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THE EXTENDED CENTER OF COPRODUCTS

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ABSTRACT. The sole purpose of this paper is to prove that if R_1 and R_2 are algebras with 1 over a common field F, with each $(R_i:F)>1$ and at least one $(R_i:F)>2$, then the extended center of the coproduct $R_1 \amalg R_2$ is equal to F.

Let R_1 and R_2 be arbitrary algebras with 1 over a common field F, with each $(R_i:F)>1$, and let $R=R_1 \amalg R_2$ denote the coproduct of R_1 and R_2 over F. We fix an F-basis $\{x_i\}\cup 1$ for R_1 and an F-basis $\{y_j\}\cup 1$ for R_2. The various finite products of x_i's and y_j's, in which the x_i's and y_j's alternate, together with 1, will be called monomials; the degree of a monomial is the total number of x_i's and y_j's in it (thus deg $1=0$). The set $\{U_k\}$ of all monomials forms an F-basis for R. For $f=\sum \beta_k U_k \in R$, $\beta_k \in F$, the support of f consists of all U_k's such that $\beta_k \neq 0$. The degree of f is the highest degree of any monomial in its support. Every $f\neq 0$ may be written uniquely as the sum of its homogeneous components $f=f_n+f_{n-1}+\cdots+f_0$, $f_n\neq 0$, where all monomials in the support of f_i have degree i. We shall generally denote the highest degree component of f by $\hat{f}$ (rather than by f_n). For $0\neq n$ even we use the suggestive notation $\hat{f}=f_{xy}+f_{yx}$, where the monomials in the support of f_{xy} (resp. f_{yx}) begin with an x_i, (resp. y_j) and end with a y_j (resp. x_i). For n odd we write $\hat{f}=f_x+f_y$, where the monomials in the support of f_x (resp. f_y) begin and end with an x_i (resp. y_j). An element $f\neq 0$ of degree $n>0$ is called pure if for n even $\hat{f}=f_{xy}$ or $\hat{f}=fgy_x$ and for n odd $\hat{f}=f_x$ or $\hat{f}=f_y$. A pair (f, g) is called pure interacting if f and g are pure and the monomials of $\hat{f}$ end in the same letter in which the monomials of $\hat{g}$ begin. It is clear that for $f, g\neq 0$, $\deg(fg)=\deg f+\deg g$ unless (f, g) is a pure interacting pair. It follows in particular that R is always a prime ring (if $fg=0$ for nonzero f and g then $fzg\neq 0$ for z approximately chosen as x_1 or y_1). For prime rings R in general the notion of the extended center C was introduced in [3]. We refer the reader to that source for the basic construction and properties, emphasizing the key property that for $\lambda \in C$ there exists a nonzero ideal I_λ of R such that $\lambda a \in R$ for all $a \in I_\lambda$. It is well known that C is a field containing the ordinary center of R.

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It is our purpose in the present note to show that if at least one of $(R_i:F)>2$ then the extended center C of R is F itself. Aside from the feeling that it should be potentially useful to know precisely what the extended center is for certain important classes of prime rings, our result here is specifically needed in a joint paper with Susan Montgomery [4], in which we determine the so-called normal closure of coproducts of domains. First, however, we mention that the determination of C for the case where $(R_1:F)=(R_2:F)=2$ follows immediately from [1], p. 26, example 12.2. Here one writes $R_1=F[x]$, $x^2=\alpha x+\gamma$, $R_2=F[y]$, $y^2=\beta y+\delta$, and shows that the center of R is the polynomial ring $F[t]$, where $t=xy+yx-\alpha y-\beta x$, and that $1, x, y, xy$ form an $F[t]$-basis for R. R is thus a prime PI-ring and so by the sharper version of Posner's Theorem the extended center is simply the field of fractions $F(t)$ of $F[t]$. We also mention in passing that in this case R is definitely not a primitive ring, whereas in case one of $(R_i:F)>2$ we have Lichtmann's very pretty result [2] that R is always primitive.

Henceforward we assume at least one of $(R_i:F)>2$, which assumption is needed to establish

LEMMA 1. *Let g and h be nonzero elements of R such that*

(a) *$gfh=hfg$ for all $f\in R$.*

Then $\deg g=\deg h$.

Proof. We set $n=\deg g$, $m=\deg h$, and first show that without loss of generality we may assume that n is odd and $m>0$. Indeed, we observe that property (a) is preserved on replacing g, h by ug, uh for any u. Thus if n is odd but $m=0$ we use $u=x_1y_1+y_1x_1$ whence by assumption $n+2=\deg(ug)=\deg(uh)=m+2$. In case n is even we use $u=x_1+y_1$, whence by assumption $n+1=\deg(ug)=\deg(uh)=m+1$. We next show that m must be odd. Indeed, if $g_x\neq 0$ we take $f=y_j$. Then the $(\ )_{xy}$ and $(\ )_{yx}$ terms of (a) respectively give $g_xy_jh_{xy}=0$ and $0=h_{yx}y_jg_x$; and so $h_{xy}=h_{yx}=0$, a contradiction to m even. Therefore we may assume both n and m are odd and, for sake of argument, $n\geq m+2$, with $g_x\neq 0$. If $h_x\neq 0$ we replace f by $y_jx_iy_j$ in (a) to obtain $g_xy_jx_iy_jh_x=h_xy_jx_iy_jg_x$. It follows that every monomial in the support of g_x has y_j in the $m+1$st position and x_i in the $m+2$nd position. But as $(R_1:F)>2$ or $(R_2:F)>2$ we can get the same result with $x_{i'}$ in place of x_i $(i\neq i')$ or $y_{j'}$ in place of y_j $(j\neq j')$, a contradiction. If $h_x=0$ then $h_y\neq 0$ which results in the contradiction $g_xh_y=0$.

LEMMA 2. *Let g and h be pure and homogeneous of degree n, let s and t be pure and homogeneous of degree m, and suppose (g,s) and (h,t) are both noninteracting pairs such that $gs=ht$. Then for some $\alpha\in F$ $h=\alpha g$ and $s=\alpha t$.*

Proof. Writing $g=\sum\alpha_iU_i$, $s=\sum\beta_jV_j$, $h=\sum\gamma_kW_k$, $t=\sum\delta_lZ_l$ we have the equation

$$\sum_{i,j} \alpha_i\beta_j U_i V_j = \sum_{k,l} \gamma_k\delta_l W_k Z_l,$$

in which $\{U_iV_j\}$ are distinct monomials of degree $n+m$ and likewise $\{W_kZ_l\}$ are distinct monomials of degree $n+m$. This forces, via a suitable reordering, $U_i = W_i$ and $V_j = Z_j$, and accordingly we may write $h = \sum \gamma_i U_i$ and $t = \sum \delta_j V_j$. Therefore from $\sum \alpha_i\beta_j U_i V_j = \sum \gamma_i\delta_j U_i V_j$ we have $\alpha_i\beta_j = \gamma_i\delta_j$ for all i, j. In other words

$$\alpha = \frac{\gamma_i}{\alpha_i} = \frac{\beta_j}{\delta_j}$$

for all i, j, whence α is a constant, with $h = \alpha g$ and $s = \alpha t$.

We are now in a position to fulfill the purpose of this note.

THEOREM. *Let R_1 and R_2 be algebras with* 1 *over a common field F, with each $(R_i : F) > 1$ and at least one $(R_i : F) > 2$. Then the extended center C of $R = R_1 \coprod R_2$ is equal to F.*

Proof. We remark that the desired conclusion $C = F$ is equivalent to the assertion that any pair of nonzero elements g, h satisfying property (a), namely,

(a) $gfh = hfg$ for all $f \in R$,

must be F-dependent. Let g, h be such a pair. By Lemma 1 we have $n = \deg g = \deg h$. We claim that without loss of generality n is odd. Indeed, if n is even we set $u = x_1 + y_1$ and note that ug, uh are of degree $n+1$ and also satisfy (a). By assumption we then have $uh = \alpha ug$ for some $\alpha \in F$, whence $u(h - \alpha g) = 0$ from which one concludes $h = \alpha g$. Our claim is thereby established and so we may assume now that n is odd. We next aim to show that there is $\alpha \in F$ such that $\hat{h} = \alpha\hat{g}$. Indeed, we may assume $g_x \neq 0$ which forces $h_x \neq 0$ (since $h_x = 0$, with $f = 1$ in (a), would force the contradiction $g_x h_y = 0$). We set $f = y_1$ in (a) to obtain $(g_x y_1)h_x = (h_x y_1)g_x$. Now by Lemma 2 there is $\alpha \in F$ such that $h_x = \alpha g_x$. If $g_y = 0$ (and hence $h_y = 0$) we are done. If $g_y \neq 0$ then the preceding argument shows there is $\beta \in F$ such that $h_y = \beta g_y$. On the other hand, setting $f = 1$ in (a) we get $g_x h_y = h_x g_y$, whence $g_x(\beta g_y) = (\alpha g_x)g_y$. It follows that $\alpha = \beta$ and so $\hat{h} = \alpha\hat{g}$. To complete the proof of the theorem we note $h - \alpha g$, if not already zero, has smaller degree than g. But it is also clear that the pair g, $h - \alpha g$ again satisfies (a), thus giving a contradiction to Lemma 1.

COROLLARY. *Let $\{R_\alpha \mid \alpha \in A, (R_\alpha : F) > 1, |A| > 1\}$ be a collection of algebras with* 1 *over a common field F, with either some $(R_\alpha : F) > 2$ or $|A| > 2$, and let $R = \coprod_\alpha R_\alpha$. Then the extended center of R is equal to F.*

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