CRITERIA FOR THE SEQUENCE OF DIFFERENCES OF A BOUNDED SEQUENCE TO BE NULL

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Abstract

Conditions are established for the sequence of differences \( \{a_n - a_{n-1}\} \) of a bounded sequence \( \{a_n\} \) of complex terms to converge to zero when a certain linear nonhomogeneous difference expression of the form \( k_0a_n + k_1a_{n-1} + \cdots + k_na_0 \) tends to zero as \( n \to \infty \).

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1. Introduction and the main results

Suppose throughout that \( K(z) := \sum_{n=0}^{\infty} K_n z^n \), where \( K_n \) is complex, and that \( k_n = K_n - K_{n-1} \) with \( K_{-1} := 0 \). Let \( D \) be the open unit disc \( \{z : |z| < 1\} \), let \( \bar{D} \) be its closure, and let \( \partial D := \bar{D} \setminus D \).

The object of this paper is to prove Theorems 1.1 and 1.2 stated below.

**Theorem 1.1.** If

\[
\sum_{n=0}^{\infty} |K_n| < \infty, \tag{1.1}
\]

\[
K(z) \neq 0 \quad \text{on } \partial D, \tag{1.2}
\]

and if

\( \{a_n\} \) is a bounded complex sequence \( \tag{1.3} \)

such that

\[
\lim_{n \to \infty} \sum_{r=0}^{n} k_r a_{n-r} = 0, \tag{1.4}
\]

then \( \lim_{n \to \infty} (a_n - a_{n-1}) = 0 \).

The next theorem shows that condition (1.2) is necessary in a sense for the validity of Theorem 1.1.
Theorem 1.2. If \( K(z) = p(z)q(z) \) where \( p(z) \) is a polynomial and \( q(z) = \sum_{n=0}^{\infty} q_n z^n \), and if

\[
\sum_{n=0}^{\infty} |q_n| < \infty, \tag{1.5}
\]

\( q(z) \neq 0 \) on \( \tilde{D} \), \( K(\zeta) = 0 \) for \( \zeta \neq 1, |\zeta| = 1 \), \( \tag{1.6}
\]

then there exist a bounded sequence \( \{a_n\} \) and a positive integer \( N \) such that

\[
\sum_{r=0}^{n} k_r a_{n-r} = 0 \quad \text{for all } n \geq N, \tag{1.8}
\]

but \( \{a_n - a_{n-1}\} \) does not converge.

Note that (1.5) in fact implies (1.1).

Theorem 1.1 generalises the following theorem proved by Stević [4].

Theorem S. If \( k_0 = -1, \sum_{n=1}^{N} k_n = 1 \) with \( k_n \) real, \( \sum_{n=0}^{N} k_n z^n \neq 0 \) on \( \partial D \setminus \{1\} \), and if \( \{a_n\} \) is a bounded real sequence such that \( \lim_{n \to \infty} \sum_{r=0}^{N} k_r a_{n-r} = 0 \), then

\[
\lim_{n \to \infty} (a_n - a_{n-1}) = 0.
\]

That Theorem S is a special case of Theorem 1.1 can be seen by taking \( K_0 = -1, K_n = 0 \) for \( n > N \), and observing that \( \sum_{n=0}^{N} k_n z^n = (1 - z)K(z) \). In [4] Stević cites many examples from mathematical biology which use results of this type, and also produces an extensive list of related results. A companion to Theorem 1.1 is the following result proved in [1].

Theorem B. If (1.1) and (1.2) hold, and if \( \{a_n\} \) is a bounded real sequence such that \( \sum_{r=0}^{n} k_r a_{n-r} \geq 0 \) for all \( n \) larger than some positive integer \( N \), then \( \{a_n\} \) is convergent.

Theorem B generalises a theorem of Copson’s [2] which in turn generalises the result that a bounded monotonic real sequence converges. Incidentally, Stević in [3] also proved a slight generalisation of Copson’s theorem, but failed to observe that his result was in fact a special case of the earlier Theorem B.

2. An auxiliary result

Our proof of Theorem 1.1 is largely modelled on the proof of Theorem B [1, Theorem 1]. We require the following lemma.

Lemma 2.1. Suppose that (1.1)–(1.4) hold, and that \( K(\alpha) = 0 \) with \( 0 < |\alpha| < 1 \). Then

\[
\frac{1}{\alpha - \bar{z}} K(z) = \sum_{n=0}^{\infty} P_n z^n \quad \text{where } \sum_{n=0}^{\infty} |P_n| < \infty,
\]
and
\[ \lim_{n \to \infty} \sum_{r=0}^{n} p_r a_{n-r} = 0 \] with \( p_r := P_r - P_{r-1}, \ P_{-1} := 0. \)

**Proof.** Since \( K(\alpha) = 0, \)
\[ \alpha P_n = \sum_{r=0}^{n} \alpha^{r-n} K_r = - \sum_{r=n+1}^{\infty} \alpha^{r-n} K_r, \]
and so, by (1.1),
\[ \sum_{n=0}^{\infty} |P_n| \leq \sum_{r=1}^{\infty} |K_r| \sum_{n=0}^{r-1} |\alpha|^{r-1-n} \leq \frac{1}{1 - |\alpha|} \sum_{r=1}^{\infty} |K_r| < \infty. \]

Now let
\[ v_n := \sum_{r=0}^{n} K_r a_{n-r}, \quad u_n := \sum_{r=0}^{n} P_r a_{n-r}, \]
\[ a(z) := \sum_{n=0}^{\infty} a_n z^n, \quad v(z) := \sum_{n=0}^{\infty} v_n z^n \] and \( u(z) := \sum_{n=0}^{\infty} u_n z^n. \)

Then, by (1.4),
\[ v_n - v_{n-1} = \sum_{r=0}^{n} k_r a_{n-r} \to 0 \quad \text{as} \quad n \to \infty. \]

Also, since \( v(z) = K(z) a(z), \) so that \( v(\alpha) = 0 \) and \( u(z) = (\alpha - z)^{-1} v(z), \)
\[ u_n = - \sum_{r=n+1}^{\infty} \alpha^{r-n-1} v_r = - \sum_{r=0}^{\infty} \alpha^{r} v_{n+1+r}, \]
and hence, by the series version of Lebesgue’s dominated convergence theorem,
\[ \sum_{r=0}^{n} p_r a_{n-r} = u_n - u_{n-1} = - \sum_{r=0}^{\infty} \alpha^{r} (v_{n+1+r} - v_{n+r}) \to 0 \quad \text{as} \quad n \to \infty. \]

3. Proofs of the theorems

**Proof of Theorem 1.1. Case 1.** \( K(0) \neq 0. \) By (1.1), \( K(z) \) is holomorphic on \( D \) and continuous on \( \bar{D}. \) Hence, by (1.2), \( K(z) \) can have at most a finite number of zeros in \( D. \) We can use Lemma 2.1 to remove the zeros, and thus we may assume without loss of generality that \( K(z) \) has no zeros on \( \bar{D}. \) Then, by the Wiener–Lévy theorem [5, p. 246],
\[ \frac{1}{K(z)} = \sum_{n=0}^{\infty} c_n z^n \quad \text{for} \quad z \in \bar{D} \quad \text{and} \quad \sum_{n=0}^{\infty} |c_n| < \infty. \]
Using the notation introduced in the previous section, we have $a(z) = v(z)c(z)$, so that $a_n = \sum_{r=0}^{n} c_r v_{n-r}$. Hence, for $w_n := \sum_{r=0}^{n} k_r a_{n-r}$, by (1.4),

$$a_n - a_{n-1} = \sum_{r=0}^{n} c_r w_{n-r} \to 0 \quad \text{as } n \to \infty.$$  

**Case 2.** $z = 0$ is a zero of order $m$ of $K(z)$. Since $K_m \neq 0$ and

$$z^{-m} K(z) = \sum_{n=0}^{\infty} K_{n+m} z^n,$$

it follows easily from Case 1 that $a_{n+m} - a_{n+m-1} \to 0$ as $n \to \infty$. □

**Proof of Theorem 1.2.** As in the proof of its companion theorem [1, Theorem 2], we define a sequence $\{a_n\}$ and a function $a(z)$ by

$$a(z) := \sum_{n=0}^{\infty} a_n z^n := \frac{1}{q(z)(\zeta - z)} \quad \text{for } z \in D. \quad (3.1)$$

Let

$$w(z) := \sum_{n=0}^{\infty} w_n z^n \quad \text{with } w_n := \sum_{r=0}^{n} k_r a_{n-r}.$$  

Then

$$w(z) = (1 - z) K(z) a(z) = \frac{(1 - z) p(z)}{\zeta - z},$$

and, by (1.6) and (1.7), $\zeta - z$ is a factor of the polynomial $p(z)$. Consequently $w(z)$ is a polynomial of degree $N - 1$ say, and (1.8) follows.

Further, by the Wiener–Lévy theorem, hypotheses (1.5) and (1.6) imply that there is a sequence $\{c_n\}$ such that, for $z \in \bar{D}$,

$$\frac{1}{q(z)} = \sum_{n=0}^{\infty} c_n z^n \quad \text{and} \quad \sum_{n=0}^{\infty} |c_n| < \infty. $$

It follows, on equating coefficients in (3.1), that

$$\zeta^{n+1} a_n = \zeta^n \sum_{r=0}^{n} c_r \zeta^{r-n} \to \frac{1}{q(\zeta)} \quad \text{as } n \to \infty.$$  

The sequence $\{a_n\}$ is bounded, and

$$\zeta^{n+1} a_n - \zeta^n a_{n-1} \to 0 \Rightarrow \zeta a_n - a_{n-1} \to 0 \Rightarrow a_n - a_{n-1} + (\zeta - 1) a_n \to 0.$$

Since $\{a_n\}$ does not converge, it follows that neither can $\{a_n - a_{n-1}\}$. □
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References


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