac{i+1}{2})$ and such that $A_1 \cap A_2 = \emptyset$ We proceed inductively obtaining a sequence of sets $\{A_k\}$ such that for each k,

(i) $A_k \cap (A_1 \cup A_2 \cup \cdots \cup A_{k-1}) = \emptyset$

and (ii) A_k is a union of thick Cantor sets, $A_k = A_k^0 \cup A_k^1 \cup \cdots \cup A_k^{k-1}$ with, for each $i = 0, 1, \cdots, k-1$, $A_k^i \subset (\frac{i}{k}, \frac{i+1}{k})$.

That such a sequence can be defined is an immediate consequence of the fact that for every k, the set $A_1 \cup A_2 \cup \cdots \cup A_{k-1}$ is nowhere dense in $[0, 1]$.

Now let $A_0 = \sim \bigcup_{k=1}^{\infty} A_k$. Define a sequence of sets $\{B_k\}$ by

$$B_1 = A_0 \cup \bigcup_{n=0}^{\infty} A_{2n+1},$$

$$B_{k+1} = \bigcup_{n=0}^{\infty} A_{2^k(2n+1)} \qquad \text{for } k \geqq 1.$$

It follows from (i) that the sequence $\{A_k\}$ and therefore the sequence $\{B_k\}$ is a disjoint sequence of sets. Clearly $[0, 1] = \bigcup_{k=1}^{\infty} B_k$. We now show that for each k, the set B_k has positive measure in every interval contained in $[0, 1]$. Let I be such an interval and let $|I|$ denote its length. Choose n_0 so that $2/n_0 < |I|$. For each $n \geqq n_0$, there exists a nonnegative integer $i_n < n$ such that the interval

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$(n^{-1} i_n, n^{-1}(i_n + 1))$ is contained in I. It follows that the set $A_n \cap I$ has positive measure for every $n \geq n_0$. Since for each k, the set B_k contains infinitely many of the sets A_n, we infer that the set $B_k \cap I$ has positive measure.

Remark. We observe that by suitably restricting the measures of the sets A_k, we can guarantee that for $k > 1$, the sets B_k have measure as small as we please.

THEOREM. Let $A = \{a_1, a_2, \cdots\}$ be any sequence of real numbers. There exists an absolutely continuous function F such that for every interval $I \subset [0,1]$,

(i) the set $\{x: DF(x) = a_k\} \cap I$ has positive measure for every $k = 1, 2, \cdots$, where DF denotes any of the six extreme derivates of F, and

(ii) the set $\{x: DF(x) = \lambda\} \cap I$ is nondenumerable for every λ satisfying $\inf\{y: y \in A\} < \lambda < \sup\{y: y \in A\}$ and for any Dini derivate DF.

Proof. Let $\{B_k\}$ be a sequence of sets satisfying the conclusion of the lemma. Let μ denote Lebesgue measure. By taking into account the remark following the lemma, we may assume that $|a_k|\mu(B_k) < K^{-2}$ for each $k > 1$. It follows that the function f defined by $f(x) = a_k$ if $x \in B_k$ is Lebesgue summable, since

$$\int_0^1 |f(x)|dx \leq |a_1|\mu(B_1) + \sum_{k=2}^{\infty} \frac{1}{k^2} < \infty .$$

Let F be defined by $F(x) = \int_0^x f(t)dt$. Then F is absolutely continuous and $F'(x) = f(x)$ a.e. In particular for each k, F', and therefore all six extreme derivates of F, takes on the value a_k at almost all points of B_k. This proves (i).

Condition (ii) for the upper right Dini derivate D^+F follows immediately from a theorem of Morse [2]. This theorem states that if F is continuous, $-\infty < \lambda < \infty$, and $D^+F \geqq \lambda$ on a dense set while $D^+F(x_o) < \lambda$ for some point x_o, then the set $\{x: D^+F(x) = \lambda\}$ has the power of the continuum. Condition (ii) for the other Dini derivates follows from the analogous versions of Morse's Theorem.

We observe that we can obtain certain special examples by varying the set A. Thus, if A denotes the rational numbers, then F is an absolutely continuous function whose Dini derivates take on every real value a nondenumerable number of times in every subinterval of [0, 1]. The Dini derivates of this function are in Borel class 2, of course.

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