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FUNCTIONS OF EXPONENTIAL TYPE ARE DIFFERENCES OF FUNCTIONS OF BOUNDED INDEX

BY
SHANTILAL N. SHAH

Introduction. The notion of entire function of *Bounded Index* is by now well established. It may be stated as follows.

DEFINITION. An entire function $f(z)$ is said to be of *Bounded Index* if for some fixed s

$$\frac{|f^{(n)}(z)|}{n!} \le \max_{0\le j\le s}\left\{\frac{|f^{(j)}(z)|}{j!}\right\} \qquad (f^{(0)}(z)=f(z))$$

for all n and all z. (See [1], [2].)

Pugh proved [3] that the sum of two functions of bounded index need not be of bounded index. It may be of interest to show that Pugh's result yields with little difficulty the following:

THEOREM. *An entire function whose growth is at most of mean type of order one may be expressed as the difference of two entire functions of bounded index.*

Proof of the Theorem. Let $W(z)$ be a given function whose growth does not exceed the mean type of order one.

We assume first $W(0)\neq 0$. The case $W(0)=0$ will be considered separately. By hypothesis

$$\limsup_{r\to\infty}\frac{\log M(r, W)}{r}=\beta<+\infty$$

Given $\varepsilon>0$, we can find r_0 such that

$$\log M(r, W)<r(\beta+\varepsilon) \qquad (r\ge r_0)$$

and, therefore,

$$M(2r, W)<e^{2r(\beta+\varepsilon)} \qquad (r\ge r_0) \tag{1.1}$$

Let

$$\delta=4(\beta+\varepsilon)+1 \tag{1.2}$$

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and

$$K = \max_{|z|=2r_0} |W(z)|$$

We now define

(1.3) $$g(z) = K(\cos \delta z + \cosh \delta z).$$

Clearly $g(z)$ satisfies the differential equation

$$g^{(4)}(z) = \delta^4 g(z),$$

and we also have

$$\frac{g^{(n)}(z)}{n!} = \frac{\delta^{n-k} \cdot k!}{n!} \left\{\frac{g^{(k)}(z)}{k!}\right\}$$

where

$$k \equiv n (\text{mod } 4) \qquad (0 \le k \le 3)$$

Let N be the smallest integer ≥ 3 such that

$$\frac{6\delta^N}{N!} \le \frac{1}{2},$$

and let

(1.4) $$\Omega(z) = \max_{0 \le j \le N} \{|g^{(j)}(z)|/j!\}$$

Then for $n \ge N+1$ and all z

(1.5) $$\frac{|g^{(n)}(z)|}{n!} = \frac{\delta^{n-k} \cdot k!}{n!} \left\{\frac{|g^{(k)}(z)|}{k!}\right\} \le \tfrac{1}{2}\Omega(z)$$

This means that $g(z)$ is of strongly bounded index and therefore of bounded index by Theorem 1 of Shah and Shah [5, p. 364].

LEMMA. *Let $g(z)$ and $\Omega(z)$ be defined by* (1.3) *and* (1.4), *respectively, then for all z*

(1.6) $$\Omega(z) \ge Be^{\delta|z|/2}$$

where $B > 0$ is a suitably chosen constant.

Proof. It is obvious that $\Omega(z)$ is a continuous function of z, and that, in view of the differential equation, $\Omega(z)$ never vanishes. Hence, by choosing $B_0 > 0$, sufficiently small, we can always assume that (1.6) holds for $\delta|z| \le 2$, that is

(1.7) $$B_0 e^{\delta|z|/2} \le \Omega(z) \qquad (\delta|z| \le 2).$$

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Now for $z = x + iy$

$$|\cosh \delta z|^2 = \tfrac{1}{2}(\cosh 2\delta x + \cos 2\delta y) > \tfrac{1}{4}(e^{2\delta x} + e^{-2\delta x} - 2) = \left(\frac{e^{\delta x} - e^{-\delta x}}{2}\right)^2$$

and, hence,

$$(1.8) \qquad |\cosh \delta z| \geq Ae^{\delta|x|} \qquad (0 < A = \text{const. } \delta|x| \geq 1)$$

From (1.8) we also deduce

$$(1.9) \qquad |\cos \delta z| = |\cosh i\delta z| \geq Ae^{\delta|y|} \qquad (\delta|y| \geq 1)$$

If $\delta|z| \geq 2$,

$$\max\{\delta\,|x|, \delta\,|y|\} \geq \delta\,|z|/2 \geq 1$$

and by (1.8) and (1.9)

$$(1.10) \qquad |\cosh \delta z| + |\cos \delta z| \geq Ae^{\delta|z|/2} \qquad (\delta|z| \geq 2)$$

From (1.3) we deduce

$$2 \cosh \delta z = \frac{g}{K} + \frac{g''}{K\delta^2}$$

$$2 \cos \delta z = \frac{g}{K} - \frac{g''}{K\delta^2}$$

and in view of (1.10)

$$(1.11) \qquad \begin{aligned} Ae^{\delta|z|/2} &\leq |\cosh \delta z| + |\cos \delta z| \\ &\leq \frac{4}{K} \max\{|g|, |g''|/2!\} \\ &\leq \frac{4}{K}\,\Omega(z) \qquad (\delta|z| \geq 2) \end{aligned}$$

take

$$B = \min\left\{B_0, \frac{AK}{4}\right\}$$

The lemma now follows from (1.7) and (1.11).

By (1.6), (1.2), and (1.1),

$$(1.12) \qquad \begin{aligned} \Omega(z) &\geq Be^{\delta|z|/2} > Be^{2(\beta+\varepsilon)r} \\ &> BM(2r, W) \qquad (r \geq r_0) \end{aligned}$$

For $r < r_0$

$$\Omega(z) \geq B \geq B \frac{M(2r, W)}{M(2r_0, W)}$$

and hence, if

$$L = \frac{1 + M(2r_0, W)}{B},$$

we have for all $r \geq 0$

$$M(2r, W) \leq L\Omega(z) \qquad (|z| = r).$$

This means that the function

$$f(z) = \frac{W(z)}{L}, \qquad \left(f(0) = \frac{W(0)}{L} \neq 0\right),$$

satisfies all the conditions of Pugh's theorem [3, p. 319] and therefore if $c \neq 0$ is small enough we know that

$$p(z) = g(z) + cf(z) = g(z) + \frac{c}{L} W(z)$$

is of bounded index.

The functions

$$Lp(z)/c \quad \text{and} \quad Lg(z)/c$$

are of bounded index because $p(z)$ and $g(z)$ are. Hence

$$W(z) = \frac{L}{c} p(z) - \frac{L}{c} g(z)$$

is the required representation if $W(0) \neq 0$.

We now consider the case $W(0) = 0$. Let

$$W_1(z) = W(z) + 1$$

Then $W_1(0) = 1 \neq 0$, and by the case already treated

(1.13) $$W_1(z) = \frac{L}{c} p(z) - \frac{L}{c} g(z)$$

where $g(z)$ (defined by 1.3) is of bounded index and $p(z)$ is of bounded index by Pugh's theorem.

From (1.13) we find

(1.14) $$W(z) = \frac{L}{c} p(z) - \frac{L}{c}\left(g(z) + \frac{c}{L}\right)$$

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Put

$$G(z) = g(z) + c/L$$

and notice that

$$G^{(4)}(z) = g^{(4)}(z) = \delta^4 g(z) = \delta^4(G(z) - c/L).$$

Differentiating this relation once again we obtain

$$G^{(5)}(z) = \delta^4 G'(z).$$

We thus see that $G(z)$ satisfies a homogeneous linear differential equation with constant coefficients. By Theorem 3 of S. M. Shah [4, p. 1017], $G(z)$ is necessarily a function of bounded index. Consequently, (1.14), which may be rewritten as

$$W(z) = \frac{L}{c}\, p(z) - \frac{L}{c}\, G(z)$$

gives (in the case $W(0) = 0$) the representation as a difference of functions of bounded index. The proof of the Theorem is now complete.

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Department of Mathematics
Hampton Institute
Hampton Va. 23668