SOME SPHERE PACKINGS IN HIGHER SPACE

JOHN LEECH

Introduction. This paper is concerned with the packing of equal spheres in Euclidean spaces $[n]$ of $n > 8$ dimensions. To be precise, a *packing* is a distribution of spheres any two of which have at most a point of contact in common. If the centres of the spheres form a lattice, the packing is said to be a *lattice packing*. The densest lattice packings are known for spaces of up to eight dimensions **(1, 2)**, but not for any space of more than eight dimensions. Further, although non-lattice packings are known in [3] and [5] which have the same density as the densest lattice packings, none is known which has greater density than the densest lattice packings in any space of up to eight dimensions, neither, for any space of more than two dimensions, has it been shown that they do not exist.

In Part 1 the densest lattice packings in [4] and [8] are generalized to packings, not all lattice packings, in $[2^m]$, in which each sphere touches

$$(2 + 2)(2 + 2^2)(2 + 2^3) \ldots (2 + 2^m)$$

others. This gives packings in [16] in which each sphere touches 4320 others, which may be the densest in this space. For $m > 4$ the corresponding packings are unlikely to be the densest, though they seem to be the densest yet constructed.

In Part 2 some different analogies to the densest lattice packing in [8] are considered, which lead to new packings in [12] and [24]. In [12] this does not lead to any packing as dense as K_{12} **(5)**, though it leads to new co-ordinates for some known packings. In [24] a dense lattice packing is found in which each sphere touches 98256 others. Other packings in up to 23 dimensions are found as sections of this packing in [24].

In Part 3 the densities of these packings are compared with Rogers' upper bound **(10)**. This comparison is also made for the known densest lattice packings in up to eight dimensions for which it has not been made before. The numbers of spheres touched are compared with Coxeter's upper bound **(4)**. For the packings in $[2^m]$ the density and the number of spheres touched are of a much smaller order of magnitude than Rogers' and Coxeter's upper bounds as $m \to \infty$. The packings in up to 24 dimensions are closer to the upper bounds, though not so close as in from 3 to 8 dimensions, that in [8] being especially close.

Received June 26, 1963.

https://doi.org/10.4153/CJM-1964-065-1 Published online by Cambridge University Press

1. Packings in $[2^m]$.

1.1. The densest lattice packings in [4] **and** [8]. The densest lattice packing in [4], in which each sphere touches 24 others, can be specified by taking as centres all those points whose co-ordinates are integers which are either all odd or all even. (Throughout this paper only integer co-ordinates are used.) This construction is less successful in producing a dense lattice in [8], because there are centres such as (2, 0, 0, 0, 0, 0, 0, 0) which are closer to the origin than any with coordinates all odd. To obtain a more dense packing we further restrict centres to those for which the sum of the co-ordinates is divisible by 4. This halves the density of centres while allowing a 16-fold increase in the content of each sphere, so that a denser packing results. It is, in fact, the densest lattice packing in [8], in which each sphere touches 240 others. This construction is less successful in producing dense lattice packings in $[2^m]$ for $m > 3$, even with this restriction on the sum of the co-ordinates. Before considering more effective restrictions, we consider alternative co-ordinates for the packings in [4] and [8].

We now consider, following Sylvester **(12)** and Paley **(8)**, the matrices of 0's and 1's, of order 2^m, defined recursively by the equations

$$\mathbf{A}_1 = 0, \qquad \mathbf{A}_{2n} = \begin{bmatrix} \mathbf{A}_n & \mathbf{A}_n \\ \mathbf{A}_n & \bar{\mathbf{A}}_n \end{bmatrix},$$

where $\bar{\mathbf{A}}$ denotes the complementary matrix having $\bar{a}_{ij} = 1 - a_{ij}$. (Paley actually has elements 1 and -1 where we have 0 and 1, so that he has $-\mathbf{A}$ where we have $\bar{\mathbf{A}}$. Sylvester's original formulation was slightly different.) Then the rows of the matrices $\mathbf{A}_n$, $\bar{\mathbf{A}}_n$, $n = 2^m$, comprise a row of 0's, a row of 1's, and $2n - 2$ rows each comprising $\frac{1}{2}n$ 0's and $\frac{1}{2}n$ 1's, and the rows differ from one another in exactly $\frac{1}{2}n$ places, except that the complement of each row occurs, from which it differs in all n places. This is easily proved inductively (and is a special case of §1.3). Stated geometrically, these rows give a set of co-ordinates for the vertices of a cross polytope β_n inscribed in a cube γ_n.

In [4] and [8] we obtain the densest lattice packings by taking as centres all points whose co-ordinates are congruent modulo 2 to rows of $\mathbf{A}_n$, $\bar{\mathbf{A}}_n$. In [4] this construction is equivalent to specifying that the sum of the co-ordinates be even. When stated thus, the same rule gives the densest lattice packings in [3] and [5] also. But in both [3] and [5] there are equally dense non-lattice packings, as we may see thus. In both spaces the densest lattice packings may be built up of layers of spheres packed according to the densest lattice packing in spaces of one dimension fewer. In both cases this is regular, and the spheres of each layer fit into alternate interstices of the adjacent layers. Any three layers may or may not be consistent with lattice packing, as the centres of the third layer may or may not lie on the lines joining centres of the first and second layers. By systematically stacking layers in one or other of these ways we obtain either the densest lattice packings or equally dense non-lattice packings.

These two constructions for [4] and [8] give the same lattices in different co-ordinate systems. If the co-ordinates in the two systems are x_i, y_i, $i = 0, 1, \ldots, 2^m - 1$, we find they are related by the equations

$$x_{2j} = y_{2j} + y_{2j+1}, \qquad y_{2j} = \tfrac{1}{2}(x_{2j} + x_{2j+1}),$$
$$x_{2j+1} = y_{2j} - y_{2j+1}, \qquad y_{2j+1} = \tfrac{1}{2}(x_{2j} - x_{2j+1}),$$

and, because of the symmetries of the lattices, by various other sets of equations.

1.2. Analogous packings in [16]. The lattice of all points whose co-ordinates are congruent (modulo 2) to rows of $\mathbf{A}_{16}$, $\bar{\mathbf{A}}_{16}$ does not give a very dense packing, for the same reason that the first construction of §1.1 was unsuccessful in [8], namely because points such as $(2, 0, 0, \ldots, 0)$ are closer to the origin than points some of whose co-ordinates are odd. Again we are more successful if we further restrict points to those whose co-ordinates have their sum divisible by 4. This gives a lattice packing in which each sphere touches 4320 others. For example, next to the origin there are 128 centres corresponding to each of the 30 rows of $\mathbf{A}_{16}$, $\bar{\mathbf{A}}_{16}$ comprising eight 0's and eight 1's, and 480 whose co-ordinates are two ± 2's and fourteen 0's.

We can also construct a number of non-lattice packings having the same density. We have only to vary the rule to prescribe that for points whose co-ordinates are congruent (modulo 2) to certain rows of $\mathbf{A}_{16}$ or $\bar{\mathbf{A}}_{16}$ the ssm of the co-ordinates shall be divisible by 2 but not 4, while for others the sum shall be divisible by 4 as usual. This does not affect the minimum distance between centres or the number of spheres which each sphere touches, and for most selections of rows of $\mathbf{A}_{16}$, $\bar{\mathbf{A}}_{16}$ the centres do not form a lattice.

For $m > 4$, congruence to rows of $\mathbf{A}$, $\bar{\mathbf{A}}$ does not give a dense packing with any such simple restriction on the co-ordinates, and we now go on to the general construction.

1.3. *k*-parity. For the general construction in $[2^m]$ we first define a generalization of parity as follows. Suppose we have a row of 2^m binary digits 0 or 1. This will be said to have simple parity, or 1-parity, if the number of 1's is even. To define k-parity, we write out the row of digits, and in columns beneath each 1 we write the binary value of its position, but under each 0 we write 0's. Here is an example:

basic row	0	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1
binary constituent rows	0	1	0	1	0	0	0	0	0	0	0	0	0	1	0	1
	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0	1
	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	1
	0	0	0	0	0	0	0	0	1	0	1	0	0	1	0	1

The top row is the basic row of (here) 16 digits. Numbering their positions from 0 to $2^m - 1$, we place a 1 in the second row under each 1 in the basic row whose position is odd, a 1 in the third row under each 1 whose position $p \equiv 2, 3 \pmod 4$, a 1 in the fourth row under each 1 whose position $p \equiv 4, 5, 6, 7 \pmod 8$, and so on, ending with a 1 in the $(m + 1)$th row under each 1 with $p \geqslant 2^{m-1}$. We call these last m rows the *binary constituent rows* for the given row. We now define recursively that *a row has k-parity* if it has 1-parity and all its binary constituent rows have $(k - 1)$-parity. When we say that a row has k-parity, we shall not, in general, exclude the possibility that it may have $(k + 1)$-parity or even higher; if it is necessary to exclude this possibility, we shall say that it has *exactly k-parity*.

Rows having k-parity have two properties of relevance here. The first is that the sum (mod 2) of any two such rows is itself a row having k-parity. This follows at once by induction, as it is true for rows having 1-parity and the binary constituent rows of the sum are the sums (mod 2) of the corresponding binary constituent rows of the given rows. The second is that in any row having k-parity and not all 0's there are at least 2^k 1's. This again follows by induction. It is obviously true for rows having 1-parity; for any row having k-parity we take a binary constituent row containing some but not all of the 1's of the given row (these obviously exist, since any two 1's of the row differ in the binary representation of their position), and the sum (mod 2) of this row and the given row, then these two rows have $(k - 1)$-parity and together contain exactly all the 1's of the given row, so they cannot both contain more than half as many 1's as the given row.

Combining these two properties, we have that any two rows having k-parity differ in at least 2^k places. In particular, there are only two rows of 2^m digits having m-parity, namely the row of all 0's and that of all 1's, and we see easily that the rows of $\mathbf{A}_{2^m}$, $\bar{\mathbf{A}}_{2^m}$ have $(m - 1)$-parity.

1.4. Numbers of rows having *k*-parity. We now investigate the number of rows of 2^m digits having k-parity, and show this to be

$$\exp_2\left(1 + m + \binom{m}{2} + \binom{m}{3} + \ldots + \binom{m}{m-k}\right).$$

Denote the digits of the binary representation of an integer between 0 and $2^m - 1$ by $a, b, c, \ldots, h$, so that each such integer is uniquely represented in the form $a + 2b + 2^2c + \ldots + 2^{m-1}h$, with $a, b, \ldots, h$ taking values 0 or 1. Let the characteristic function of a row of digits be that function $f(a, b, \ldots, h)$ whose value is the $(a + 2b + \ldots + 2^{m-1}h)$th digit of the row. For a row with a digit 1 in the $(a_1 + 2b_1 + \ldots + 2^{m-1}h_1)$th place and 0's elsewhere, this function is the product

$$(a + a_1 + 1)(b + b_1 + 1) \ldots (h + h_1 + 1)$$

evaluated (mod 2). Inserting the values of $a_1, b_1, \ldots, h_1$ for the position of the 1 and multiplying out, we express this as a sum of products of one or more

https://doi.org/10.4153/CJM-1964-065-1 Published online by Cambridge University Press

of the letters $a, b, \ldots, h$ and perhaps the empty product 1. Call this expression the *characteristic sum* for the row, and define the characteristic sum for an arbitrary row to be the sum (mod 2) of the characteristic sums of rows comprising one of the 1's of the arbitrary row and 0's elsewhere.

We shall now identify rows having k-parity with rows whose characteristic sums include no product of more than $m - k$ letters. First, to each row there corresponds a characteristic sum defined as above, and as we have additive bases of 2^m elements both for the rows (rows with a single 1 and $2^m - 1$ 0's) and for the characteristic sums (products of all possible subsets of the m letters $a, b, \ldots, h$), the correspondence is one-to-one. Next, the characteristic sums for the binary constituent rows for a given row are found by multiplying the characteristic sum for the given row by $a, b, \ldots, h$ respectively. If the characteristic sum for the given row includes products of $m - k$ letters but none of more, then the sums for some of the binary constituent rows will include products of $m - k + 1$ letters but none will include products of more. Now the full product $abc \ldots h$ occurs in the characteristic sums for rows having only a single 1, and so, since the sums are added (modulo 2), it is present in the sums for just those rows that do not have 1-parity. We thus have the basis for an induction that rows having exactly k-parity are precisely those rows whose characteristic sums include products of $m - k$ letters but none of more.

Thus the rows having k-parity (or higher) are just those rows whose characteristic sums include products of at most $m - k$ letters. An additive basis for these sums is the set of products of up to $m - k$ letters, of which there are

$$1 + m + \binom{m}{2} + \ldots + \binom{m}{m-k},$$

so the total number of rows having k-parity is

$$\exp_2\left(1 + m + \binom{m}{2} + \ldots + \binom{m}{m-k}\right).$$

This result generalizes the special cases for $k = 0, 1, m - 1, m$ ($k = m - 1$ giving the rows of the matrices $\mathbf{A}$, $\bar{\mathbf{A}}$ obtained above).

1.5. Numbers of minimal rows having k-parity. We now investigate the number of rows of 2^m digits having k-parity which have the minimum possible number 2^k of 1's. This we find to be

$$\frac{2^m(2^m - 2^0)(2^m - 2^1)(2^m - 2^2) \ldots (2^m - 2^{k-1})}{2^k(2^k - 2^0)(2^k - 2^1)(2^k - 2^2) \ldots (2^k - 2^{k-1})}.$$

For $k = 0$ there are obviously 2^m rows, comprising any single 1 and $2^m - 1$ 0's. For $k = 1$ there are clearly $\frac{1}{2}.2^m(2^m - 1)$ rows, comprising any two 1's and $2^m - 2$ 0's.

For $k = 2$ we may begin by choosing any three 1's in arbitrary positions. The fourth 1 is then uniquely determined to be that whose position makes all

https://doi.org/10.4153/CJM-1964-065-1 Published online by Cambridge University Press

the binary constituent rows have 1-parity; thus the binary representation of its position has a 1 in each binary constituent row which did not already have 1-parity. Since any three of such a set of four determines the fourth, these sets of four form a Steiner system $S(3, 4, 2^m)$. Dividing the number of ways of choosing the first three 1's by the number of choices leading to the same set of four, we find the number of distinct sets of four 1's having 2-parity to be

$$\frac{2^m(2^m - 1)(2^m - 2)}{4.3.2}.$$

For $k = 3$ we choose any three 1's as above, find the fourth 1 which these determine, and choose any fifth 1. The remaining three 1's are the fourth members of sets of four determined by this fifth 1 and pairs from the first four 1's, complementary pairs giving the same result. Dividing the number of ways of choosing the first three and fifth 1's by the number of choices leading to the same set of eight, we find that the number of distinct sets of eight 1's having 3-parity is

$$\frac{2^m(2^m - 1)(2^m - 2)(2^m - 4)}{8.7.6.4}.$$

Continuing similarly, we construct sets of 2^k 1's by means of constructing sets of 2^{k-1} 1's, adjoining a further 1, and completing sets of four 1's determined by this further 1 and pairs from the first 2^{k-1} 1's, of which only $2^{k-1} - 1$ give distinct results. Thus, we find the number of such sets of 2^k 1's having k-parity to be

$$\frac{2^m(2^m - 2^0)(2^m - 2^1)(2^m - 2^2)\ldots(2^m - 2^{k-1})}{2^k(2^k - 2^0)(2^k - 2^1)(2^k - 2^2)\ldots(2^k - 2^{k-1})},$$

concluding, for $k = m$, with just one set of 2^m 1's having m-parity.

1.6. Co-ordinates for sphere packings in $[2^m]$. The concept of k-parity enables us to obtain an effective generalization to $[2^m]$ of the densest lattice packings in [4] and [8]. Each integer occurring as a co-ordinate will be expressed in binary form, using complementary representation for negative integers, so that non-negative and negative integers have only finite numbers of 1's and 0's respectively (the integers 0 and -1 having no 1's and no 0's respectively). Two sets of co-ordinates will be given, corresponding to the two sets given for [4] and [8] in §1.1.

First we give co-ordinates which are either all odd or all even, so that the ones digits form rows having m-parity. We add the requirements that the twos digits form rows with $(m - 2)$-parity, the fours digits form rows with $(m - 4)$-parity, and generally the 2^r digits form rows with $(m - 2r)$-parity as long as $m > 2r$, with no restriction on more significant digits. Any two such points have co-ordinates which differ at least either by 1 in all 2^m places or by 2 in 2^{m-2} places or generally by 2^r in 2^{m-2r} places for some $r < \frac{1}{2}m$ or by at least $2^{\frac{1}{2}m}$ in some one co-ordinate. Hence, the distance between centres is at least $2^{\frac{1}{2}m}$, and we have a packing of spheres of radius $2^{\frac{1}{2}(m-2)}$.

For the other set of co-ordinates, we prescribe that the ones digits of the co-ordinates form rows having $(m - 1)$-parity, i.e. they form rows of the matrices $\mathbf{A}$, $\bar{\mathbf{A}}$. We add the requirements that the twos digits form rows having $(m - 3)$-parity, the fours digits form rows having $(m - 5)$-parity, and generally the 2^r digits form rows having $(m - 2r - 1)$-parity as long as $m > 2r + 1$, with no restriction on more significant digits. Any two such points have co-ordinates differing at least either by 1 in 2^{m-1} places or by 2 in 2^{m-3} places or generally by 2^r in 2^{m-2r-1} places for some $r < \frac{1}{2}(m - 1)$ or by at least $2^{\frac{1}{2}(m-1)}$ in some one co-ordinate. Hence, the distance between centres is at least $2^{\frac{1}{2}(m-1)}$, and we have a packing of spheres of radius $2^{\frac{1}{2}(m-3)}$.

For $m \geqslant 4$ there are a large number of non-lattice packings which have the same density as these packings. This is because the rows of 2^m digits having k-parity determine a set of

$$\exp_2 \left(1 + m + \binom{m}{2} + \ldots + \binom{m}{m-k}\right)$$

rows differing in at least 2^k places from each other, but in a manner which is very far from unique. The rows are co-ordinates of a subset of the vertices of a unit cube in $[2^m]$, and any symmetry operation of the cube which does not take this subset into itself takes the rows into another set having the same property of differing from one another in at least 2^k places. Thus for $m = 4$, with all co-ordinates odd or all even, we may specify that the twos digits of odd co-ordinates form a set of rows which is quite different from that formed by the twos digits of even co-ordinates. For larger values of m the number of possibilities is enormous.

Further, the sets of co-ordinates given above do not themselves give lattices for $m > 6$. For if we have two centres of the packing whose non-zero co-ordinates are 2^k values 2^r, forming minimal sets having k-parity, the point whose co-ordinates are the sum of the co-ordinates of these points will have co-ordinates 2^{r+1} in just those places where both the centres have co-ordinates 2^r. The characteristic sum for this row will be the product of the characteristic sums for the given rows, and it may contain products of up to $\min(m, 2(m-k))$ letters. If both k and $m - k$ exceed 2, this may exceed the maximum of $m - k + 2$ necessary for these digits 2^{r+1} to form a row with $(k - 2)$-parity. This means that the lattice generated by centres contains points which are not centres, so that the set of centres is not itself a lattice.

For $m = 6$ the set of centres whose co-ordinates are either all odd or all even does not form a lattice, while the other set does. It is only for $m \leqslant 5$ that the two sets of co-ordinates give the same (lattice) packing in co-ordinates related by the equations at the end of §1.1. For $m > 6$ our construction has not exhibited a lattice packing at all; though there seems no reason to doubt that there will be a lattice packing of the same density, I have not proved this.

1.7. Numbers of spheres which each touches. We now go on to calculate the number of centres in these packings at the minimum distance from the origin, which will give the number of spheres each touches in these packings in $[2^m]$. It is convenient to take both sets of co-ordinates together here. In both cases we are concerned with finding the numbers of centres whose co-ordinates are 2^k of value $\pm 2^r$ and the rest 0, where $2k = r$ for the first set of co-ordinates and $2k = r + 1$ for the second set, the non-zero co-ordinates being in sets of positions having k-parity. The total number of spheres touched is then found by summing over the relevant values of k.

First, we need to know the number of permissible sign combinations for the values $\pm 2^r$. This is determined by the patterns for the 2^{r+1} digit, which is 0 for $+2^r$ and 1 for -2^r. These have to form rows having $(k - 2)$-parity, as we have seen. Thus (since this digit is 0 for every co-ordinate 0) the number of possible sign combinations is the number of rows of 2^k digits having $(k - 2)$-parity, namely

$$\exp_2\left(1 + k + \binom{k}{2}\right) = \exp_2(1 + \tfrac{1}{2}k + \tfrac{1}{2}k^2).$$

Thus the total number of centres whose co-ordinates are 2^k values $\pm 2^r$ and the rest 0 is

$$T_{m,k} = \exp_2(1 + \tfrac{1}{2}k + \tfrac{1}{2}k^2).\frac{2^m(2^m - 2^0)(2^m - 2^1)\ldots(2^m - 2^{k-1})}{2^k(2^k - 2^0)(2^k - 2^1)\ldots(2^k - 2^{k-1})},$$

and the total number of spheres which each touches in the packings in $[2^m]$ is found by summing $T_{m,k}$ for alternate values of k, all odd for one set and all even for the other.

For notational convenience, define $T_{m,k} = 0$ for $k < 0$ and for $k > m$. With this notation we find that the recurrence relation

$$T_{m+1,k+1} = 2^{k+1}T_{m,k} + 2^{k+2}T_{m,k+1}$$

is valid for all k for each $m \geqslant 0$, by straightforward calculation, noticing that, except for the factor $(2^{k+1} - 2^0)$, each factor of the denominator of $T_{m,k+1}$ is twice a factor of that of $T_{m,k}$.

We now define a generating function

$$g_m(x) = \sum T_{m,k}\, x^k,$$

the summation, here and later, being over all values of k for fixed m. Then, we find that

$$\begin{aligned} g_{m+1}(x) &= \sum T_{m+1,k+1}\, x^{k+1} \\ &= \sum (2^{k+1}T_{m,k} + 2^{k+2}T_{m,k+1})x^{k+1} \\ &= 2x \sum 2^k\, T_{m,k}\, x^k + 2 \sum 2^{k+1}\, T_{m,k+1}\, x^{k+1} \\ &= (2x + 2)\, g_m(2x). \end{aligned}$$

Repeated application of this relation yields

https://doi.org/10.4153/CJM-1964-065-1 Published online by Cambridge University Press

$$\begin{aligned} g_m(x) &= (2 + 2x)\, g_{m-1}(2x) \\ &= (2 + 2x)(2 + 2^2x)\, g_{m-2}(2^2x) \\ &\ \ \ldots \\ &= (2 + 2x)(2 + 2^2x) \ldots (2 + 2^m x).2, \end{aligned}$$

since $g_0(x) \equiv 2$. Putting $x = 1$, we obtain

$$g_m(1) = \sum T_{m,k} = 2(2 + 2)(2 + 2^2) \ldots (2 + 2^m), \qquad m \geqslant 0,$$

and, putting $x = -1$, we establish that the two sums of alternate terms are equal (for $m > 0$), as was expected for geometrical reasons. Thus the number of spheres touched by each in these packings in $[2^m]$ is

$$\tfrac{1}{2} g_m(1) = (2 + 2)(2 + 2^2) \ldots (2 + 2^m).$$

For $m = 0$ the last step is invalid, and we get the correct figure $g_0(1) = 2$ for the obvious packing in [1]. For $m = 1$ we get 4 for the number of circles each touches in the square packing (not, of course, the densest in [2]). For $m = 2, 3$ we get 24, 240 for the densest lattice packings in [4] and [8]. For $m = 4$ we get 4320, as anticipated in §1.2, and the next two values are 146880 and 9694080, in [32] and [64] respectively, which is as far as we have explicitly obtained lattice packings. The asymptotic behaviour of the number of spheres touched as $m \to \infty$ is discussed in §3.2.

2. Packings in up to 24 dimensions.

2.1. Some packings in [12]. We have used the matrices $\mathbf{A}_n$, $\bar{\mathbf{A}}_n$ to obtain packings in $[n]$ for $n = 2^m$. Paley **(8)** has shown that similar matrices exist for many (conjecturally all) other values of n which are multiples of 4. For $n > 16$ these matrices do not lead to dense packings by our present constructions, as the simple condition on the sum of the co-ordinates is not sufficiently restrictive (there are still points closer to the origin with two co-ordinates ± 2 and the rest 0 than any with some co-ordinates odd), and higher parity conditions are inapplicable when n is not a power of 2. But for $n = 12$ we are more successful. Following Paley, we consider the matrix

$$\mathbf{A}_{12} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&1&0&0&0&1&1&1&0&1 \\ 0&0&1&0&0&0&1&1&1&0&1&1 \\ 0&1&0&0&0&1&1&1&0&1&1&0 \\ 0&0&0&0&1&1&1&0&1&1&0&1 \\ 0&0&0&1&1&1&0&1&1&0&1&0 \\ 0&0&1&1&1&0&1&1&0&1&0&0 \\ 0&1&1&1&0&1&1&0&1&0&0&0 \\ 0&1&1&0&1&1&0&1&0&0&0&1 \\ 0&1&0&1&1&0&1&0&0&0&1&1 \\ 0&0&1&1&0&1&0&0&0&1&1&1 \\ 0&1&1&0&1&0&0&0&1&1&1&0 \end{bmatrix},$$

https://doi.org/10.4153/CJM-1964-065-1 Published online by Cambridge University Press

where $a_{ij} = 0$ if $i = 1$ or $j = 1$ or if $i + j - 4$ is a quadratic residue of 11, and $a_{ij} = 1$ otherwise. We obtain a packing by taking as centres all points whose co-ordinates are congruent (mod 2) to rows of $\mathbf{A}_{12}$ or $\bar{\mathbf{A}}_{12}$ (where $\bar{a}_{ij} = 1 - a_{ij}$, as before) and have their sum divisible by 4. In this packing each sphere touches $22.2^5 = 704$ others. This figure is intermediate between those for the lattice packings J_{12} and K_{12} of Coxeter and Todd **(5)**, in which each sphere touches 648 and 756 others respectively, and it is also intermediate in density (see §3.1). For convenience of reference, I call this packing L_{12}.

L_{12} is not a lattice packing, and it seems to be "strictly non-lattice" in the sense that unlike other non-lattice packings mentioned in this paper it does not seem to be a variant of an equally dense lattice packing. The sum (mod 2) of any two rows of $\mathbf{A}_{12}$, $\bar{\mathbf{A}}_{12}$ has at least six 1's, in fact exactly six 1's unless the rows are complementary, but there are sets of three rows whose sum (mod 2) has only four 1's, and it will follow from our analysis below that the lattice generated by rows of $\mathbf{A}_{12}$, $\bar{\mathbf{A}}_{12}$ reduces (mod 2) to the lattice D_{12} of all points whose co-ordinates have their sum even. The lattice generated by rows of $\mathbf{A}_{12}$, $\bar{\mathbf{A}}_{12}$ turns out to be another lattice D_{12}, of a larger size and in a different orientation.

To see this, we consider the lattice generated (mod 3) by rows of the matrix

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & 1 & 0 \end{bmatrix} = (\mathbf{J}_6\mathbf{D}_6), \text{ say,}$$

where the elements of $\mathbf{D}_6$ are $d_{11} = 0$, $d_{ij} = 1$ if i or j is 1 but not both, otherwise d_{ij} is the quadratic residue symbol $(\frac{i-j}{5})$, and where the signs of the elements of $\mathbf{J}_6$ are chosen to make the row sums divisible by 3. Each row has six 0's, and so have the sums and differences of pairs of rows. Since $\mathbf{D}_6{}^2 \equiv -\mathbf{I}_6 \pmod 3$, the rows of

$$\mathbf{D}^* = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

are linear combinations (mod 3) of those of $\mathbf{D}$ and vice versa. Similarly, the rows of $\mathbf{D}^*$ and the sums and differences of pairs of rows all have just six 0's. We thus see that any linear combination (mod 3) of rows of $\mathbf{D}$ having four or five 0's in either its first six elements or its last six elements has just six 0's

https://doi.org/10.4153/CJM-1964-065-1 Published online by Cambridge University Press

in all. As any other combination has at most three 0's in each of its first and last sets of six elements, it follows that every (non-null) linear combination (mod 3) of rows of **D** has at most six 0's.

Among linear combinations (mod 3) of the rows of **D** we find (including the null row) the rows of the matrices $\mathbf{B}_{12}$, $\bar{\mathbf{B}}_{12}$, where

$$\mathbf{B}_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

and $\bar{b}_{ij} = 1 - b_{ij}$. Since linear combinations of rows of **D** have at most six 0's, rows of $\mathbf{B}_{12}$, $\bar{\mathbf{B}}_{12}$ differ in at least six places, and hence, like the rows of $\mathbf{A}_{12}$, $\bar{\mathbf{A}}_{12}$, they give the co-ordinates of vertices of a cross polytope β_{12} inscribed in a cube γ_{12}. Thus, the matrices differ only in arrangement of rows and columns.

We now show that the rows of $\mathbf{A}_{12}$, $\bar{\mathbf{A}}_{12}$ (and hence those of $\mathbf{B}_{12}$, $\bar{\mathbf{B}}_{12}$) generate a lattice D_{12}. If we multiply these rows by the matrix $\mathbf{T}_{12} = \frac{1}{3}(\bar{\mathbf{A}}_{12} - \mathbf{A}_{12})$, they are transformed into rows having the first element 2, one other element ± 2, and ten elements 0, except that the first row of $\mathbf{A}_{12}$ remains null and that of $\bar{\mathbf{A}}_{12}$ goes into $(4, 0, \ldots, 0)$. These rows generate the lattice D_{12} in a co-ordinate system in which the co-ordinates of the centres are even with their sum divisible by 4. Since $\frac{1}{2}\sqrt{3}\,\mathbf{T}_{12}$ is orthogonal, the rows of $\mathbf{A}_{12}$, $\bar{\mathbf{A}}_{12}$ also generate a lattice D_{12}, and so do those of $\mathbf{B}_{12}$, $\bar{\mathbf{B}}_{12}$. It is easily seen that these are those of the points of the lattice generated (mod 3) by rows of **D** which have the sum of their co-ordinates even (and so divisible by 6). Those with sum odd are transformed by $\mathbf{T}_{12}$ into sets of points whose co-ordinates are odd and have their sum divisible by 4, forming with the lattice D_{12} a lattice $D_{12}{}^2$ having twice the density.

It is now easy to see that if this lattice is reduced (mod 2), all integer points belong to it, while that generated by the rows of $\mathbf{A}_{12}$, $\bar{\mathbf{A}}_{12}$ reduces to the lattice of points whose co-ordinates have their sum even, which is another lattice D_{12}.

In the lattice D_{12} as generated by rows of $\mathbf{A}_{12}$, $\bar{\mathbf{A}}_{12}$, the nearest centres to the origin are those whose co-ordinates are six ± 1's and six 0's. We now show that the sets of six 0's form a Steiner system $S(5, 6, 12)$. First, no two of these rows of co-ordinates have just five 0's in common, as otherwise either their sum or their difference would have at least eight 0's, which we know does not happen. Again, no two rows have all six 0's in common unless they are identical or one row is minus the other; otherwise again their sum or difference would have more than six 0's without being null. Finally if we take any two rows of $\mathbf{A}_{12}$, $\bar{\mathbf{A}}_{12}$, not being exact complements, their difference has six 0's, and, except that the complements of two rows have the same difference as the given rows, these differences are all different (they are transformed by $\mathbf{T}_{12}$ into obviously different rows). There are $\frac{1}{2}.24.22 = 264$ such pairs, and so there are 132 different sets of six 0's occurring, no two of which have five 0's in common. Each contains six subsets of five 0's, which accounts for all the 792 possible sets of five out of twelve elements, showing that they form a Steiner system $S(5, 6, 12)$. Each set occurs in the co-ordinates of centres of two distinct spheres nearest to the origin, each with co-ordinates minus those of the other, giving 264 as the number of spheres each touches in the packings. The spheres of each set in the packing $D_{12}{}^2$ have no contact with those of the other set, so the figure 264 for the number each touches is the same for both the packings D_{12} and $D_{12}{}^2$.

Matrices similar to our $\mathbf{D}_6$ are used by Coxeter **(3)** and Todd **(13)** in connection with geometrical representations of the Mathieu group M_{12} (the group of automorphisms of the Steiner system $S(5, 6, 12)$), and by Golay **(6)** in connection with ternary digital coding. Paige **(7)** mentions a similar possibility in relation to the Steiner system $S(4, 5, 11)$ but gives no matrix.

Convenient co-ordinates for the lattice J_{12} **(5)** are those in which either they are all even, or only the first four, or only the middle four, or only the last four, with their sum divisible by 4. Nearest the origin there are $3.2^7 = 384$ centres whose co-ordinates are eight ± 1's and four 0's, and $\frac{1}{2}.12.11.2^2 = 264$ whose co-ordinates are two ± 2's and ten 0's, a total of 648. The spheres are the same size as those of D_{12} in the system in which the co-ordinates are all even with their sum divisible by 4, and we see at once that the packing J_{12} is four times as dense as D_{12} and thus twice as dense as $D_{12}{}^2$.

I know of no convenient integer co-ordinates for K_{12} **(5)**.

2.2. Neighbourhoods of vertices of a cube. We now give another derivation of the densest lattice packing in [8] with a view to considering its analogues. In [7] each vertex of the unit cube has seven neighbours distant 1 from it, which form with it a neighbourhood of eight vertices. Since this is an exact submultiple of the total number of vertices, we may ask whether we can find a subset of the vertices which are centres of a set of neighbourhoods which exactly exhaust the vertices of the cube. This is in fact possible. The rows of the matrices $\mathbf{A}_8$, $\bar{\mathbf{A}}_8$, with any one column deleted throughout, give the co-ordinates of a set of such centres of neighbourhoods of vertices. Restoring the deleted co-ordinate by parity and extending to a lattice by congruence (mod 2), we arrive at the densest lattice packing in [8] as above.

The corresponding construction is possible in any space $[2^m]$. We may take as centres of neighbourhoods of 2^m vertices of the cube in $[2^m - 1]$ those points which, when the 2^mth co-ordinate is inserted by parity, have co-ordinates forming a row with 2-parity (§1.3). For $m > 3$ this yields a packing which is a subset of that obtained in $[2^m]$ in §1.6, a packing of little interest. For example, in [16] it gives the centres with even co-ordinates in the representation with co-ordinates all odd or all even.

We consider next the possibility of neighbourhoods of vertices distant greater than 1 on the unit cube γ_n. The number of such vertices within distance r (measured along the edges) is

$$1 + n + \binom{n}{2} + \ldots + \binom{n}{r},$$

and we have to find such sums which are powers of 2. There are trivial solutions with $r \geqslant n$ (useless for our purposes) and with $r = \frac{1}{2}(n-1)$, leading to the packing with all co-ordinates odd or all even, which is of low density for $n > 4$.

For $r = 2$ we have to satisfy

$$2^h = 1 + n + \tfrac{1}{2}n(n-1) = \tfrac{1}{8}((2n+1)^2 + 7),$$

so we require solutions of

$$(2n+1)^2 + 7 = 2^{h+3}.$$

This is known to have solutions only for $h = 0, 1, 2, 4, 12$. Only the last is non-trivial for our purposes, and Golay **(6)** and Paige **(7)** have shown that it does not lead to a covering of the cube in [90].

For $r = 3$ we have to satisfy

$$2^h = 1 + n + \binom{n}{2} + \binom{n}{3} = \frac{1}{24}(n+1)((2n-1)^2 + 23),$$

which requires solutions of

$$(n+1)((2n-1)^2 + 23) = 3.2^{h+3}.$$

It is impossible to satisfy the requirements that one factor be a power of 2 and the other three times a power of 2 for n large, since the second factor is asymptotic to the square of twice the first, and they can have no other common factor since $(2n-1)^2 + 23 = (n+1)(4n-8) + 32$, so we have only to examine fairly small values of n. We find solutions (with $n > 0$) only for $h = 1, 2, 3, 6, 11$, of which only the last is non-trivial for our purposes. We shall see in §2.3 that it leads to a covering of the unit cube in [23] and to a sphere packing in [24].

It seems unlikely that any non-trivial solutions exist for values of $r > 3$. It is still more unlikely that any such solution would lead to a covering of the unit cube, as it would lead to the discovery of Steiner systems $S(r+2, 2r+2, n+1)$ of improbably high order, analogous to the system $S(5, 8, 24)$ as constructed in §2.3 below. The possibility of constructing other sphere packings by this method is therefore extremely remote.

2.3. A lattice packing in [24]**.** Golay **(6)** and Paige **(7)** have shown that there is a covering of the cube in [23] by 2^{12} neighbourhoods of

$$1 + 23 + \binom{23}{2} + \binom{23}{3} = 2^{11}$$

vertices each. They give matrix formulations for the co-ordinates of centres, but though their matrices are rather similar to each other and have essentially the same properties as that given below, they do not have a form similar to that below and it is less easily shown that they have the relevant properties. Part of this difference is one of purpose: Golay is concerned with binary digital coding, while Paige is concerned with representation of the Steiner system $S(4, 7, 23)$.

We consider the matrix $\mathbf{C} = (\mathbf{I}_{12}\ \mathbf{C}_{12})$,

$$\text{where} \qquad \mathbf{C}_{12} = \begin{bmatrix} 0&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&0&1&0&0&0&1&1&1&0&1\\ 1&0&1&0&0&0&1&1&1&0&1&1\\ 1&1&0&0&0&1&1&1&0&1&1&0\\ 1&0&0&0&1&1&1&0&1&1&0&1\\ 1&0&0&1&1&1&0&1&1&0&1&0\\ 1&0&1&1&1&0&1&1&0&1&0&0\\ 1&1&1&1&0&1&1&0&1&0&0&0\\ 1&1&1&0&1&1&0&1&0&0&0&1\\ 1&1&0&1&1&0&1&0&0&0&1&1\\ 1&0&1&1&0&1&0&0&0&1&1&1\\ 1&1&1&0&1&0&0&0&1&1&1&0 \end{bmatrix}.$$

$\mathbf{C}_{12}$ is adapted from $\mathbf{A}_{12}$ by replacing the first row and column of 0's by 1's except for the leading element. We shall show that the rows of $\mathbf{C}$ form an additive basis (mod 2) for a set of centres on the cube in [24], any two of which differ in at least eight co-ordinates. Then if any one co-ordinate is deleted throughout, we shall have co-ordinates for a set of centres on the cube in [23], any two of which differ in at least seven co-ordinates, so that the neighbourhoods of 2^{11} points each differing from the centre in at most three co-ordinates exactly exhaust the vertices of the cube in [23] as required.

We require to show that no sum (mod 2) of rows of $\mathbf{C}$ has fewer than eight 1's in it (except for the empty sum). This we show by proving that every row having at most three 1's in either its first twelve places or in its last twelve places has at least eight 1's in all. (The similarity of the present argument to that for $\mathbf{D}$ in §2.1 will be apparent.) Rows having a single 1 in their first twelve places are rows of $\mathbf{C}$ and have eight or twelve 1's in all. Rows having two 1's in their first twelve places are sums (mod 2) of pairs of rows of $\mathbf{C}$. If one of them is the first row, the sum will have a 1 in its thirteenth place, and a 1 in place of each of the five 0's in the last eleven places of the other row, a total of eight 1's. If not, then the sum has 1's in those six places in which the corresponding rows of $\mathbf{A}_{12}$ differ (any two of which differ in six places), and again we have a total of eight 1's.

Rows having three 1's in their first twelve places are sums (mod 2) of three rows, one of which may be the first row while the other two are certainly not. As we have seen, the sum of the two latter rows has six 1's and five 0's in its last eleven places, so if the first row is added, these five 0's become 1's and the row has eight 1's in all. If not, then at most four of the six 1's in the last eleven places of the sum coincide with 1's of the other row; otherwise the corresponding rows formed from $\mathbf{A}_{12}$ would have five 0's in common. There are, therefore, at least four 1's in the sum corresponding to places of discrepancy, which with the 1 in the thirteenth place make a total of at least eight 1's again. Thus no row having three or fewer 1's in its first twelve places has fewer than eight 1's in all.

To deal with rows having three or fewer 1's in the last twelve places, we notice that $\mathbf{C}_{12}{}^2 \equiv \mathbf{I}_{12}$ (mod 2), and so the rows of $\mathbf{C}^* = (\mathbf{C}_{12}\, \mathbf{I}_{12})$ are linear combinations (mod 2) of those of $\mathbf{C}$ and vice versa. Thus, without further consideration, we can deduce that rows having at most three 1's in their last twelve places also have at least eight 1's in all. This completes the proof that all non-null sums (mod 2) of rows of $\mathbf{C}$ have at least eight 1's in them. It follows at once that any two such sums differ in at least eight places.

Since the neighbourhoods on the cube in [23] are exactly exhaustive, any vertex having just four co-ordinates 1 is in the neighbourhood of just one centre having seven co-ordinates 1. Hence, the sets of seven 1's forming co-ordinates of centres nearest to the origin form a Steiner system $S(4, 7, 23)$, as found by Paige. We extend this to show that the sets of eight co-ordinates 1 of centres nearest to the origin on the cube in [24] form a Steiner system $S(5, 8, 24)$. To find the centre nearest to any point whose co-ordinates include just five 1's, we delete a co-ordinate corresponding to one of these 1's throughout. Then the four 1's remaining determine a centre on the cube in [23] with just seven co-ordinates 1, and when the deleted co-ordinate is restored it adds an eighth 1 to form the co-ordinates of a centre on the cube in [24] (since none has only seven co-ordinates 1), and these eight 1's include the five 1's with which we began.

To obtain the lattice packing of spheres in [24], we take as centres all points whose co-ordinates are congruent (mod 2) to sums of rows of $\mathbf{C}$ and have their sum divisible by 4. Any two such points differ at least either by 1 in eight co-ordinates or by 2 in two co-ordinates, and so we have a packing of spheres of radius $\sqrt{2}$. Since there are 759 sets of eight 1's in the Steiner system $S(5, 8, 24)$, the sphere with centre the origin touches $759.2^7 = 97152$ others with centres whose co-ordinates are eight ± 1's and sixteen 0's, and $\frac{1}{2}.24.23.2^2 = 1104$ with centres whose co-ordinates are two ± 2's and twenty-two 0's, a total of 98256 spheres.

A similar construction based on modifying the first row and column of $\mathbf{A}_4$

leads again to the densest packing in [8]. It is only when n is an odd multiple of 4 that the matrix $\mathbf{A}_n$ becomes modified into a matrix which is orthogonal (mod 2). For other values of n examined (I have examined Paley's matrices for $n = 20, 28$), the matrices are otherwise unsuitable, having sums (mod 2) of rows with too few 1's in them. As in these cases the number of vertices in the neighbourhoods is not a power of 2, a covering of the cube is impossible for these, and probably for all higher, values of n.

2.4. Cross-sections in fewer than 24 dimensions. To obtain packings in spaces of up to 23 dimensions, we may examine sections of the packing in [24] obtained in §2.3. The following are the densest I have found. In [23], [22], [21], [20], [19], take the sections in which any one, two, three, four, or five co-ordinates are equated to 0. (Because of the fivefold transitivity of the Steiner system, the choice of co-ordinates is arbitrary.) The numbers of spheres which each touches in these packings are 65780, 43164, 27720, 17400, and 10668 respectively. In [19] (again), [18], [17], [16], equate to 0 the sum of eight co-ordinates forming a Steiner set and also any four, five, six, or all of them respectively. The numbers of spheres which each touches in these packings are 10668, 7398, 5346, and 4320 respectively. This last section in [16] is the lattice packing obtained in §1.2 (with the co-ordinates in a different order).

For cross-sections in fewer than 16 dimensions, we use co-ordinates based on rows of the matrices $\mathbf{A}_{16}$, $\bar{\mathbf{A}}_{16}$ as in §1.2. Define a *tetrad* to be a set of four co-ordinates such that a 1 in each of them forms a set having 2-parity. The densest sections I have found are as follows (the co-ordinates may be given in several ways, corresponding to symmetries of the lattice). In [15] equate to 0 either any one co-ordinate, or the sum of a tetrad, or the sum of all sixteen co-ordinates; each sphere touches 2340 others. In [14] equate to 0 the sum of a tetrad and either any one of them or the sum of all sixteen co-ordinates; each sphere touches 1422 others. In [13] equate to 0 either the sum of a tetrad and any two of them, or the sum of all sixteen co-ordinates and of any two tetrads making up a row of $\mathbf{A}_{16}$ or $\bar{\mathbf{A}}_{16}$; each sphere touches 906 others. In [12] I have found no denser section than J_{12} (cf. §2.1), with each sphere touching 648 others. This may be found by equating to 0 any four co-ordinates forming a tetrad, or the sums of any four tetrads such that every pair of them forms a row of $\mathbf{A}_{16}$ or $\bar{\mathbf{A}}_{16}$.

A similarity in the sections from [20] downwards and from [16] downwards is worth comment. Define a tetrad in the packing in [20] to be a set of four co-ordinates which form Steiner sets with the four co-ordinates already equated to 0 (the co-ordinates fall into five such tetrads). Then for $n = 16$ or 20, our sections in $[n-1]$, $[n-2]$, $[n-3]$, $[n-4]$ are found by equating to 0 any one co-ordinate or the sum of a tetrad, the sum of a tetrad and any one of them, the sum of a tetrad and any two of them, and any four co-ordinates forming a tetrad, respectively.

For sections in fewer than 12 dimensions we use the co-ordinates of §2.1, which is what we get if the last four of the sixteen co-ordinates, which form a tetrad, are equated to 0. We continue to call the first four, the middle four, and the last four co-ordinates tetrads. The densest sections I have found are as follows. In [11] equate any two co-ordinates from the same tetrad to each other; each sphere touches 438 others. In [10] equate any three co-ordinates from the same tetrad to each other; each sphere touches 336 others. In [9] equate the four co-ordinates of a tetrad to each other, or any three of them to 0; each sphere touches 272 others. In [8] we obtain the densest lattice packing by equating to 0 the four co-ordinates of any tetrad; each sphere touches 240 others. The co-ordinates are those which are all odd or all even.

To complete the picture, we remark that the densest lattice packings in [7], [6], [5], [4], in which each sphere touches 126, 72, 40, 24 others respectively, can be obtained as sections of that in [8] by a sequence verbally identical with that just given for sections of J_{12} in [12], deeming any four co-ordinates of the eight to be a tetrad. The final co-ordinates in [4] are the doubles of those of §1.1 based on rows of $\mathbf{A}_4$, $\bar{\mathbf{A}}_4$. Regarding these four co-ordinates as a tetrad, we may continue the sequence using verbally the same sections as were used for both [16] and [20]. In fact, these two sequences of sections, as used for [20], [16], [4] and for [12], [8] respectively, are closely related, being interchanged if the co-ordinates are transformed by the equations at the end of §1.1 (after rearranging the co-ordinates in [20] so that the tetrads are consecutive sets of four co-ordinates). A repeating pattern in the density of these sections is described in §3.1.

To all these lattice packings from [24] down to [9] there correspond equally dense non-lattice packings. Instead of uniformly applying the rule that the sum of the co-ordinates is to be divisible by 4, we specify (as in §1.2) that for certain arrangements of the odd co-ordinates the sum is divisible by 2 but not by 4, while for others it is divisible by 4 as usual. As long as there are more than two patterns of odd co-ordinates, i.e. in nine or more dimensions, the choice can be made in ways resulting in non-lattice packings.

The densest lattice packing known in [12] is K_{12}, mentioned in §2.1; I have not found a section of it in [11] as dense as that of J_{12} given above, but the co-ordinates available **(5)** are inconvenient from this aspect. However, the non-lattice packing L_{12} (§2.1) has a denser section, obtained by equating the sum of the co-ordinates to 0. Each sphere touches 440 others, as compared with 438 for the section of J_{12}, and this packing is also denser by a factor $3^6/2^{9\frac{1}{2}} \doteqdot 1.0068$. Thus the densest packing I have found in [11] is one which seems to be "strictly non-lattice," not a variant of an equally dense lattice packing.

The densest known packings for $n \leqslant 12$ are listed in Table I with particulars of density and numbers of spheres touched. All the packings derived from that in [24], which include the densest known for $n > 12$, are listed in Table II with particulars of density (see §3.1) and numbers of spheres touched.

TABLE I

DENSEST KNOWN PACKINGS FOR $n \leqslant 12$

	Centre density			No. of spheres touched		
n	Bound	Best achieved	Ratio	Bound	Best achieved	Ratio
1	2^{-1}	2^{-1}	1	2	2	1
2	$2^{-1}3^{-\frac{1}{2}}$	$2^{-1}3^{-\frac{1}{2}}$	1	6	6	1
3	0·1861	$2^{-2\frac{1}{2}}$ = 0·1767	0·949	13·397	12	0·895
4	0·1312	2^{-3} = 0·125	0·952	26·440	24	0·907
5	0·09987	$2^{-3\frac{1}{2}}$ = 0·08838	0·885	48·702	40	0·821
6	0·08112	$2^{-3}3^{-\frac{1}{2}}$ = 0·07216	0·889	85·814	72	0·839
7	0·06981	2^{-4} = 0·0625	0·895	146·57	126	0·859
8	0·06326	2^{-4} = 0·0625	0·987	244·62	240	0·981
9	0·06007	$2^{-4\frac{1}{2}}$ = 0·04419	0·735	401·03	272	0·678
10	0·05953	$2^{-4}3^{-\frac{1}{2}}$ = 0·03608	0·606	648·13	336	0·518
11	0·06137	$2^{-14\frac{1}{2}}3^{6}$ = 0·03146	0·512	1035·3	440	0·424
12	0·06559	3^{-3} = 0·03703	0·564	1637·8	756	0·461

Note: Decimal quantities have been truncated, so that the last figure given has not been raised where the next figure exceeds 5.

TABLE II

PACKINGS DERIVED FROM THAT IN [24] (§2.4)

n	Relative density (see §3.1)		Numbers of spheres touched
1	1	$\frac{9}{32}$	2
2	$1+c$	$c-\frac{3}{8}$	6
3	$2\frac{1}{2}$	$\frac{17}{32}$	12
4	3	$\frac{1}{2}$	24
5	$3\frac{1}{2}$	$\frac{17}{32}$	40
6	$3+c$	$c-\frac{3}{8}$	72
7	4	$\frac{9}{32}$	126
8	4	0	240
9	$4\frac{1}{2}$	$\frac{9}{32}$	272
10	$4+c$	$c-\frac{3}{8}$	336
11	5	$\frac{17}{32}$	438
12	5	$\frac{1}{2}$	648
13	5	$\frac{17}{32}$	906
14	$4+c$	$c-\frac{3}{8}$	1422
15	$4\frac{1}{2}$	$\frac{9}{32}$	2340
16	4	0	4320
17	4	$\frac{9}{32}$	5346
18	$3+c$	$c-\frac{3}{8}$	7398
19	$3\frac{1}{2}$	$\frac{17}{32}$	10668
20	3	$\frac{1}{2}$	17400
21	$2\frac{1}{2}$	$\frac{17}{32}$	27720
22	2	$\frac{5}{8}$	43164
23	$1\frac{1}{2}$	$\frac{25}{32}$	65780
24	1	1	98256

3. Bounds and limits.

3.1. Rogers' upper bound for the density. Rogers **(10)** has proved that the average density of a packing in $[n]$, i.e. the average proportion of space which is interior to spheres of the packing, cannot exceed the proportion of the interior of a regular simplex of side 2 which is interior to unit spheres centred at its vertices. It is of interest to evaluate this upper bound and to compare it with the densities actually achieved by the known densest lattice packings for $n \leqslant 8$ and by the packings of this paper.

Schläfli has defined a function $F_n(\alpha)$ (discussed by Coxeter **(4)**), which is defined in relation to the $(n-1)$-dimensional content of a regular spherical simplex of dihedral angle 2α on a unit sphere in $[n]$ in such a way that the content of the simplex is

$$2^{-n}\, n!\, F_n(\alpha)\, H_n,$$

where $H_n = 2\pi^{\frac{1}{2}n}/\Gamma(\frac{1}{2}n)$ is the total $(n-1)$-dimensional "surface" content of the sphere. For convenience we redefine the function in terms of a new argument and write $f_n(\sec 2\alpha) = F_n(\alpha)$. Since the dihedral angle of a regular (Euclidean) simplex in $[n]$ is arcsec n, the solid angular content at each vertex of a regular simplex is $2^{-n}\, n!\, f_n(n) H_n$. Thus, the interior content of those parts of unit spheres centred at the $n+1$ vertices of a regular simplex of side 2 which are interior to the simplex is $2^{-n}(n+1)!\, f_n(n)\, J_n$, where $J_n = \pi^{\frac{1}{2}n}/\Gamma(\frac{1}{2}n+1)$ is the total interior content of a unit sphere. The total interior content of the simplex is $2^{\frac{1}{2}n}(n+1)^{\frac{1}{2}}/n!$, so that the proportion of the simplex occupied is

$$2^{-3n/2}(n+1)^{\frac{1}{2}}(n!)^2 f_n(n)\, J_n.$$

This then is Rogers' upper bound for the density of sphere packings in $[n]$.

In practice, there is a certain convenience in omitting the factor J_n; what remains is the bound for the average number of centres of unit spheres of the packings per unit n-dimensional content of the space, or, for brevity, the bound for the *centre density*. For the lattice and other packings of this paper, this centre density is a convenient quantity, either rational or with rational square, and of course its ratio to its bound is the same as the ratio of the density to its bound. For example, the centre densities for the packings in [12] are $3^{-3} = 0.03703\ldots$ for K_{12}, $3^7/2^{16} = 0.03337\ldots$ for L_{12}, and $2^{-5} = 0.03125$ for J_{12}, while Rogers' upper bound is $0.06559\ldots$.

I have calculated the values of $f_n(n)$ for $n \leqslant 8$ for Coxeter (who gives details in **(4)**), and have extended this to $n = 9, 10$ by a further stage of integration and to $n = 11, 12$ by extrapolation. The corresponding figures for the bounds for the centre density are given in Table I in comparison with the densities achieved by the known densest lattice packings for $n \leqslant 8$ and the densest packings of this paper for $8 < n \leqslant 12$. They are also shown graphically in Figure 1, where the curve is based on these figures and extended by an empirical asymptotic formula of the form

https://doi.org/10.4153/CJM-1964-065-1 Published online by Cambridge University Press

$$f_n(n) = \frac{(n+1)^{\frac{1}{2}}}{2^{\frac{1}{2}}en!}\left(\frac{2e}{\pi n}\right)^{\frac{1}{2}n}(1 + an^{-1} + bn^{-2} + cn^{-3} + \ldots).$$

Values of the coefficients were determined by fitting for $n \leqslant 12$, the leading term being the known asymptotic expression for $f_n(n)$. I do not give details as the accuracy is somewhat uncertain, but the accuracy should be at least that to which the curve can be drawn. For $n > 8$ the density achieved is not close to the bound, so high accuracy is of less interest than for $n \leqslant 8$. The curve in Figure 1 is drawn through the calculated values, while the points show the best values achieved.

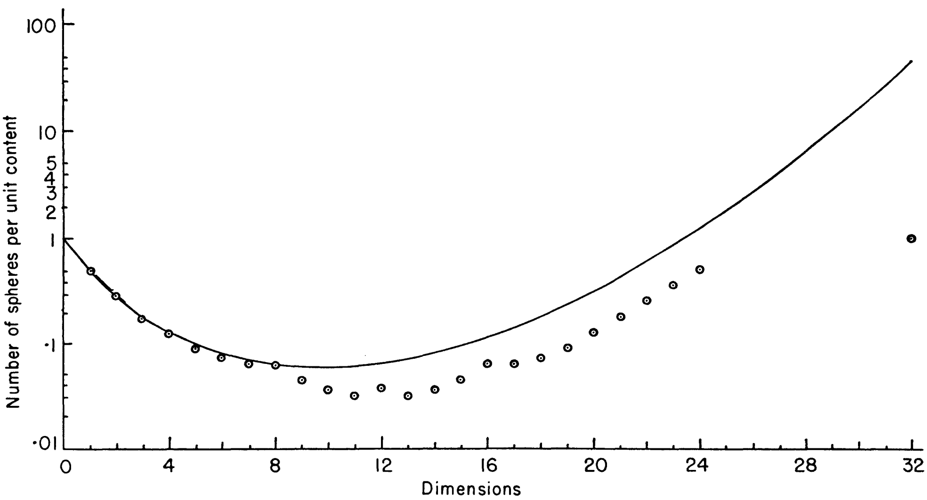

FIGURE 1

The remarkable closeness of the density of the densest lattice packing in [8] to its bound will be noted. To quote Coxeter's remark **(4)** in the context of comparing the number of spheres touched with its bound (cf. §3.3), this "seems to be a manifestation of the extraordinary 'near-regularity' of the honeycomb 5_{21}, whose vertices are the centres in the lattice packing. Any simplicial cell of the honeycomb indicates a set of nine spheres, perfectly packed. Of every 137 cells, 128 are simplexes and only 9 are cross polytopes." This closeness defeats representation on Figure 1, where the point appears to be on the curve instead of microscopically below it.

A feature of the densities of packings derived from that in [24] is worthy of notice. It is apparent from Figure 1 that the centre densities of these packings form a pattern with approximate symmetry about $n = 12$. If we ignore the known denser packings in [11] and [12], and consider only those derived from the packing in [24], we get the second column of numbers in Table II, where the tabulated quantity is $-\log_2$(centre density), and $c = \log_2 \sqrt{3} = 0.792\ldots$.

From this it will be seen that the symmetry is exact for $3 \leqslant n \leqslant 21$. Noting that the general trend of points in Figure 1 is parabolic (on the logarithmic scale used), we add to each entry in the second column of Table II the quantity $2^{-5}(n-12)^2 - 4\frac{1}{2}$, obtaining the entries in the third column. These are seen to follow a precisely regular symmetrical periodic pattern for $n \leqslant 21$. This is presumably related to the regularity in the sections used in §2.4 to obtain these packings.

Before analogy is pressed too far, it should be remarked that continuation of this regular pattern would lead one to conjecture the existence of packings in [28] upwards with a density greater than Rogers' upper bound. For example in [32] the packing of §1.6 is less dense than Rogers' bound by a factor of about 46, but it is less dense than one would conjecture by continuing the regular pattern of densities by a factor of $2^{7\frac{1}{2}} \doteqdot 181$. Thus, the departure from the regular pattern for our packings for $n \geqslant 22$ cannot be made a serious ground for conjecture that denser packings remain to be found in accordance with the pattern.

3.2. Limits for the density. The asymptotic formula quoted in §3.1 for $f_n(n)$ leads to the asymptotic formula

$$2^{-3n/2}(n+1)^{\frac{1}{2}}(n!)^{2}f_n(n)\,J_n \sim \frac{(n+1)!\,e^{\frac{1}{2}n-1}}{2^{n+\frac{1}{2}}n^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+1)} \sim \frac{n}{2^{\frac{1}{2}n}e}$$

for the density, using Stirling's formula for the factorial and gamma function, as given by Rogers **(10)**.

The densest packings we have found for large values of n are those for $n = 2^m$ (§1.6). To evaluate the density of these packings, we count the proportion of the integer points which are accepted after the various parity conditions have been imposed on their co-ordinates, and scale the results corresponding to the actual size of the spheres of the packings. We use the first set of co-ordinates, all odd or all even; the other set gives the same result.

Of all the integer points, we accept a proportion

$$\exp_2 - \left(m + \binom{m}{2} + \ldots + \binom{m}{m}\right)$$

because their ones digits have to be all 0 or all 1; i.e. two out of

$$\exp_2\left(1 + m + \binom{m}{2} + \ldots + \binom{m}{m}\right) = \exp_2 2^m.$$

Of these we accept a proportion

because their twos digits have to be in one of the

$$\exp_2\left(1 + m + \binom{m}{2}\right)$$

https://doi.org/10.4153/CJM-1964-065-1 Published online by Cambridge University Press

arrangements with $(m-2)$-parity; of these we accept a proportion

$$\exp_2 - \left(\binom{m}{5} + \binom{m}{6} + \ldots + \binom{m}{m}\right)$$

because their fours digits have to be in one of the

$$\exp_2\left(1 + m + \binom{m}{2} + \binom{m}{3} + \binom{m}{4}\right)$$

arrangements having $(m-4)$-parity; and so on. The final proportion of integer points which are accepted as centres of the packing is therefore

$$\begin{aligned}\exp_2 - \Bigg(m + \binom{m}{2} + \binom{m}{3} + \binom{m}{4} + \binom{m}{5} + \binom{m}{6} + \ldots + \binom{m}{m}& \\ + \binom{m}{3} + \binom{m}{4} + \binom{m}{5} + \binom{m}{6} + \ldots + \binom{m}{m}& \\ + \binom{m}{5} + \binom{m}{6} + \ldots + \binom{m}{m}& \\ + \ldots\Bigg)&\end{aligned}$$

$$= \exp_2 - \left(m + \binom{m}{2} + 2\binom{m}{3} + 2\binom{m}{4} + 3\binom{m}{5} + \ldots\right)$$

$$\begin{aligned}= \exp_2 - \Bigg(\tfrac{1}{2}m + \binom{m}{2} + 1\tfrac{1}{2}\binom{m}{3} + 2\binom{m}{4} + 2\tfrac{1}{2}\binom{m}{5} + \ldots + \tfrac{1}{2}m\binom{m}{m}& \\ + \tfrac{1}{2}m \qquad + \tfrac{1}{2}\binom{m}{3} \qquad + \tfrac{1}{2}\binom{m}{5} + \ldots\Bigg)&\end{aligned}$$

$$= \exp_2 - (m.2^{m-2} + 2^{m-2})$$

$$= \exp_2(-\tfrac{1}{4}.2^m(m+1)).$$

The radius of the spheres of this packing is $2^{\frac{1}{2}m-1}$; hence the number of centres of unit spheres per unit content of the space, i.e. the centre density, is

$$\begin{aligned}(2^{\frac{1}{2}m-1})^{2^m} \;.\; &\exp_2(-\tfrac{1}{4}.2^m(m+1)) \\ &= \exp_2(2^m(\tfrac{1}{2}m - 1) - 2^m.\tfrac{1}{4}(m+1)) \\ &= \exp_2(\tfrac{1}{4}.2^m(m-5)).\end{aligned}$$

This formula agrees with our figures in Table II for $m = 2, 3, 4$. Replacing 2^m by n, we get

$$2^{-5n/4}\, n^{n/4}$$

for those values of n which are powers of 2. Thus the density of the packing is

$$\begin{aligned}2^{-5n/4}\, n^{n/4}\, J_n &= 2^{-5n/4}\, n^{n/4}\, \pi^{n/2}/\Gamma(\tfrac{1}{2}n + 1) \\ &\sim n^{-n/4}(\pi n)^{-\frac{1}{2}}\, (2\sqrt{2}/\pi e)^{n/2}.\end{aligned}$$

https://doi.org/10.4153/CJM-1964-065-1 Published online by Cambridge University Press

This is of a much smaller order of magnitude than the limit $n\, e^{-1}\, 2^{-\frac{1}{2}n}$ for Rogers' upper bound, because of the overriding factor $n^{-n/4}$. Thus, whereas our packings for $n \leqslant 24$ are of a density smaller than Rogers' bound by a factor not exceeding 3, this factor is about 46 for our packing in [32], and there seems to be no reason to suppose that these packings are the densest in $[2^m]$ for any $m \geqslant 5$.

3.3. Coxeter's bound for the number of spheres which one may touch. Coxeter **(4)** has discussed the number of spheres which one of a packing may touch. In terms of Schläfli's function as modified to $f_n(x)$ (§3.1), he suggests that an upper bound for this number of spheres is $2f_{n-1}(n)/f_n(n)$. I have calculated this quantity for $n \leqslant 8$, as given in **(4)**, and have extended the calculation to $n = 10$ by integration and to $n = 12$ by extrapolation. The values are given in Table I, including for convenience those already published in **(4)**, in comparison with the greatest numbers of spheres touched in any known packing. In Figure 2 the curve is drawn through these values of $2f_{n-1}(n)/f_n(n)$, extended by an empirical formula of the form

$$2^{\frac{1}{2}(n-1)}\, e^{-1}\, \pi^{\frac{1}{2}}\, n^{3/2}\, (1 + an^{-1} + bn^{-2} + cn^{-3} + \ldots),$$

where the leading term is the known asymptotic expression for $2f_{n-1}(n)/f_n(n)$ **(4)**, and the coefficients were determined by fitting to the values for $n \leqslant 12$. I do not give details, as the accuracy is somewhat uncertain, but the accuracy should be at least that to which the curve is drawn.

The points in Figure 2 give the greatest numbers of spheres touched in any of the packings mentioned in this paper. As in Figure 1, the point for [8] is below the curve by an amount too small to show. For $n \geqslant 11$ we have no packing in which the number of spheres touched is within a factor 2 of the upper bound given by Coxeter, so one's interest in exact values is diminished. Exact calculation is increasingly difficult, and I have carried it out with sufficient precision to give the bound to the nearest integer only for $n \leqslant 12$. As with the density, it will be seen that the number of spheres touched in the packings in [32] is relatively greatly inferior to the numbers for $n \leqslant 24$, and there is no reason to suppose that this is the best possible packing in [32].

3.4. Limits for the number of spheres touched. Coxeter **(4)** has shown that, as $n \to \infty$,

$$2f_{n-1}(n)/f_n(n) \sim 2^{\frac{1}{2}(n-1)}\, e^{-1}\, \pi^{\frac{1}{2}}\, n^{3/2}.$$

The best we have found for unlimited values of n is for $n = 2^m$, where the number of spheres touched is (§1.7)

$$(2 + 2)(2 + 2^2)(2 + 2^3)\ldots(2 + 2^m)$$
$$= 2^{\frac{1}{2}m(m+1)}\,(1 + 1)(1 + 2^{-1})(1 + 2^{-2})\ldots(1 + 2^{-m}).$$

https://doi.org/10.4153/CJM-1964-065-1 Published online by Cambridge University Press

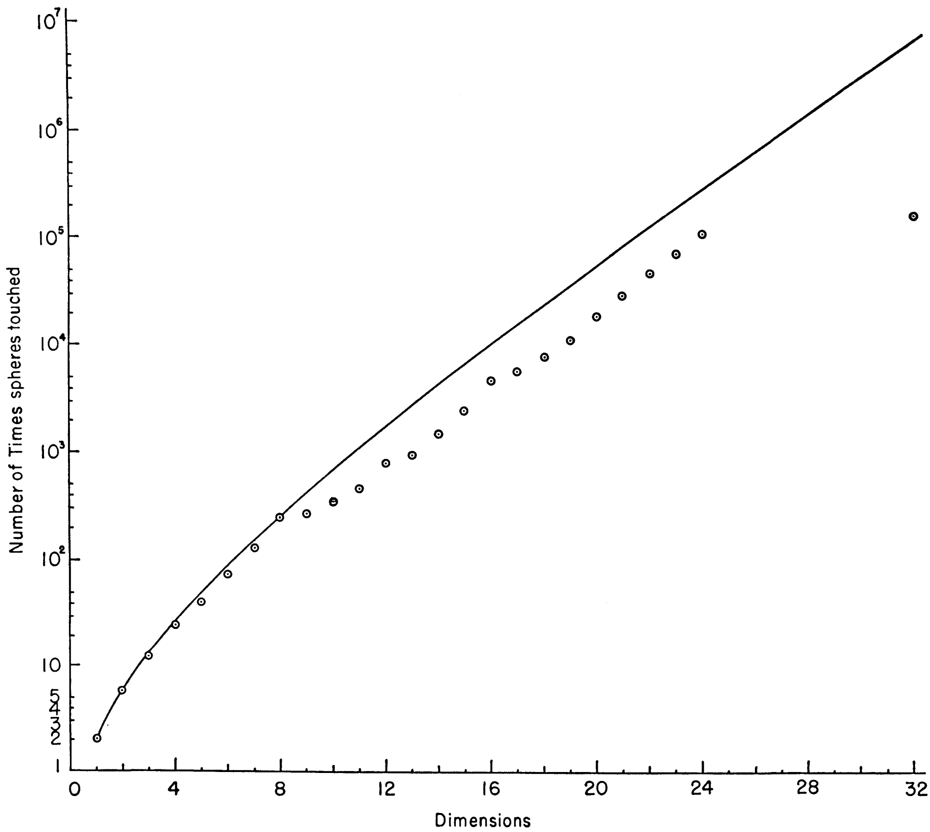

FIGURE 2

The limit as $m \to \infty$ of

$$(1 + 1)(1 + 2^{-1})(1 + 2^{-2}) \ldots (1 + 2^{-m})$$

does not seem to have been given explicitly or numerically, but the product is obviously rapidly convergent and I find the limit to be

$$l = 4.768462 \ldots .$$

Thus the asymptotic form for the numbers of spheres touched is

$$l.2^{\frac{1}{2}m(m+1)}.$$

Changing the variable to $n = 2^m$, we express this as

$$l.n^{\frac{1}{2}(\log_2 n+1)}$$

for those values of n which are powers of 2. For intermediate values of n, we

https://doi.org/10.4153/CJM-1964-065-1 Published online by Cambridge University Press

can claim only the figure for the next smaller power of 2, which leads in the worst case to

$$l.n^{\frac{1}{2}(\log_2 n-1)}.$$

These expressions are of a much smaller order of magnitude than Coxeter's limit

$$2^{\frac{1}{2}(n-1)}\, e^{-1}\, \pi^{\frac{1}{2}}\, n^{3/2},$$

and as with the density we cannot suppose that these packings in $[2^m]$ for $m \geqslant 5$ give the maximum numbers of spheres touched.

NOTES ADDED NOVEMBER 2, 1963.

1. My attention has been drawn to a paper by Rogers **(11)** from which it may be deduced that the asymptotic series for $f_n(n)$ (§3.1) begins

$$1 + \frac{31}{12} n^{-1} + \ldots,$$

while that for $2f_{n-1}(n)/f_n(n)$ (§3.3) begins

$$1 + \frac{21}{4} n^{-1} + \ldots.$$

These differ only slightly from my empirical values, and lead to no perceptible change in the graphs.

2. Leo Moser, in a letter of February 1959 to Coxeter, has given independently a set of co-ordinates for the lattice packing in [15]. In the terminology of this paper, the cycle of 15 binary digits

$$1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0,$$

given by Perron **(9)**, can be used to form an A_{16} by cyclic permutation exactly analogously to our use of the cycle of quadratic residues of 11 to construct A_{12} (§2.1). This can be used similarly to give co-ordinates for a packing in [16]. Actually Moser gave only the packing in [15] as obtained by equating the first co-ordinate to 0. This cycle shares with the quadratic residue cycles for 3 and 7 the property that if it is added (mod 2) to any cyclic permutation of itself, the result is another cyclic permutation of itself. The corresponding matrices thus give us the lattice packings in [4], [8], [16]. But, for example, matrices based on the quadratic residue cycles for 11 or 31 do not give us lattice packings in [12] or [32].

3. W. W. Peterson's book *Error-Correcting Codes* (Cambridge, Mass., 1961) has just come to my notice. I find from this that the material in §§1.3, 1.4 has been anticipated (Reed-Muller codes, pp. 73–77). Another derivation of a matrix equivalent to the construction of §2.3 is also given (p. 140), but this does not seem to lead to a proof of its properties as simple as the present proof. See the quoted book for full references to original sources.

References

1. H. F. Blichfeldt, *The minimum values of quadratic forms and the closest packing of spheres,* Math. Ann., *101* (1929), 605–608.

2. ——— *The minimum values of positive quadratic forms in six, seven and eight variables,* Math. Z., *39* (1935), 1–15.

3. H. S. M. Coxeter, *Twelve points in* PG(5, 3) *with* 95040 *self-transformations,* Proc. Roy. Soc. (London), A, *247* (1958), 279–293.

4. ——— *An upper bound for the number of equal non-overlapping spheres that can touch another of the same size,* Proc. Symposia Pure Math., vol. VII (Providence, 1963), 53–71.

5. H. S. M. Coxeter and J. A. Todd, *An extreme duodenary form,* Can. J. Math., *5* (1953), 384–392.

6. M. J. E. Golay, *Notes on digital coding,* Proc. Inst. Radio Engrs., *37* (1949), 657.

7. L. J. Paige, *A note on the Mathieu groups,* Can. J. Math. *9* (1957), 15–18.

8. R. E. A. C. Paley, *On orthogonal matrices,* J. Math. & Phys., *12* (1933), 311–320.

9. O. Perron, *Bemerkungen über die Verteilung der quadratischen Reste,* Math. Z., *56* (1952), 122–130.

10. C. A. Rogers, *The packing of equal spheres,* Proc. London Math. Soc. (3), *8* (1958), 609–620.

11. ——— *An asymptotic expansion for certain Schläfli functions,* J. London Math. Soc., *36* (1961), 78–80.

12. J. J. Sylvester, *Thoughts on inverse orthogonal matrices . . . ,* Phil. Mag. (4), *34* (1867), 461–475.

13. J. A. Todd, *On representations of the Mathieu groups as collineation groups,* J. London Math. Soc., *34* (1959), 406–416.

Glasgow University

https://doi.org/10.4153/CJM-1964-065-1 Published online by Cambridge University Press