Glasgow Math. J. **46** (2004) 363–369. © 2004 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S0017089504001843. Printed in the United Kingdom

ON NON-CO-HOPFIAN p-GROUPS WITH FINITE DERIVED SUBGROUP

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(Received 6 June, 2003; accepted 27 November, 2003)

Abstract. In this article the following are proved: 1. Let G be an infinite p-group of cardinality either $\aleph_0$ or greater than $2^{\aleph_0}$. If G is center-by-finite and non-Černikov, then it is non-co-Hopfian; that is, G is isomorphic to a proper subgroup of itself. 2. Let G be a nilpotent p-group of class 2 with G/G' a non-Černikov group of cardinality $\aleph_0$ or greater than $2^{\aleph_0}$. If G' is of order p, then G is non-co-Hopfian.

2000 *Mathematics Subject Classification.* 20E07, 20F18, 20F24.

1. Introduction. A group G is *co-Hopfian* if every injective endomorphism of G is an isomorphism; that is, G has no proper subgroup isomorphic to itself. Non-co-Hopfian groups in some classes of groups are considered by many authors. The class of abelian p-groups is well known. R. A. Beaumont and R. S. Pierce [**4**] showed that if the cardinality of a reduced abelian p-group G is either $\aleph_0$ or greater than $2^{\aleph_0}$, then G is non-co-Hopfian. But P. Crawley [**7**] constructed an infinite abelian p-group without elements of infinite height that is co-Hopfian. Also J. M. Irwin and T. Ito [**11**] considered some co-Hopfian abelian p-groups with some additional properties. See also [**1**], [**3**], [**12**] for more details. In view of these results it seems natural to ask which infinite nilpotent p-groups are non-co-Hopfian. As a first step it seems suitable to study infinite nilpotent p-groups with finite derived subgroup as the results with some restrictions show in the sequel.

In [**2**], [**5**], [**6**] the class **K** of locally finite groups all of whose Sylow subgroups are Černikov is considered. In [**6**], V. V. Belyaev showed that if G is a countable locally soluble **K**-group then G is co-Hopfian if and only if G is hyperfinite, which was conjectured by R. Baer. This result is proved by S. D. Bell in [**5**] for locally finite groups independently, using character theoretic ideas. See [**9**] for more details.

In recent years, H. Smith and J. Wiegold [**14**], [**15**], [**16**] considered some non-co-Hopfian groups which are isomorphic to their non-nilpotent or non-abelian subgroups.

See also [**8**], [**18**] for some other results.

In this article we consider the class of center-by-finite p-groups (which is a subclass of the class of all locally finite FC-groups) and nilpotent p-groups of class two with derived subgroup of order p and prove the following results.

THEOREM 1. *Let G be an infinite p-group of cardinality either* $\aleph_0$ *or greater than* $2^{\aleph_0}$. *If G is center-by-finite and non-Černikov then it is non-co-Hopfian.*

THEOREM 2. *Let G be a nilpotent p-group of class* 2 *with* G/G' *a non-Černikov group of cardinality* $\aleph_0$ *or greater than* $2^{\aleph_0}$. *If* G' *is of order p then G is non-co-Hopfian.*

2. Center-by-finite p-groups. We start with a Lemma which may be helpful to see the structures of the groups in some classes. Here we consider *center-by-finite* p-groups since these groups are always considered as an important subclass of the class of all locally finite FC-groups.

LEMMA 1. *Let $G = FA$ be an infinite p-group with A an abelian group of cardinality $\aleph_0$ or greather than $2^{\aleph_0}$ and F a subgroup of G. If A is non-Černikov, $A \cap F$ is finite and $[A, F] = 1$, then G is non-co-Hopfian.*

Proof. By Theorem 21.3 of **[10]** there exist radicable and reduced subgroups D and R respectively such that $A = D \times R$. First suppose that D is a direct product of infinitely many p^∞-type subgroups. Hence there exists a subgroup U of D that is a direct product of countably many p^∞-type subgroups $A_1, A_2, \ldots$ such that $D = U \times V$ for some subgroup V of D and $F \cap A$ is contained in $A_1 \ldots A_r$ for a natural number r. If A_i $(i = 1, 2, \ldots)$ is generated by $a_{i,k}$ for $k = 1, 2, \ldots$ then put

$$X = RV\langle A_1, \ldots, A_r, A_{r+2}, \ldots\rangle$$

and define ϕ from A to X by $\phi(a_{i,k}) = a_{i,k}$ for $i = 1, \ldots, r$ and for all k, $\phi(a_{i,k}) = a_{i+1,k}$ for $i > r$ and $\phi(x) = x$ for all $x \in RV$. Now it is not difficult to see that ϕ is an isomorphism. If we define Φ from G to FX as $\Phi(fg) = f(\phi(g))$ for all $f \in F$ and $g \in A$ then since $[A, F] = 1$, Φ is a homomorphism. Let fg be in the kernel of Φ. Then $\phi(g) = f^{-1}$ is contained in $X \cap F$ and, since ϕ fixes $X \cap F$ elementwise, g is contained in $X \cap F$. This implies that $g = f^{-1}$; that is, $fg = 1$. Hence Φ is a monomorphism. It is obvious that Φ is onto and since $F \cap A$ is contained in X, FX is a proper subgroup of G and thus G is non-co-Hopfian.

Now suppose D is a direct product of finitely many p^∞-type subgroups. Hence R must be infinite. If R has finite exponent then R has an infinite direct factor which is a direct product of countably many cyclic subgroups. If R is countable of infinite exponent then by Exercise 8(a) on page 67 of **[10]** R again has an infinite direct factor which is the product of countably many cyclic subgroups. If R is of cardinality greater than $2^{\aleph_0}$ then by the proof of the Theorem (Case 3) [**11**, p 152] R has an infinite direct factor as in the previous two cases. Hence in every case there are subgroups M, N of R such that $R = M \times N$ and M is the direct product of countably many cyclic subgroups $\langle y_i\rangle$ of order p^{k_i} such that $p^{k_i} \leq p^{k_{i+1}}$ for $i = 1, 2, \ldots$. Now there exists a natural number n such that

$$(F \cap A) \leq \left(Dr_{i=1}^{n-1}\langle y_i\rangle\right).$$

Put

$$Y = ND\langle y_1, \ldots, y_{n-1}, y_{n+1}, \ldots\rangle$$

and define θ from A to Y as $\theta(y_i) = y_i$ for $i = 1, \ldots, n-1$, $\theta(y_i) = y_{i+1}^{p^{k_{i+1}-k_i}}$ for $i \geq n$ and $\theta(x) = x$ for all $x \in ND$. Now θ is an isomorphism. Define γ from G to FY as $\gamma(fg) = f(\theta(g))$ for all $f \in F$, $g \in A$. Now, γ is an isomorphism and since $F \cap A$ is contained in Y, FY is a proper subgroup of G and again G is non-co-Hopfian, as desired. □

Proof of Theorem 1. There is a finite subgroup F such that $G = ZF$ where Z is the center of G. Hence G satisfies the hypothesis of Lemma 1 and so G is non-co-Hopfian. □

3. Nilpotent groups of class two with derived subgroup of order p. Now we give a technical Lemma which concentrates on the derived factor group of a group and plays a crucial role in the proof of Theorem 2.

LEMMA 2. *Let G be a countably infinite nilpotent p-group of class 2 and let K/G' be a subgroup of G/G' that is a direct product of cyclic subgroups $\langle y_iG'\rangle$ such that y_iG' is of order p^{k_i} and $p^{k_i} < p^{k_{i+1}}$ for $i = 1, 2, \ldots$. If G' is of order p then K is non-co-Hopfian.*

Proof. Let z be a generator of G'. If $y_i^{p^{ki}} = z^v$ where $0 < v < p$ then by considering the multiplicative inverse l of v in the field of elements modulo p we take y_i^l in place of y_i. Now $y_i^{p^{ki}}$ can be 1 or z. By reordering the y_i we have that $K = HL$ and $H \cap L = G'$, where H and L are subgroups of K such that if y_i is in H then $y_i^{p^{ki}} = 1$ and if it is in L then $y_i^{p^{ki}} = z$, and the $\langle y_i\rangle$ are ordered separately in H and L such that y_iG' is of order p^{k_i} and $p^{k_i} < p^{k_{i+1}}$ for $i = 1, 2, \ldots$, as in the hypothesis. Clearly H or L is infinite. We first assume that both subgroups are infinite and show that they are non-co-Hopfian.

Every element k of H (and of L) can be written in the form

$$k = \left(\Pi_{i=1}^r y_i^{t_i}\right)u$$

where $0 \leq t_i < p^{k_i}$, u is in G' and in this expression u and $y_i^{t_i}$ are uniquely determined whenever t_i is non-zero. Let M denote either of H, L and define ϕ from M to Y as

$$\phi(k) = \left[\Pi_{i=1}^r \left(y_i y_{i+1}^{(p^{k_{i+1}-k_i})} \ldots y_{i+p}^{(p^{k_{i+p}-k_i})}\right)^{t_i}\right]u$$

where

$$Y = \left\langle y_i y_{i+1}^{(p^{k_{i+1}-k_i})} \ldots y_{i+p}^{(p^{k_{i+p}-k_i})}, G' \mid i = 1, 2, \ldots\right\rangle.$$

Since G' has order p, G^p is contained in the center of G and we have

$$\phi(k) = \left(\Pi_{i=1}^r y_i^{t_i} y_{i+1}^{t_i(p^{k_{i+1}-k_i})} \ldots y_{i+p}^{t_i(p^{k_{i+p}-k_i})}\right)u.$$

We shall show in detail that ϕ is an isomorphism.

Since K/G' is a direct product of cyclic groups $\langle y_iG'\rangle$ (for $i = 1, 2, \ldots$), the set $\{y_iG' : i = 1, \ldots, r\}$ is linearly independent and thus ϕ is well-defined. Let

$$l = \left(\Pi_{i=1}^r y_i^{s_i}\right)v,$$

where $0 \leq s_i < p^{k_i}$ and v is in G'. Then

$$kl = \left(\Pi_{i=1}^r y_i^{t_i}\right)\left(\Pi_{i=1}^r y_i^{s_i}\right)uv = \Pi_{i=1}^r \left(y_i^{t_i+s_i}\right)uvw$$

for an element w of G' (w can be calculated in terms of the commutators). First assume that $(t_i + s_i) < p^{k_i}$ for all $i = 1, \ldots, r$ and put $n_i = t_i + s_i$ for $i = 1, \ldots, r$. Now

$$kl = \left(\Pi_{i=1}^r y_i^{n_i}\right)uvw$$

and it is in the form stated above. Thus

$$\phi(kl) = \left[\Pi_{i=1}^r \left(y_i^{n_i} y_{i+1}^{n_i(p^{k_{i+1}-k_i})} \ldots y_{i+p}^{n_i(p^{k_{i+p}-k_i})}\right)\right]uvw,$$

https://doi.org/10.1017/S0017089504001843 Published online by Cambridge University Press

since $uvw \in G'$ and ϕ fixes uvw by the definition of ϕ. We also see that

$$\begin{aligned}\phi(k)\phi(l) &= [\Pi_{i=1}^{r} y_i^{t_i} y_{i+1}^{t_i(p^{k_{i+1}-k_i})} \dots y_{i+p}^{t_i(p^{k_{i+p}-k_i})}][\Pi_{i=1}^{r} y_i^{s_i} y_{i+1}^{s_i(p^{k_{i+1}-k_i})} \dots y_{i+p}^{s_i(p^{k_{i+p}-k_i})}]uv \\ &= [\Pi_{i=1}^{r} y_i^{s_i+t_i} y_{i+1}^{(s_i+t_i)(p^{k_{i+1}-k_i})} \dots y_{i+p}^{(s_i+t_i)(p^{k_{i+p}-k_i})}]uvw \\ &= [\Pi_{i=1}^{r} y_i^{n_i} y_{i+1}^{n_i(p^{k_{i+1}-k_i})} \dots y_{i+p}^{n_i(p^{k_{i+p}-k_i})}]uvw,\end{aligned}$$

since G^p is contained in $Z(G)$. Now assume that there exist $m_1, \dots, m_c \in \{1, \dots, r\}$ such that $t_{m_j} + s_{m_j} \geq p^{k_{m_j}}$ for $j = 1, \dots, c$. Then there exist d_{m_j} and w_{m_j} such that $t_{m_j} + s_{m_j} = p^{k_{m_j}} w_{m_j} + d_{m_j}$ and $0 \leq d_{m_j} < p^{k_{m_j}}$. Since $t_{m_j} + s_{m_j} < 2p^{k_{m_j}}$ we have $w_{m_j} = 1$ and thus $t_{m_j} + s_{m_j} = p^{k_{m_j}} + d_{m_j}$. Define

$$n_i = \begin{cases} t_i + s_i & \text{if } t_i + s_i < p^{k_i}, \\ d_i & \text{if } t_i + s_i \geq p^{k_i}. \end{cases}$$

Now $0 \leq n_i < p_{k_i}$ for $i = 1, \dots, r$ and

$$kl = (\Pi_{i=1}^{r} y_i^{n_i})(\Pi_{j=1}^{c} y_{m_j}^{p^{k_{m_j}}})uvw.$$

Put $g = (\Pi_{j=1}^{c} y_{m_j}^{p^{k_{m_j}}})$. Then $g = 1$ or z^c according to whether M is H or L, respectively. It follows that

$$\phi(kl) = [\Pi_{i=1}^{r} y_i^{n_i} y_{i+1}^{n_i(p^{k_{i+1}-k_i})} \dots y_{i+p}^{n_i(p^{k_{i+p}-k_i})}]guvw,$$

since $guvw \in G'$ and ϕ fixes $guvw$ by the definition of ϕ. We also have

$$\begin{aligned}\phi(k)\phi(l) &= [\Pi_{i=1}^{r} y_i^{t_i} y_{i+1}^{t_i(p^{k_{i+1}-k_i})} \dots y_{i+p}^{t_i(p^{k_{i+p}-k_i})}][\Pi_{i=1}^{r} y_i^{s_i} y_{i+1}^{s_i(p^{k_{i+1}-k_i})} \dots y_{i+p}^{s_i(p^{k_{i+p}-k_i})}]uv \\ &= [\Pi_{i=1}^{r} y_i^{s_i+t_i} y_{i+1}^{(s_i+t_i)(p^{k_{i+1}-k_i})} \dots y_{i+p}^{(s_i+t_i)(p^{k_{i+p}-k_i})}]uvw \\ &= [\Pi_{i=1}^{r} y_i^{n_i} y_{i+1}^{n_i(p^{k_{i+1}-k_i})} \dots y_{i+p}^{n_i(p^{k_{i+p}-k_i})}][\Pi_{j=1}^{c} y_{m_j}^{p^{k_{m_j}}} y_{m_j+1}^{p^{k_{m_j+1}}} \dots y_{m_j+p}^{p^{k_{m_j+p}}}]uvw \\ &= [\Pi_{i=1}^{r} y_i^{n_i} y_{i+1}^{n_i(p^{k_{i+1}-k_i})} \dots y_{i+p}^{n_i(p^{k_{i+p}-k_i})}]guvw,\end{aligned}$$

since $[\Pi_{j=1}^{c} y_{m_j}^{p^{k_{m_j}}} y_{m_j+1}^{p^{k_{m_j+1}}} \dots y_{m_j+p}^{p^{k_{m_j+p}}}]$ is either equal to 1 or $(z^{p+1})^c = z^c$. This yields that ϕ is a homomorphism.

Let bars denote subgroups and elements modulo G'. If k is in $ker\phi$ then we have

$$\Pi_{i=1}^{r}(\bar{y}_i^{t_i} \bar{y}_{i+1}^{t_i(p^{k_{i+1}-k_i})} \dots \bar{y}_{i+p}^{t_i(p^{k_{i+p}-k_i})}) = 1.$$

Now $p^{k_i} \mid t_i$ for $i = 1, 2, \dots$ and this implies that $k = 1$. Consequently, we see that ϕ is an isomorphism and M is non-co-Hopfian.

Let y_1 be an element of Y. Then

$$\bar{y}_1 = [\bar{y}_1 \bar{y}_2^{(p^{k_2-k_1})} \dots \bar{y}_{1+p}^{(p^{k_{p+1}-k_1})}]^{m_1} \dots [\bar{y}_r \bar{y}_{r+1}^{(p^{k_{r+1}-k_r})} \dots \bar{y}_{r+p}^{(p^{k_{r+p}-k_r})}]^{m_r},$$

for some $r \geq 1$. If $r \geq 2$, then $p^{k_r} \mid m_r$, since $p^{k_{r+p}} \mid (p^{k_{r+p}} m_r)$. This implies that

$$\bar{y}_1 = [\bar{y}_1 \bar{y}_2^{(p^{k_2-k_1})} \dots \bar{y}_{1+p}^{(p^{k_{p+1}-k_1})}]^{m_1}.$$

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Now we see that m_1 is of the form $p^{k_1}t+1$ and $p^{k_2} \mid [(p^{k_1}t+1)(p^{k_2-k_1})]$. But then $p^{k_1} \mid 1$, a contradiction.

Finally, we will show that $K = HL$ is non-co-Hopfian assuming that H is infinite (if L is infinite then the same method can be used). Define θ from G to YL as $\theta(hl) = \phi(h)l$. Since ϕ fixes G' elementwise, θ is well-defined. If

$$h = \left(\Pi_{i=1}^{r} y_i^{s_i}\right)u,$$

then

$$l^{\phi(h)} = l^{(\Pi_{i=1}^{r} y_i^{s_i} y_{i+1}^{s_i(p^{k_{i+1}-k_i})} \ldots y_{i+p}^{s_i(p^{k_{i+p}-k_i})})u} = l^{(\Pi_{i=1}^{r} y_i^{s_i})u} = l^h,$$

since $y_{i+1}^{s_i(p^{k_{i+1}-k_i})} \ldots y_{i+p}^{s_i(p^{k_{i+p}-k_i})}$ is contained in the center of G. This shows that θ is a homomorphism. It is not difficult to see that the kernel of θ is trivial and YL is proper in K. Therefore K is non-co-Hopfian. □

Proof of Theorem 2. Case 1. G/G' is reduced. Suppose first $card(G/G') = \aleph_0$ and G/G' is of infinite exponent. Then it has a direct factor K/G' which is a direct product of cyclic subgroups with strictly increasing order by Exercise 8(a) on page 67 of [**10**]. Now there is a subgroup N of G such that $G = KN$ and $N \cap K = G'$. By Lemma 2, K is non-co-Hopfian. Let θ and ϕ be as in the proof of Lemma 2 and let $K_1 = \theta(K)$. Define γ from G to K_1N as $\gamma(kn) = \theta(k)n$. Since $K = HL$ where $H$and L are as in the proof of Lemma 2,

$$n^{\theta(k)} = n^{\theta(hl)} = n^{\phi(h)l} = n^{hl}.$$

Hence we obtain that γ is a homomorphism. Since $K \cap N = G'$ and θ fixes G' elementwise, $ker\gamma$ is trivial. Consequently, γ is an isomorphism and thus G is non-co-Hopfian.

Now let bars denote subgroups and elements modulo G' and suppose $\bar{G}$ has finite exponent. Then $\bar{G} = \bar{B}_1 \times \ldots \times \bar{B}_m$ where $\bar{B}_k$ is a (possibly trivial) direct product of cyclic subgroups of order p^k for $k = 1, \ldots, m$ and some $\bar{B}_j$ is infinite. Suppose $j > 1$. $\bar{B}_j$ has an infinite subgroup $\bar{U}$ which is a direct product of countably many cyclic subgroups $\langle \bar{y}_i \rangle$ of order p^j such that $\bar{B}_j = \bar{U} \times \bar{V}$ for some subgroup $\bar{V}$ of $\bar{B}_j$. Put

$$M = B_1 \ldots B_{j-1} V B_{j+1} \ldots B_m \left\langle y_i y_{i+1}^p \mid i = 1, 2, \ldots \right\rangle$$

and define γ from G to M as $\gamma(y_i) = y_i y_{i+1}^p$ for $i = 1, 2, \ldots$ and $\gamma(x) = x$ for all $x \in B_1 \ldots B_{j-1} V B_{j+1} \ldots B_m$. Using an argument as in the proof of Lemma 2 and the fact that $y_{j+1}^{p^{j+1}} = 1$ we can show that γ is an isomorphism and $M \neq G$. Hence G is non-co-Hopfian. If $j = 1$, then $\bar{B}_1$ is an infinite elementary abelian p-group. Now $\bar{B}_1$ has a direct factor $\bar{H}$ which is a direct product of countably many cyclic subgroups such that $\bar{B}_1 = \bar{H} \times \bar{L}$ for a subgroup $\bar{L}$ of $\bar{G}$. Put $G' = \langle z \rangle$ and define $\phi : \bar{H} \times \bar{H} \to GF(p)$ as $\phi(\bar{a}, \bar{b}) = k$ where $[a, b] = z^k$. It is easy to show that ϕ is a non-degenerate, alternating bilinear form. Since $dim_F(\bar{H})$ is countable where $F = GF(p)$, $\bar{H}$ is an orthogonal direct product of hyperbolic planes $\bar{H}_1, \bar{H}_2 \ldots$. If we consider the pre-image of each one of these hyperbolic planes then we can find a_i, b_i for $i = 1, 2, \ldots$ such that $H = \langle a_1, b_1 \rangle \langle a_2, b_2 \rangle \ldots$ where $[a_1, b_1] = [a_2, b_2] = \ldots$ and $[\langle a_i, b_i \rangle, \langle a_j, b_j \rangle] = 1$ if $i \neq j$. Now define γ from G to Y as $\gamma(a_i) = a_{i+1}$, $\gamma(b_i) = b_{i+1}$ and $\gamma(x) = x$ for all $x \in L$

where

$$Y = L\langle a_i, b_i \mid i = 2, 3, \ldots\rangle.$$

Then γ is an isomorphism, $Y \neq G$ and thus G is non-co-Hopfian.

Finally suppose $\bar{G}$ is of infinite exponent and cardinality greater than $2^{\aleph_0}$. Then

$$\bar{G} = \bar{B}_1 \times \ldots \times \bar{B}_j \times \bar{G}_j$$

for a subgroup $\bar{G}_j$ of $\bar{G}$. By the proof of the Theorem (Case 3) on page 152 of [**11**] some $\bar{B}_j$ must be infinite. Arguing the same way as in the previous case we see that G is non-co-Hopfian for $j = 1$ and for $j > 1$ with few changes. If $\bar{G}$ is of finite exponent then again the above arguments work.

Case 2. G/G' is not reduced. Now if D/G' is the radicable part of G/G' then $G/G' = D/G' \times R/G'$ for a reduced subgroup R/G' of G/G' by Theorem 21.3 of [**10**]. If D is non-abelian then $D' = G'$ and hence D is radicable by Theorem 9.23 of [**13**]. This implies that D is abelian by Corollary 2 to Theorem 9.23 of [**13**], a contradiction. If D is not radicable then $D = G' \times D^p$ and thus D^p is radicable. Put $A = D^p$ then $G = AR$ and A centralizes R. Hence R cannot be abelian, since G is nilpotent of class two. Now $card(R)$ or $card(A)$ is either $\aleph_0$ or greater than $2^{\aleph_0}$; otherwise $card(G/G')$ would equal $2^{\aleph_0}$. If $card(A)$ has that property, then G is non-co-Hopfian by Lemma 1. Hence we may assume that $card(R)$ has the above property. If we define Φ as $\Phi(ar) = a\gamma(r)$ for all $a \in A$ and $r \in R$ then Φ is an isomorphism and thus G is non-co-Hopfian. This completes the proof. □

ACKNOWLEDGMENT. The author is grateful to Professor Martyn Russell Dixon for many helpful discussions and to the referee for valuable comments and suggestions.

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