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ELEMENTARY PROOFS OF PARITY RESULTS FOR 5-REGULAR PARTITIONS

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Abstract

In a recent paper, Calkin *et al.* [N. Calkin, N. Drake, K. James, S. Law, P. Lee, D. Penniston and J. Radder, 'Divisibility properties of the 5-regular and 13-regular partition functions', *Integers* **8** (2008), #A60] used the theory of modular forms to examine 5-regular partitions modulo 2 and 13-regular partitions modulo 2 and 3; they obtained and conjectured various results. In this note, we use nothing more than Jacobi's triple product identity to obtain results for 5-regular partitions that are stronger than those obtained by Calkin and his collaborators. We find infinitely many Ramanujan-type congruences for $b_5(n)$, and we prove the striking result that the number of 5-regular partitions of the number n is even for at least 75% of the positive integers n.

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1. Introduction

In the seven-author paper [1], Calkin *et al.* examined the parity of 5-regular partitions which are defined by

$$\sum_{n\geq 0} b_5(n)q^n = \frac{(q^5;q^5)_\infty}{(q;q)_\infty}.$$

Using the theory of modular forms, they proved results equivalent to the following:

$$b_5(2n) \text{ is odd if and only if } 12n+1 \text{ is a square;}$$

and, for all $n \geq 0$,

$$b_5(20n+5) \text{ is even} \quad \text{and} \quad b_5(20n+13) \text{ is even.}$$

Combining these two results, one can deduce that $b_5(n)$ is even for at least 60% of the positive integers n.

In this note we use nothing more than Jacobi's triple product identity to prove the two results above.

We also prove infinitely many new Ramanujan-type congruences for $b_5(n)$, of which the 'smallest' is

$$b_5(1156n+65) \equiv 0 \pmod 2,$$

and we prove that

$$b_5(n) \text{ is even for at least } 75\% \text{ of the positive integers } n.$$

This theorem is striking, because it is believed that the unrestricted partition function, $p(n)$, is even for half of the positive integers n.

2. The proofs

We begin with a fundamental theorem which provides the 2-dissection of the generating function of $b_5(n)$.

THEOREM 2.1.

$$\sum_{n\geq 0} b_5(n)q^n = \frac{(q^8;q^8)_\infty (q^{20};q^{20})_\infty^2}{(q^2;q^2)_\infty^2 (q^{40};q^{40})_\infty} + q\frac{(q^4;q^4)_\infty^3 (q^{10};q^{10})_\infty (q^{40};q^{40})_\infty}{(q^2;q^2)_\infty^3 (q^8;q^8)_\infty (q^{20};q^{20})_\infty},$$

where $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)(1-aq^3)\cdots$.

PROOF. We start by noting that

$$\begin{aligned}\sum_{n\geq 0} b_5(n)q^n &= \frac{(q^5;q^5)_\infty}{(q;q)_\infty}\\ &= \frac{1}{(q,q^2,q^3,q^4;q^5)_\infty}\\ &= \frac{1}{(q,q^2,q^3,q^4,q^6,q^7,q^8,q^9;q^{10})_\infty}\\ &= \frac{(-q,-q^3,-q^7,-q^9,q^{10},q^{10};q^{10})_\infty}{(q^2,q^2,q^4,q^6,q^6,q^8,q^{10},q^{10},q^{12},q^{14},q^{14},q^{16},q^{18},q^{18},q^{20},q^{20};q^{20})_\infty}\\ &= \frac{(q^4;q^4)_\infty(-q,-q^3,-q^7,-q^9,q^{10},q^{10};q^{10})_\infty}{(q^2;q^2)_\infty^2(q^{20};q^{20})_\infty},\end{aligned}$$

where $(a_1,a_2,\ldots,a_k;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_k;q)_\infty$.
Now,

$$\begin{aligned}&(-q,-q^3,-q^7,-q^9,q^{10},q^{10};q^{10})_\infty\\ &\quad= \sum_{m,n=-\infty}^{\infty} q^{5m^2-4m+5n^2-2n} \quad \text{(by Jacobi's triple product identity)}\\ &\quad= \sum_{r,s=-\infty}^{\infty} q^{5(r+s)^2-4(r+s)+5(r-s)^2-2(r-s)}\\ &\qquad+ \sum_{r,s=-\infty}^{\infty} q^{5(r+s+1)^2-4(r+s+1)+5(r-s)^2-2(r-s)}\end{aligned}$$

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$$
\begin{aligned}
&= \sum_{r,s=-\infty}^{\infty} q^{10r^2-6r+10s^2-2s} + q \sum_{r,s=-\infty}^{\infty} q^{10r^2+4r+10s^2+8s} \\
&= (-q^4, -q^8, -q^{12}, -q^{16}, q^{20}, q^{20}; q^{20})_\infty \\
&\quad + q(-q^2, -q^6, -q^{14}, -q^{18}, q^{20}, q^{20}; q^{20})_\infty \\
&= \frac{(-q^4; q^4)_\infty (q^{20}; q^{20})_\infty^2}{(-q^{20}; q^{20})_\infty} + q \frac{(-q^2; q^2)_\infty (-q^{20}; q^{20})_\infty (q^{20}; q^{20})_\infty^2}{(-q^4; q^4)_\infty (-q^{10}; q^{10})_\infty} \\
&= \frac{(q^8; q^8)_\infty (q^{20}; q^{20})_\infty^3}{(q^4; q^4)_\infty (q^{40}; q^{40})_\infty} + q \frac{(q^4; q^4)_\infty^2 (q^{10}; q^{10})_\infty (q^{40}; q^{40})_\infty}{(q^2; q^2)_\infty (q^8; q^8)_\infty}.
\end{aligned}
$$

Therefore,

$$
\sum_{n \geq 0} b_5(n) q^n = \frac{(q^8; q^8)_\infty (q^{20}; q^{20})_\infty^2}{(q^2; q^2)_\infty^2 (q^{40}; q^{40})_\infty} + q \frac{(q^4; q^4)_\infty^3 (q^{10}; q^{10})_\infty (q^{40}; q^{40})_\infty}{(q^2; q^2)_\infty^3 (q^8; q^8)_\infty (q^{20}; q^{20})_\infty}
$$

as claimed. □

THEOREM 2.2. [1, Theorem 1] *For all $n \geq 0$, $b_5(2n)$ is odd if and only if $12n+1$ is a perfect square.*

PROOF. Thanks to Theorem 2.1 above, we know that

$$
\begin{aligned}
\sum_{n \geq 0} b_5(2n) q^n &= \frac{(q^4; q^4)_\infty (q^{10}; q^{10})_\infty^2}{(q; q)_\infty^2 (q^{20}; q^{20})_\infty} \\
&\equiv \frac{(q^4; q^4)_\infty (q^{20}; q^{20})_\infty}{(q^2; q^2)_\infty (q^{20}; q^{20})_\infty} \pmod 2 \\
&= (-q^2; q^2)_\infty \\
&\equiv (q^2; q^2)_\infty \pmod 2 \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+n} \\
&\equiv \sum_{n=-\infty}^{\infty} q^{3n^2+n} \pmod 2.
\end{aligned}
$$

Thus,

$$
\sum_{n \geq 0} b_5(2n) q^{12n+1} \equiv \sum_{n=-\infty}^{\infty} q^{(6n+1)^2} \pmod 2,
$$

from which the result follows. □

THEOREM 2.3. *For all $n \geq 0$, $b(4n+1)$ is even unless $24n+7 = 2x^2 + 5y^2$ for some integers x and y.*

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PROOF. From Theorem 2.1, we know that

$$\sum_{n\geq 0} b_5(2n+1)q^n = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^3}\cdot\frac{(q^5;q^5)_\infty(q^{20};q^{20})_\infty}{(q^4;q^4)_\infty(q^{10};q^{10})_\infty}.$$

Now,

$$\begin{aligned}\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^3} &= \prod_{n\geq 1}\left(\frac{1-q^{2n}}{1-q^n}\right)^3 = \prod_{n\geq 1}(1+q^n)^3 \\ &\equiv \prod_{n\geq 1}(1+q^n+q^{2n}+q^{3n}) \pmod 2 \\ &= \prod_{n\geq 1}\frac{1-q^{4n}}{1-q^n} \\ &= \frac{(q^4;q^4)_\infty}{(q;q)_\infty}.\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{n\geq 0} b_5(2n+1)q^n \\ &\quad\equiv \frac{(q^5;q^5)_\infty(q^{20};q^{20})_\infty}{(q;q)_\infty(q^{10};q^{10})_\infty} \pmod 2 \\ &\quad= \prod_{n\geq 1}(1+q^{10n})\sum_{n\geq 0} b_5(n)q^n \\ &\quad\equiv \sum_{n=-\infty}^{\infty} q^{10(3n^2+n)/2}\left(\sum_{n=-\infty}^{\infty} q^{2(3n^2+n)} + \sum_{n\geq 0} b_5(2n+1)q^{2n+1}\right) \pmod 2.\end{aligned}$$

This means that

$$\sum_{n\geq 0} b_5(4n+1)q^n \equiv \sum_{m,n=-\infty}^{\infty} q^{(3m^2+m)+5(3n^2+n)/2} \pmod 2,$$

which implies that

$$\sum_{n\geq 0} b_5(4n+1)q^{24n+7} \equiv \sum_{m,n=-\infty}^{\infty} q^{2(6m+1)^2+5(6n+1)^2} \pmod 2.$$

The result follows. □

THEOREM 2.4. [1, Theorem 3] *For all* $n \geq 0$,

$$b_5(20n+5) \equiv 0 \pmod 2$$
$$\textit{and}\quad b_5(20n+13) \equiv 0 \pmod 2.$$

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PROOF. From Theorem 2.3, we know $b(20n+5)$ is even unless $24(5n+1)+7 = 2x^2+5y^2$ for some integers x and y. Consideration of this equation modulo 5 yields $x^2 \equiv 3 \pmod 5$. Since 3 is a quadratic nonresidue modulo 5, we know that there can be no such solutions. This proves the first congruence. A proof of the second congruence can be obtained from the fact that 2 is the other quadratic nonresidue modulo 5. □

THEOREM 2.5. *Suppose that p is any prime greater than 3 such that -10 is a quadratic nonresidue modulo p, u is the reciprocal of 24 modulo p^2, and $r \not\equiv 0 \pmod p$. Then, for all m,*

$$b_5(4p^2m + 4u(pr-7)+1) \equiv 0 \pmod 2.$$

PROOF. If we set $n = p^2m + u(pr-7)$, then

$$24n+7 \equiv 24p^2m + pr = p(24pm+r) \pmod{p^2}$$

is divisible by p but not by p^2. If $24n+7 = 2x^2+5y^2$, then $2x^2+5y^2 \equiv 0 \pmod p$ but $2x^2+5y^2 \not\equiv 0 \pmod{p^2}$. This is impossible; so, by Theorem 2.3, $b_5(4n+1) \equiv 0 \pmod 2$. □

EXAMPLES. With $p = 17$, we find that for $r \not\equiv 0 \pmod{17}$ and for all m,

$$b_5(1156m + 340r + 337) \equiv 0 \pmod 2.$$

In particular, with $r = 6$ (and m replaced by $m-2$),

$$b_5(1156m + 65) \equiv 0 \pmod 2.$$

We close with one last observation about the parity of $b_5(n)$.

THEOREM 2.6. *$b_5(n)$ is even for at least 75% of the positive integers n.*

PROOF. By Theorem 2.2, $b_5(2n)$ is almost always even; and, by Theorem 2.3, $b_5(4n+1)$ is almost always even. The latter statement is true because in the prime factorization of $24n+7 = 2x^2+5y^2$, primes congruent to

$$3,\ 17,\ 21,\ 27,\ 29,\ 31,\ 33 \text{ or } 39 \pmod{40},$$

those for which -10 is a quadratic nonresidue, necessarily occur to an even power (3 itself does not occur). The density of such numbers is

$$\frac{1}{\prod_{\text{such } p>3}(1+\frac{1}{p})} = 0. \qquad \square$$

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Reference

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