J. Austral. Math. Soc. (Series A) **66** (1999), 90–103

DIRECT AND CONVERSE INEQUALITIES FOR POSITIVE LINEAR OPERATORS ON THE POSITIVE SEMI-AXIS

JOSÉ A. ADELL and CARMEN SANGÜESA

(Received 3 April 1997; revised 4 September 1998)

Communicated by R. A. Maller

Abstract

We consider positive linear operators of probabilistic type L_tf acting on real functions f defined on the positive semi-axis. We deal with the problem of uniform convergence of L_tf to f, both in the usual sup-norm and in a uniform L_p type of norm. In both cases, we obtain direct and converse inequalities in terms of a suitable weighted first modulus of smoothness of f. These results are applied to the Baskakov operator and to a gamma operator connected with real Laplace transforms, Poisson mixtures and Weyl fractional derivatives of Laplace transforms.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 44A10, 41A35, 60J30.
Keywords and phrases: positive linear operator, direct and converse inequalities, weighted modulus of smoothness, Baskakov operator, gamma operator, Laplace transform, Poison mixture, Weyl fractional derivative.

1. Introduction

Let $Z := (Z_t(x),\ t > 0,\ x \geq 0)$ be a double-indexed family of nonnegative random variables such that, for each $x \geq 0$, $Z_t(x)/t$ converges weakly to x, as $t \to \infty$. We consider the family $\mathbb{L} := (L_t,\ t > 0)$ of positive linear operators of the form

$$L_tf\,(x) := Ef\left(\frac{Z_t(x)}{t}\right), \qquad t > 0, \quad x \geq 0, \tag{1.1}$$

where f is any real measurable function defined on $[0, \infty)$ for which the right-hand side in (1.1) makes sense. Classical examples of operators of the form (1.1) can be found in the book by Ditzian and Totik [8] (see also [3, 6, 9]).

Research supported by the Grant PB95-0809 of the Spanish DGICYT.

Let $(S(u),\ u \geq 0)$ be a gamma process, that is, a process starting at the origin, having independent stationary increments and such that, for each $u > 0$, $S(u)$ has the gamma density

$$d(u,\theta) := \frac{1}{\Gamma(u)}\theta^{u-1}e^{-\theta}, \quad \theta > 0.$$

In this paper, we introduce two families $\mathbb{G}$ and $\mathbb{G}^*$ of gamma-type operators having the form (1.1) and given, respectively, by

$$\begin{aligned} G_t f(x) :&= Ef\left(\frac{S(tx+1)}{t}\right) \\ &= \frac{1}{\Gamma(tx+1)}\int_0^\infty f\left(\frac{\theta}{t}\right)\theta^{tx}e^{-\theta}\,d\theta, \qquad t > 0, \quad x \geq 0 \end{aligned} \tag{1.2}$$

and

$$G_t^* f(x) := G_t f\left(\frac{[tx]}{t}\right), \quad t > 0, \quad x \geq 0, \tag{1.3}$$

where $[\cdot]$ stands for integral part. These operators differ from other gamma-type operators considered in the literature, such as the operator introduced by Lupas and Müller [11] and investigated in subsequent papers (see, for instance, [10, 14]) and the operator introduced by Khan [9] (see also [3]).

We shall firstly mention some examples in which both operators arise in a natural way (we refer to [1, 2] for more details).

(a) Real Laplace transforms of signed measures. Let μ be a signed measure concentrated on $[0,\infty)$. Denote by $F(x) := \mu([0,x])$, $x \geq 0$, and assume that $|F|(x) = O(e^{\gamma x})$ as $x \to \infty$, for some $\gamma \geq 0$, where $|F|$ stands for the total variation of F. Then the real Laplace transform of μ, that is,

$$\phi(t) := \int_0^\infty e^{-t\theta}\,dF(\theta)$$

is well-defined for $t > \gamma$ and we have the well-known inversion formula (see [15])

$$\lim_{t\to\infty}\sum_{k\leq tx}\frac{(-t)^k}{k!}\phi^{(k)}(t) = F(x), \quad \text{for every continuity point } x \text{ of } F.$$

A simple integration by parts gives us for any $k = 1, 2, \ldots$ and $t > \gamma$

$$G_t F\left(\frac{k}{t}\right) - G_t F\left(\frac{k-1}{t}\right) = \frac{t^k}{k!}\int_0^\infty \theta^k e^{-t\theta}\,dF(\theta) = \frac{(-t)^k}{k!}\phi^{(k)}(t)$$

and, therefore,

$$G_t^* F(x) = \sum_{k\leq tx}\frac{(-t)^k}{k!}\phi^{(k)}(t), \qquad x \geq 0.$$

(b) Normalized Poisson mixtures. Mixtures of probability distributions play an important role in applied probability (see, for instance, [2, 16] and the references therein). Here, we consider the following example: Let T be a nonnegative random variable with probability distribution function $F(x) := P(T \le x)$, $x \ge 0$, and let $(N(t),\ t \ge 0)$ be a standard Poisson process independent of T. The random variable $t^{-1}N(tT)$ is called a normalized Poisson mixture with mixing distribution T. Using the well-known formula

$$P(N(t) \le n) = \frac{1}{\Gamma(n+1)} \int_t^\infty \theta^n e^{-\theta}\, d\theta, \qquad t \ge 0, \quad n = 0, 1, 2, \ldots$$

and then integrating by parts, we obtain for any $t > 0$ and $x \ge 0$

$$\begin{aligned} P\left(\frac{N(tT)}{t} \le x\right) &= \int_0^\infty P(N(t\theta) \le [tx]\, dF(\theta) \\ &= \frac{1}{\Gamma([tx]+1)} \int_0^\infty F\left(\frac{\theta}{t}\right) \theta^{[tx]} e^{-\theta}\, d\theta \\ &= G_t^* F(x). \end{aligned}$$

(c) Weyl fractional derivatives of real Laplace transforms. Let $f \in C[0, \infty)$ be such that $|f(x)| = O(e^{\gamma x})$, as $x \to \infty$, for some $\gamma \ge 0$. Consider the real Laplace transform of f given by

$$\mathscr{L} f(t) := \int_0^\infty e^{-t\theta} f(\theta)\, d\theta, \qquad t > \gamma \tag{1.4}$$

and denote by D the differential operator $D := -d/dt$. Differentiating under the integral sign in (1.4), we obtain

$$G_t^* f(x) = \frac{t^{[tx]+1}}{\Gamma([tx]+1)} D^{[tx]} \mathscr{L} f(t), \qquad x \ge 0, \quad t > \gamma. \tag{1.5}$$

If we replace usual derivatives by fractional derivatives on the right-hand side in (1.5), the resulting expression is representable by the operator G_t. To see this, let $g \in C(a, \infty)$ with $a \ge 0$. Let $\tau > 0$ and denote by $\nu = [\tau] + 1 - \tau$. Recall that (see [13, Chapter VII]) the Weyl fractional integral of g of order ν is defined by

$$W^{-\nu} g(t) := \frac{1}{\Gamma(\nu)} \int_0^\infty \theta^{\nu-1} g(t+\theta)\, d\theta, \qquad t > a,$$

whenever this integral exists. If we assume, further, that such integral has $[\tau] + 1$ continuous derivatives, then the Weyl fractional derivative of g of order τ is defined by

$$W^\tau g(t) := D^{[\tau]+1} W^{-\nu} g(t), \qquad t > a.$$

In the case at hand, it is not hard to see that for all $0 < \nu < 1$ (or, equivalently, if τ is not a positive integer) the Weyl fractional integral $W^{-\nu}\mathscr{L}f\,(t)$ is well-defined for all $t > \gamma$. Moreover, $W^{-\nu}\mathscr{L}f\,(t)$ has $[\tau]+1$ continuous derivatives and we have

$$\begin{aligned} W^{\tau}\mathscr{L}f\,(t) &= D^{[\tau]+1}W^{-\nu}\mathscr{L}f\,(t) \\ &= W^{-\nu}D^{[\tau]+1}\mathscr{L}f\,(t) \\ &= \frac{\Gamma(\tau+1)}{t^{\tau+1}}G_tf\left(\frac{\tau}{t}\right), \qquad t > \gamma, \end{aligned}$$

as it follows from Fubini's theorem. Thus, if tx is not a positive integer, we have

$$G_tf\,(x) = \frac{t^{tx+1}}{\Gamma(tx+1)}W^{tx}\mathscr{L}f\,(t), \qquad x > 0, \quad t > \gamma. \tag{1.6}$$

We therefore conclude from (1.5) and (1.6) that usual derivatives and Weyl fractional derivatives of real Laplace transforms can be represented in a unified way by means of the operator G_t.

In the setting of (1.1), the main purpose of this paper is to investigate the problem of uniform convergence of L_tf to f, as measured by

$$\|L_tf - f\| := \sup_{x\geq 0}|L_tf\,(x) - f\,(x)|$$

and

$$N_p(f\,;t) := \sup_{x\geq 0} E^{1/p}\left|f\left(\frac{Z_t(x)}{t}\right) - f\,(x)\right|^p, \qquad p \geq 1. \tag{1.7}$$

Rates of convergence are given in terms of suitable weighted first moduli of smoothness of f which satisfy the subadditivity property (see Section 2). The main results are stated in Section 3. We firstly obtain direct inequalities (upper bounds) under certain integrability assumptions on Z (Theorem 1). Converse inequalities (lower bounds) are given in Theorems 2 and 3. With respect to the L_p type of norm in (1.7), our results apply to the largest possible set of continuous functions, whilst, with respect to the sup-norm, they apply to a smaller set of functions. In any case, to obtain converse inequalities, we need to make some smoothness assumptions on the marginal distributions of Z which imply, in particular, that L_tf is differentiable for any bounded function f. Obviously, this condition is not satisfied by the operator G_t^* defined in (1.3). However, since G_t^* is asymptotically close to G_t, G_t^* inherits from G_t its approximation properties (Theorem 4 in Section 4). Finally, the last section is devoted to illustrating the preceding results by considering the aforementioned gamma operators and the Baskakov operator.

2. Weight functions

For any real numbers x and y, denote by $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$ and $x_+ = x \vee 0$. A nondecreasing function φ defined on $[0, \infty)$ is called a *weight function* if $\varphi(x) > 0$, whenever $x > 0$. The *first modulus of smoothness* of $f \in C[0, \infty)$ with step-weight function φ is defined by

$$\omega(f; \varphi; \delta) := \sup\{|f(x + \varphi(x)h) - f(x)| : x \geq 0,\ 0 \leq h \leq \delta\}, \qquad \delta \geq 0.$$

These definitions are more restrictive than those considered in [7, 8]. However, they guarantee the subadditivity of $\omega(f; \varphi; \cdot)$, as it is stated in the following

LEMMA 1. *Let $f \in C[0, \infty)$ and let φ and ψ be weight functions. Then*

(a) *$\omega(f; \varphi; \cdot)$ is subadditive. Therefore, if $\varphi(0) > 0$, then*

$$|f(y) - f(x)| \leq \left(1 + \frac{|y - x|}{\delta\varphi(x \wedge y)}\right) \omega(f; \varphi; \delta), \qquad x, y \geq 0, \quad \delta > 0.$$

(b) $\omega(f; \varphi \vee \psi; \cdot) = \omega(f; \varphi; \cdot) \vee \omega(f; \psi; \cdot)$.

Let σ be a weight function. We define the weight functions

$$\sigma_t(x) := \sqrt{t}\sigma(x) \vee 1, \qquad x \geq 0, \quad t > 0 \tag{2.1}$$

and the set

$$\mathcal{M}(\sigma) := \{f \in C[0, \infty) : \omega(f; \sigma \vee 1; 1) < \infty\}.$$

As an immediate consequence of Lemma 1, we have

LEMMA 2. *If $f \in \mathcal{M}(\sigma)$, then*

(a) $|f(x)| \leq C(x \vee 1)$, $\quad x \geq 0$, *for some constant* $C > 0$.

(b) $\omega(f; \sigma_t; \delta) = \omega\left(f; \sigma; \delta\sqrt{t}\right) \vee \omega(f; 1; \delta) < \infty, \qquad \delta > 0, \quad t > 0.$

If σ is a weight function such that $1/\sigma$ is locally integrable at the origin, we define

$$g(x) := \int_0^x \frac{1}{\sigma(\theta)}\, d\theta, \qquad x \geq 0. \tag{2.2}$$

With these notations, we state

LEMMA 3. *For any $\delta > 0$ and $t > 0$, we have*

$$\left(\delta\sqrt{t}\, r(\delta\sqrt{t})\right) \vee g(\delta) \leq \omega(g; \sigma_t; \delta) \leq \delta\sqrt{t} \vee g(\delta),$$

where

$$r(\theta) = \sup_{x \geq 0} \frac{\sigma(x)}{\sigma(x + \sigma(x)\theta)}, \qquad \theta \geq 0.$$

Therefore, $r(\theta) \to 1$ as $\theta \to 0$.

3. Direct and converse inequalities

Let $\mathbb{L}$ be a family of positive linear operators represented by a double-indexed family Z of random variables as in (1.1). For any $p \geq 1$ and $t > 0$, we introduce the constants

(3.1)
$$c_t(p) := \sup_{x\geq 0} E^{1/p}\left(|Z_t(x) - tx|\left[\sigma_t\left(\frac{Z_t(x)}{t}\wedge x\right)\right]^{-1}\right)^p, \quad c(p) := \sup_{t\geq 1} c_t(p),$$

where $\sigma_t(x)$ is defined in (2.1). Observe that, if $c_t(p)$ is finite, then

$$\sup_{x\geq 0} E^{1/p}\left(\frac{|Z_t(x) - tx|}{\sigma_t(x)}\right)^p < \infty.$$

This, together with Lemma 2(a), implies that, for any $f \in \mathcal{M}(\sigma)$, $L_t|f|(x) < \infty$, $x \geq 0$.

Recalling the notations in (1.7), we state

THEOREM 1. (Direct inequalities). *Let* $f \in \mathcal{M}(\sigma)$, $p \geq 1$ *and* $t > 0$.

(a) *If* $c_t(1) < \infty$, *then*

$$\|L_t f - f\| \leq (1 + c_t(1))\omega\left(f; \sigma_t; \frac{1}{t}\right).$$

(b) *If* $c_t(p) < \infty$, *then*

$$N_p(f; t) \leq 2^{(p-1)/p}(1 + c_t(p))\omega\left(f; \sigma_t; \frac{1}{t}\right).$$

PROOF. Let φ be a weight function such that $\varphi(0) > 0$. Let $x \geq 0$ and $\delta > 0$. Applying Lemma 1(a) and the inequality

(3.2) $$(a + b)^q \leq 2^{(q-1)_+}(a^q + b^q), \qquad a, b \geq 0, \quad q > 0,$$

we have for any $p \geq 1$

$$\left|f\left(\frac{Z_t(x)}{t}\right) - f(x)\right|^p \leq 2^{p-1}\left(1 + \left(\frac{|Z_t(x) - tx|}{\delta t}\left[\varphi\left(\frac{Z_t(x)}{t}\wedge x\right)\right]^{-1}\right)^p\right)\omega^p(f; \varphi; \delta).$$

Statements (a) and (b) follow by taking expectations in this last inequality, with $\delta = t^{-1}$ and $\varphi = \sigma_t$ and then applying (3.2). □

REMARK 1. Assume that the constants $c(p)$ defined in (3.1) are finite for all $p \geq 1$. Then, for any $t \geq 1$, we can restate Theorem 1 in such a way that the upper constants only depend upon p.

To show converse inequalities, we need to make additional assumptions on Z and σ. Let $t > 0$ be fixed. Denote by μ the measure on $[0, \infty)$ defined by $d\mu(\theta) := (\theta \vee 1) d\lambda(\theta)$, where λ stands for the Lebesgue measure. Let $B_t \subseteq [0, \infty)$ be a Borel set such that $\lambda([0, \infty) \setminus B_t) = 0$. For any $x > 0$, let I_x be a neighborhood of x and let $h_{t,x}$ be a nonnegative μ-integrable function. We make the following assumption on Z.

(H) For each $x > 0$, the random variable $Z_t(x)$ has a density $d_t(x, \cdot)$ such that for any $\theta \in B_t$, $d_t(x, \theta)$ has a continuous partial derivative $d_t'(x, \theta)$ with respect to x and

$$|d_t'(y, \theta)| \leq h_{t,x}(\theta), \qquad y \in I_x, \quad \theta \in B_t.$$

By Lemma 2(a) and [4, p.215], assumption (H) implies the following. For any $f \in \mathscr{M}(\sigma)$, $L_t f$ has a continuous derivative $L_t' f$ given by

$$L_t' f(x) = \int_0^\infty f\left(\frac{\theta}{t}\right) d_t'(x, \theta)\, d\theta, \qquad x > 0.$$

This formula can be rewritten as

$$L_t' f(x) = E\left(f\left(\frac{Z_t(x)}{t}\right) - f(x)\right) V_t(x), \qquad x > 0, \tag{3.3}$$

where $V_t(x)$ is a zero-mean random variable defined by

$$V_t(x) := \frac{d_t'(x, Z_t(x))}{d_t(x, Z_t(x))}, \qquad x > 0. \tag{3.4}$$

Observe that, for each $x > 0$, the set where the quotient in (3.4) is not defined is a null set with respect to the probability measure of $Z_t(x)$.

From now on, we assume that the weight function σ is such that

$$\liminf_{\theta \to 0} \frac{\sigma(\theta)}{\sqrt{\theta}} = C > 0, \tag{3.5}$$

for some (possibly infinite) constant C. Finally, for any $q \geq 1$ and $t > 0$, we consider the constants

$$k_t(q) := \sup_{x>0} E^{1/q}\left(\frac{\sigma(x)|V_t(x)|}{\sqrt{t}}\right)^q, \qquad k(q) := \sup_{t \geq 1} k_t(q). \tag{3.6}$$

We are in a position to state

THEOREM 2. (Converse inequality). *Let $f \in \mathcal{M}(\sigma)$ and $t > 0$. Assume that Z satisfies* (H) *and that the weight function σ satisfies* (3.5). *Assume, further, that the constants $c_t(p)$ defined in* (3.1) *are all finite, for any $p \geq 1$, and that $k_t(q)$ is finite for some $q > 1$. Then*

$$C_t\, \omega\left(f\,;\sigma_t;\frac{1}{t}\right) \leq N_1(f\,;t) \leq N_p(f\,;t), \qquad p \geq 1,$$

for some positive constant C_t not depending on f.

PROOF. Let $x > 0$, $t > 0$ and $0 \leq h \leq t^{-1}$. By the triangular inequality and (3.3), we have

$$\begin{aligned} &|f\,(x+\sigma_t(x)h) - f\,(x)| \\ &\leq 2N_1(f\,;t) + \int_x^{x+\sigma_t(x)h} E\left|f\left(\frac{Z_t(\theta)}{t}\right) - f\,(\theta)\right| |V_t(\theta)|\, d\theta. \end{aligned} \tag{3.7}$$

Choose $q_* \in (1, q)$ and denote by p_* the conjugate exponent of q_*. Denote by $I(\cdot)$ the indicator function. By Hölder's inequality and (3.6), the integrand in (3.7) is bounded above, for any arbitrary $a > 0$, by

$$\begin{aligned} &\frac{\sqrt{t}}{a\sigma(\theta)}N_1(f\,;t) + N_{p_*}(f\,;t)E^{1/q_*}|V_t(\theta)|^{q_*}I\left(|V_t(\theta)| > \frac{\sqrt{t}}{a\sigma(\theta)}\right) \\ &\leq \frac{\sqrt{t}}{a\sigma(\theta)}N_1(f\,;t) + \frac{\sqrt{t}}{\sigma(\theta)}\left(a^{q-q_*}(k_t(q))^q\right)^{1/q_*} N_{p_*}(f\,;t). \end{aligned} \tag{3.8}$$

Let g be the function defined in (2.2). From (3.7) and (3.8), we have

$$\begin{aligned} &\omega\left(f\,;\sigma_t;\frac{1}{t}\right) \leq 2N_1(f\,;t) \\ &\quad + \left(\frac{N_1(f\,;t)}{a} + \left(a^{q-q_*}(k_t(q))^q\right)^{1/q_*} N_{p_*}(f\,;t)\right)\sqrt{t}\omega\left(g;\sigma_t;\frac{1}{t}\right). \end{aligned} \tag{3.9}$$

By (3.5) and Lemma 3, we see that

$$\sqrt{t}\omega\left(g;\sigma_t;\frac{1}{t}\right) < \infty. \tag{3.10}$$

Therefore, the conclusion follows by applying Theorem 1(b) and choosing a in (3.9) small enough. □

REMARK 2. In the setting of Theorem 2, assume, in addition, that $c(p) < \infty$, for all $p \geq 1$, and that $k(q) < \infty$, for some $q > 1$. Then

$$\sup_{t \geq 1} C_t < \infty,$$

thus obtaining a constant not depending upon t. This follows from (3.9) by observing that $N_1(f;t)$, $N_{p_*}(f;t)$ and the term in (3.10) can be uniformly bounded in $t \geq 1$, using Theorem 1(b) and Lemma 3, respectively.

Let σ be a weight function satisfying (3.5). For any $f \in \mathcal{M}(\sigma)$, we denote by

$$l(f;t) = \sqrt{t}\,\omega\left(f;\sigma_t;\frac{1}{t}\right), \qquad t > 0.$$

Observe that by Lemma 2(b), if f is non-constant, then $\liminf_{t\to\infty} l(f;t) = a \in (0,\infty]$. Therefore, it follows from Theorems 1 and 2 that the best possible rate of convergence of $N_p(f;t)$ is $t^{-1/2}$. On the other hand, for smooth functions $f \in \mathcal{M}(\sigma)$, we cannot expect a converse inequality in the sup-norm similar to that given in Theorem 2. Indeed, taking $f(x) = e^{-x}$, $x \geq 0$, it can be seen in many examples that

$$\omega\left(f;\sigma_t;\frac{1}{t}\right) \approx \frac{1}{\sqrt{t}} \qquad \text{and} \qquad \|L_t f - f\| \approx \frac{1}{t}, \quad \text{as } t \to \infty.$$

For this reason, we restrict the set of functions under consideration in the following sense. For any $f \in \mathcal{M}(\sigma)$, denote by

$$l_*(f;\lambda) = \liminf_{t\to\infty} \frac{l(f;\lambda t)}{l(f;t)}, \qquad \lambda \geq 1.$$

We consider the set of functions

$$\mathcal{M}_*(\sigma) := \left\{ f \in \mathcal{M}(\sigma) : \quad \sup_{\lambda\geq 1} l_*(f;\lambda) = \infty \right\}.$$

Observe that every function $f \in \mathcal{M}(\sigma)$ such that the lower Karamata index of $l(f;\cdot)$ is strictly positive is in $\mathcal{M}_*(\sigma)$ (see Bingham, Goldie and Teugels [5]). However, $\mathcal{M}_*(\sigma)$ does not contain the functions $f \in \mathcal{M}(\sigma)$ for which $\omega(f;\sigma;\delta) \approx \omega(f;1;\delta) \approx \delta$, as $\delta \to 0$.

THEOREM 3. (Converse inequality). *Let $f \in \mathcal{M}_*(\sigma)$. Assume that Z satisfies* (H) *and that σ satisfies* (3.5). *Also, assume that the constants $c(p)$ defined in* (3.1) *are finite, for any $p \geq 1$, and that $k(q)$, as defined in* (3.6), *is finite for some $q > 1$. Then there exist positive constants K and $t_0 \geq 1$, depending on f, such that*

$$K\omega\left(f;\sigma_t;\frac{1}{t}\right) \leq \|L_t f - f\|, \qquad t \geq t_0.$$

PROOF. Let $s \geq t \geq 1$, $x > 0$, $t > 0$ and $0 \leq h \leq s^{-1}$. As in the proof of Theorem 2, we have

$$|f(x+\sigma_s(x)h) - f(x)| \leq 2\|L_t f - f\| + \int_x^{x+\sigma_s(x)h} E\left|f\left(\frac{Z_t(\theta)}{t}\right) - f(\theta)\right| |V_t(\theta)|\, d\theta. \tag{3.11}$$

Let p_0 be the conjugate exponent of q. By Hölder's inequality and Theorem 1(b), we can bound the integrand in (3.11) by

$$2(1+c(p_0))\omega\left(f;\sigma_t;\frac{1}{t}\right)E^{1/q}|V_t(\theta)|^q \leq 2k(q)(1+c(p_0))\omega\left(f;\sigma_t;\frac{1}{t}\right)\frac{\sqrt{t}}{\sigma(\theta)}.$$

Therefore, we obtain from (3.11)

$$\frac{\sqrt{s}\omega(f;\sigma_s;1/s)}{\sqrt{t}\omega(f;\sigma_t;1/t)} \leq 2\sqrt{\frac{s}{t}}\frac{\|L_t f - f\|}{\omega(f;\sigma_t;1/t)} + K_1\sqrt{s}\omega\left(g;\sigma_s;\frac{1}{s}\right), \tag{3.12}$$

where $K_1 = K_1(p_0, q)$ is an absolute constant and g is defined in (2.2). Choosing $s = \lambda t$ in (3.12), with $\lambda \geq 1$, and recalling Lemma 3 and (3.5), we obtain

$$l_*(f;\lambda) \leq 2\sqrt{\lambda}\liminf_{t\to\infty}\frac{\|L_t f - f\|}{\omega(f;\sigma_t;1/t)} + K_2, \tag{3.13}$$

for some absolute constant K_2. Since $f \in \mathscr{M}_*(\sigma)$, the conclusion follows from (3.13) by choosing λ in such a way that $l_*(f;\lambda) > K_2$. □

Finally, as an immediate consequence of Theorems 1-3 and Remarks 1 and 2, we state

COROLLARY 1. *Let $f \in \mathscr{M}(\sigma)$. Assume that Z satisfies* (H) *and that σ satisfies* (3.5). *Also, assume that $c(p) < \infty$, for any $p \geq 1$, and that $k(q) < \infty$, for some $q > 1$. Then, for any $\alpha \in (0, 1]$, the following assertions are equivalent:*

(a) $\omega\left(f;\sigma_t;\dfrac{1}{t}\right) \approx \dfrac{1}{t^{\alpha/2}}$, *as* $t \to \infty$.

(b) $N_p(f;t) \approx \dfrac{1}{t^{\alpha/2}}$, *as* $t \to \infty$, *for any* $p \geq 1$.

If $\alpha \in (0, 1)$, statement (a) is also equivalent to

(c) $f \in \mathscr{M}_*(\sigma)$ *and* $\|L_t f - f\| \approx \dfrac{1}{t^{\alpha/2}}$, *as* $t \to \infty$.

REMARK 3. Let $\mathbb{N}$ be the set of nonnegative integers. Suppose that, for each $x > 0$ and $t > 0$, the random variable $Z_t(x)$ takes values in $\mathbb{N}$ with $P(Z_t(x) = k) = p_t(x, k)$, $k = 0, 1, 2, \ldots$. Then, all the preceding results hold true if in assumption (H) we replace the Lebesgue measure λ by the counting measure on $\mathbb{N}$ and the density $d_t(x, \cdot)$ by $p_t(x, \cdot)$.

4. Examples

To illustrate the preceding results, we consider the family $\mathbb{G}$ of gamma operators defined in (1.2) and the family $\mathbb{B}$ of Baskakov operators. Their corresponding random variables are absolutely continuous in the first case and discrete in the second. We check that all the assumptions made in Section 3 are satisfied in both cases. The weight functions considered are closely related to the standard deviations of the random variables involved. We also consider the family $\mathbb{G}^*$ of gamma operators defined in (1.3) and show that the behavior of $\mathbb{G}^*$ is similar to that of $\mathbb{G}$, although $\mathbb{G}^*$ does not satisfy the aforementioned assumptions.

(a) The gamma operator G_t. Let G_t be the operator defined in (1.2) and represented by a gamma process $(S(u),\, u \geq 0)$. Denote by Ψ the psi function (see [12, p. 11]), that is,

$$\Psi(u) := \frac{\Gamma'(u)}{\Gamma(u)} = E \log S(u), \qquad u > 0.$$

It follows from [12, p. 13] that

$$\Psi'(u) = \sum_{n=0}^{\infty} \frac{1}{(u+n)^2} \leq \frac{1}{u^2} + \frac{1}{u}, \qquad u > 0. \tag{4.1}$$

Consider the weight function $\sigma(x) = \sqrt{x}$, $x \geq 0$. In this case, the constant $c(p)$ in (3.1) is given by

$$c(p) = \sup_{u \geq 0} E^{1/p} \left(\frac{|S(u) - u|}{1 \vee \sqrt{S(u+1) \wedge u}} \right)^p, \qquad p \geq 1.$$

As it follows by calculus, $c(p)$ is finite for any $p \geq 1$. On the other hand, the random variable $V_t(x)$ defined in (3.4) has the form

$$V_t(x) = t\left(\log S(tx+1) - \Psi(tx+1)\right), \qquad x > 0, \quad t > 0.$$

Hence, the constant $k(2)$ in (3.6) is given by

$$\begin{aligned} k(2) &= \sup_{u>0} \sqrt{u}\, E^{1/2} \left(\log S(u+1) - \Psi(u+1)\right)^2 \\ &= \sup_{u>0} \sqrt{u \Psi'(u+1)} \\ &< \infty, \end{aligned}$$

as it follows for (4.1).

https://doi.org/10.1017/S1446788700036296 Published online by Cambridge University Press

(b) The gamma operator G_t^*. Since the operator G_t^* defined in (1.3) does not produce differentiable functions, we cannot directly apply Theorems 2 and 3 in this case. However, G_t^* is asymptotically close to G_t, so that G_t^* inherits from G_t its approximation properties.

Let σ be as in example (a). According to (1.7), we denote for any $f \in \mathscr{M}(\sigma)$

$$N_p^*(f;t) := \sup_{x\geq 0} E^{1/p}\left|f\left(\frac{S([tx]+1)}{t}\right) - f(x)\right|^p, \qquad p \geq 1, \quad t > 0.$$

The following result shows that all the statements in Theorems 1-3 and Corollary 1 hold true for G_t^*.

THEOREM 4. *Let $f \in \mathscr{M}(\sigma)$, $p \geq 1$ and $t > 0$. Then*

(a) $\|G_t f - f\| \leq 9\|G_t^* f - f\| \leq 9\left(\|G_t f - f\| + \omega\left(f;1;\frac{1}{t}\right)\right)$;

(b) $N_p(f;t) \leq K_p N_p^*(f;t) \leq 2K_p\left(N_p(f;t) + \omega\left(f;1;\frac{1}{t}\right)\right)$,

where $K_p = 18 + 16\Gamma^{1/p}(p+1)$.

PROOF. Let $x \geq 0$ and $t > 0$. We claim that

$$\omega\left(f;1;\frac{1}{t}\right) \leq 4\|G_t^* f - f\|. \tag{4.2}$$

Indeed, if $[tx] \leq ty \leq [tx]+1$, it follows from the triangular inequality and the continuity of f that $|f(y) - f(x)| \leq 2\|G_t^* f - f\|$. If $[tx]+1 < ty < [tx]+2$, then (4.2) follows from the triangular inequality and the preceding estimate.

On the other hand, the subadditivity of $\omega(f;1;\cdot)$ implies that

$$\begin{aligned} |G_t f(x) - f(x)| &\leq \|G_t^* f - f\| + (1 + ES(tx - [tx]))\,\omega\left(f;1;\frac{1}{t}\right) \\ &\leq \|G_t^* f - f\| + 2\omega\left(f;1;\frac{1}{t}\right), \end{aligned} \tag{4.3}$$

where we have used that the gamma process has nondecreasing paths and stationary increments. Thus, the first inequality in (a) follows from (4.2) and (4.3). The remaining inequalities are shown in a similar way. □

(c) The Baskakov operator B_t. The family $\mathbb{B} := (B_t, t > 0)$ of Baskakov operators is defined by (see, for instance [3, 6])

$$\begin{aligned} B_t f(x) &= Ef\left(\frac{N(xS(t))}{t}\right) \\ &= \sum_{k=0}^{\infty} f\left(\frac{k}{t}\right)\binom{t+k-1}{k}\frac{x^k}{(1+x)^{t+k}}, \qquad x \geq 0, \quad t > 0, \end{aligned}$$

https://doi.org/10.1017/S1446788700036296 Published online by Cambridge University Press

where $(N(u),\ u \geq 0)$ is a standard Poisson process independent of the gamma process $(S(u),\ u \geq 0)$. We consider the weight function $\sigma(x) = \sqrt{x(1+x)}$, $x \geq 0$. Taking into account Remark 3, it is readily seen that

$$V_t(x) = \frac{N\,(xS(t)) - tx}{x(1+x)}, \qquad x > 0, \quad t > 0.$$

On the other hand, it follows by calculus that

$$k(2) = k_t(2) = \sup_{x>0} E^{1/2} \left(\frac{\sigma(x)|V_t(x)|}{\sqrt{t}} \right)^2 = 1, \qquad t > 0.$$

In this case, the constants defined in (3.1) are not finite. However, it can be checked that the constants $\sup_{t \geq 2\lceil p \rceil + 1} c_t(p)$, where $\lceil p \rceil$ denotes the first integer not less than p, are finite for any $p \geq 1$ and, therefore, all the results in Section 3 hold true with minor modifications.

References

[1] J. A. Adell and J. de la Cal, 'On the uniform convergence of normalized Poisson mixtures to their mixing distribution', *Statist. Probab. Lett.* **18** (1993), 227–232.

[2] ———, 'Approximating gamma distributions by normalized negative binomial distributions', *J. Appl. Probab.* **31** (1994), 391–400.

[3] ———, 'Bernstein-type operators diminish the ϕ-variation', *Constr. Approx.* **12** (1996), 489–507.

[4] P. Billingsley, *Probability and measure* (Wiley, New York, 1986).

[5] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular variation*, Encyclopedia Math. Appl. 27 (Cambridge Univ. Press, Cambridge, 1987).

[6] W. Z. Chen and M. Zhou, 'Limiting operators for some operator sequences of probabilistic type', *Math. Proc. Cambridge Philos. Soc.* **121** (1997), 547–554.

[7] B. Della Vecchia, 'Direct and converse results by rational operators', *Constr. Approx.* **12** (1996), 271–285.

[8] Z. Ditzian and V. Totik, *Modulus of smoothness* (Springer, New York, 1987).

[9] R. A. Khan, 'Some probabilistic methods in the theory of approximation operators', *Acta Math. Hungar.* **39** (1980), 193–203.

[10] A. Lupas, D. H. Mache and M. W. Müller, 'Weighted L_p-approximation of derivatives by the method of gamma operators', *Result. Math.* **16** (1995), 277–286.

[11] A. Lupas and M. W. Müller, 'Approximationseigenschaften der Gammaoperatoren', *Math. Z.* **98** (1967), 208–226.

[12] A. M. Mathai, *A handbook of generalized special functions for statistical and physical sciences* (Oxford Univ. Press, New York, 1993).

[13] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations* (Wiley, New York, 1993).

[14] V. Totik, 'The gamma operators in L^p spaces', *Publ. Math. Debrecen* **32** (1985), 43–55.

[15] D. V. Widder, *The Laplace transform* (Princeton Univ. Press, Princeton, 1946).

[16] G. E. Willmot, 'Asymptotic tail behavior of Poisson mixtures with applications', *Adv. in Appl. Probab.* **22** (1990), 147–159.

Departamento de Métodos Estadísticos
Facultad de Ciencias
Universidad de Zaragoza
50009 Zaragoza
Spain
e-mail: adell@posta.unizar.es, csangues@posta.unizar.es

https://doi.org/10.1017/S1446788700036296 Published online by Cambridge University Press