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POSITIVE ENERGY REPRESENTATIONS AND CONTINUITY OF PROJECTIVE REPRESENTATIONS FOR GENERAL TOPOLOGICAL GROUPS

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Abstract. Let G and T be topological groups, $\alpha\colon T \to \mathrm{Aut}(G)$ a homomorphism defining a continuous action of T on G and $G^\sharp := G \rtimes_\alpha T$ the corresponding semidirect product group. In this paper, we address several issues concerning irreducible continuous unitary representations $(\pi^\sharp, \mathcal{H})$ of $G^\sharp$ whose restriction to G remains irreducible. First, we prove that, for $T = \mathbb{R}$, this is the case for any irreducible positive energy representation of $G^\sharp$, i.e. for which the one-parameter group $U_t := \pi^\sharp(\mathbf{1}, t)$ has non-negative spectrum. The passage from irreducible unitary representations of G to representations of $G^\sharp$ requires that certain projective unitary representations are continuous. To facilitate this verification, we derive various effective criteria for the continuity of projective unitary representations. Based on results of Borchers for W^*-dynamical systems, we also derive a characterization of the continuous positive definite functions on G that extend to $G^\sharp$.

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1. Introduction. Let G and T be topological groups and $\alpha\colon T \to \mathrm{Aut}(G)$ be a homomorphism defining a continuous action of T on G. Then the semidirect product $G^\sharp := G \rtimes_\alpha T$ is also a topological group. In this note, we take a closer look at irreducible continuous unitary representations of $G^\sharp$ whose restriction to G remains irreducible. This is motivated by the concrete class of examples where G is a Banach–Lie group, such as the group of H^1-maps $\mathbb{S}^1 \to K$, where K is a Lie group, and $\alpha\colon \mathbb{R} \to \mathrm{Aut}(G)$ correspond to rotations of the circle. Then, $G^\sharp$ is not a Lie group, but it still is a topological group (cf. [**14**]).

For $T = \mathbb{R}$, a particularly interesting class of representations $(\pi^\sharp, \mathcal{H})$ of $G^\sharp$ are those for which the infinitesimal generator of the one-parameter group $U_t := \pi^\sharp(\mathbf{1}, t)$ has non-negative spectrum, the so-called *positive energy representations*. Our first main result (Theorem 2.2) asserts that every irreducible positive energy representation of $G^\sharp$ remains irreducible when restricted to G. This is quite remarkable because for non-trivial actions, there are many situations where irreducible representations of $G^\sharp$ are not irreducible when restricted to G. The simplest example are the irreducible representations of the orientation preserving affine group of $\mathbb{R}$, which is of the form $G^\sharp = \mathbb{R} \rtimes_\alpha \mathbb{R}$ for $\alpha_t(x) = e^t x$. We derive Theorem 2.2 as a consequence of the Borchers–Arveson Theorem on covariant representations of von Neumann algebras (cf. [**6**, Theorem 3.2.46]).

In the light of Theorem 2.2, it becomes a natural question which irreducible unitary representations of G extend to a positive energy representation of $G^\sharp = G \rtimes_\alpha \mathbb{R}$. This appears to be a difficult problem, and already for the rather concrete class of loop groups the answer becomes rather involved because it turns out that, for $G = C^\infty(\mathbb{T}, K)$ and $(\alpha_z f)(w) := f(zw)$ (the rotation action), smooth positive energy representations of $G^\sharp$ are trivial in the sense that they factor through representations of products $A \times \mathbb{R}$, where A is an abelian quotient of K. This is shown in Section 3, where we also derive a similar result for groups of the form $G = C^\infty(\mathbb{R}, K)$ and $(\alpha_t f)(s) = f(s + t)$. It is well known from the classical theory of loop groups that the problem of the triviality of positive energy representations of $C^\infty(\mathbb{T}, K) \rtimes \mathbb{T}$ can be resolved by passing from the loop group $C^\infty(\mathbb{T}, K)$ to a suitable central extension $\widetilde{G}$ for which $\widetilde{G} \rtimes \mathbb{T}$ has plenty of positive energy representations ([**17**, **14**]). One obtains a similar picture for the case where G is abelian, where one has to pass to Heisenberg groups to obtain positive energy representations (see [**16**] and [**20**] for details).

Given an irreducible unitary representation $(\pi, \mathcal{H})$ of G, a necessary condition for the existence of an extension to $G^\sharp = G \rtimes_\alpha T$ is that, for every $t \in T$, the representation $\pi \circ \alpha_t$ is equivalent to π. Then there exists, up to a phase factor in $\mathbb{T}$ uniquely determined, unitary operators $(U_t)_{t \in T}$ with

$$U_t \pi(g) U_t^* = \pi(\alpha_t(g)) \quad \text{for} \quad t \in T, g \in G, \tag{1}$$

and the question is whether these operators can be chosen in such a way that $U \colon T \to \mathrm{U}(\mathcal{H})$ is a continuous group homomorphism. For any group homomorphism $\chi : T \to \mathbb{T}$, the operators $\widetilde{U}_t := U_t \chi(t)$ also satisfy the relation (1) and since groups such as $T = \mathbb{R}$ have many discontinuous homomorphisms $\chi : T \to \mathbb{T}$, one cannot hope for an arbitrarily chosen family (U_t) to be continuous. However, what is uniquely determined by π is the corresponding projective representation $\overline{U} \colon T \to \mathrm{PU}(\mathcal{H})$, so that the main issue is whether this homomorphism is continuous or not.

To facilitate the verification of the continuity of a projective unitary representation of a topological group, we provide in Section 5 some useful criterion (Theorem 5.15). It asserts that, for a connected topological group G and a projective unitary representation $\pi : G \to \mathrm{PU}(\mathcal{H})$, continuity already follows from the existence of a cyclic ray $[v] = \mathbb{C}v$ in the projective space $\mathbb{P}(\mathcal{H})$ of $\mathcal{H}$ whose orbit map $G \to \mathbb{P}(\mathcal{H}), g \mapsto \pi(g)[v]$ is continuous. In particular, for irreducible representations, it suffices that one ray has a continuous orbit map. The main point of this criterion is that it is easier to check continuity for one vector than for all of them. This result has an analogue for unitary representations $\pi : G \to \mathrm{U}(\mathcal{H})$ whose proof is much easier, namely that a unitary representation generated by a vector with continuous orbit map is continuous. What makes the projective version more difficult is the lack of ‘addition’ on projective space. We overcome this problem by using the midpoint operation for the natural Riemannian metric on $\mathbb{P}(\mathcal{H})$ as a replacement. Our Theorem 5.15 refines the criterion by Rieffel ([**18**]) that a projective unitary representation is continuous if and only if all its orbit maps in $\mathbb{P}(\mathcal{H})$ are continuous.

Since the continuity of a projective unitary representation is characterized in terms of continuity of orbit maps in $\mathbb{P}(\mathcal{H})$, it is also important to have effective tools to verify such continuity properties. In Section 4 we provide such a tool by giving sufficient conditions for a subset $E \subseteq \mathcal{H}$ such that the topology on the projective space $\mathbb{P}(\mathcal{H})$ is initial with respect to the functions $h_v([w]) := |\langle v, w\rangle|$, $v \in E$, where we represent $[w] \in \mathbb{P}(\mathcal{H})$ by a unit vector w. In particular, it suffices that the corresponding functions

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$\ell_v(A) := \langle Av, v\rangle$ separate the points on the space of hermitian trace class operators. For any such set, the continuity of an orbit map of a ray $[w] \in \mathbb{P}(\mathcal{H})$ for a projective unitary representation can now be verified in terms of the continuity of the scalar-valued functions $G \to \mathbb{R}$, $g \mapsto h_v(\pi(g)[w])$.

The homomorphism $\alpha\colon T \to \mathrm{Aut}(G)$ is a group theoretic variant of the structure of a C^*-dynamical system, where $\alpha\colon T \to \mathrm{Aut}(\mathcal{A})$ is a homomorphism into the group of automorphisms of a C^*-algebra $\mathcal{A}$. For covariant representations (π, U) of a C^*-algebra $\mathcal{A}$ with respect to α, Borchers gives in **[3]** a characterization of those states in $\mathcal{A}^*$ (the topological dual of $\mathcal{A}$) occurring for covariant representations (for which U is continuous) as those transforming continuously under the natural T-action on the Banach space $\mathcal{A}^*$. For covariant representations satisfying the spectral condition, i.e. the one-parameter group $(U_t)_{t\in\mathbb{R}}$ has non-negative spectrum, extra conditions on the states have to be imposed (cf. [**5**, Section II.5]). In Section 6, we explain how Borchers result can be applied to the problem to extend continuous representations of G to continuous representations of $G^\sharp = G \rtimes_\alpha T$. For the special case of pure states, this follows easily from our continuity criteria for projective representations.

For representations on separable Hilbert spaces and separable topological groups which are completely metrizable, so-called polish groups, one can also derive the continuity of projective unitary representations from rather weak measurability requirements. This has been shown by Cattaneo in **[7]** based on the results that for a separable Hilbert space $\mathcal{H}$, its projective space $\mathbb{P}(\mathcal{H})$ is a polish space ([**7**, Proposition 4]) and $\mathrm{PU}(\mathcal{H})$ is a polish group ([**7**, Proposition 5]).

Notation. $\mathcal{H}$ denotes a complex Hilbert space, $B(\mathcal{H})$ denotes the C^*-algebra of bounded operators on $\mathcal{H}$ and $\mathrm{U}(\mathcal{H})$ denotes its unitary group. The *strong topology* on $\mathrm{U}(\mathcal{H})$ is the coarsest topology for which all functions $f_v(U) := \langle Uv, v\rangle$, $v \in \mathcal{H}$, are continuous. It defines on $\mathrm{U}(\mathcal{H})$ a group topology and we write $\mathrm{U}(\mathcal{H})_s$ for this topological group. The centre $\mathbb{T}\mathbf{1} = Z(\mathrm{U}(\mathcal{H}))$ is a closed subgroup, so that we also obtain the structure of a (Hausdorff) topological group on the projective unitary group $\mathrm{PU}(\mathcal{H}) = \mathrm{U}(\mathcal{H})_s/\mathbb{T}\mathbf{1}$.

The space of trace class operators on $\mathcal{H}$ is denoted $B_1(\mathcal{H})$, and we write $\mathrm{Herm}_1(\mathcal{H})$ for the subspace of hermitian trace class operators. For $v, w \in \mathcal{H}$ we write $P_{v,w}(x) := \langle x, w\rangle v$ and $P_v := P_{v,v}$ for the corresponding operators on $\mathcal{H}$.

2. Irreducible positive energy representations.

Let G and T be topological groups and $\alpha\colon T \to \mathrm{Aut}(G)$, $t \mapsto \alpha_t$, be a homomorphism defining a continuous T-action on G. We write $G^\sharp := G \rtimes_\alpha T$ for the corresponding semidirect product.

DEFINITION 2.1. We call a pair (π, U) of a continuous unitary representation $(\pi, \mathcal{H})$ of G and a continuous unitary representation $(U, \mathcal{H})$ of T *covariant* if

$$U_t\pi(g)U_t^* = \pi(\alpha_t g) \quad \text{for} \quad g \in G, t \in T.$$

This is equivalent to $\pi^\sharp(g, t) := \pi(g)U_t$ to define a continuous unitary representation of the semidirect product $G^\sharp$. For $T = \mathbb{R}$, a covariant representation (π, U) is said to be a *positive energy representation*[1] if the infinitesimal generator $A = -i\frac{d}{dt}|_{t=0}U_t$ of the

[1] In the context of covariant representations of operator algebras, this condition is simply called the *spectral condition* or the *spectrum condition*.

unitary one-parameter group $(U_t)_{t\in\mathbb{R}}$ has non-negative spectrum. We then also say that the unitary one-parameter group $(U_t)_{t\in\mathbb{R}}$ has *non-negative spectrum*.

The following theorem is a central result on covariant representations of operator algebras.

THEOREM 2.2. (Borchers–Arveson Theorem; [**6**, Theorem 3.2.46]) *Let* $(\alpha_t)_{t\in\mathbb{R}}$ *be a* σ*-weakly continuous one-parameter group of automorphisms of a von Neumann algebra* $\mathcal{M} \subseteq B(\mathcal{H})$*, i.e. for each* $\beta \in \mathcal{M}_*$ *(the predual of* $\mathcal{M}$*) and* $M \in \mathcal{M}$*, the function* $t \mapsto \beta(\alpha_t M)$ *is continuous. Then the following are equivalent:*

(i) *There exists a strongly continuous unitary one-parameter group* $(U_t)_{t\in\mathbb{R}}$ *in* $\mathrm{U}(\mathcal{H})$ *with non-negative spectrum such that*

$$\alpha_t(M) = U_t M U_t^* \quad \textit{for} \quad t \in \mathbb{R}, M \in \mathcal{M}.$$

(ii) *There exists a strongly continuous unitary one-parameter group* $(U_t)_{t\in\mathbb{R}}$ *in* $\mathrm{U}(\mathcal{M})$ *with non-negative spectrum such that*

$$\alpha_t(M) = U_t M U_t^* \quad \textit{for} \quad t \in \mathbb{R}, M \in \mathcal{M}.$$

REMARK 2.3. If $(U_t)_{t\in\mathbb{R}}$ is a strongly continuous unitary one-parameter group, $A \in B_1(\mathcal{H}) \cong B(\mathcal{H})_*$ is a trace class operator and $B \in B(\mathcal{H})$, then the function

$$t \mapsto \mathrm{tr}(AU_tBU_t^*) = \mathrm{tr}(U_t^*AU_tB)$$

is continuous because the action of $\mathbb{R}$ on $B_1(\mathcal{H})$ defined by $(t, A) \mapsto U_tAU_t^*$ is strongly continuous. Therefore, $\alpha_t(B) := U_tBU_t^*$ defines a σ-weakly continuous one-parameter group of automorphisms of $B(\mathcal{H})$, and hence on every von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ which is invariant under conjugation with the operators U_t. Therefore, the σ-weak continuity of α is necessary for the conclusion of the Borchers–Arveson Theorem to hold.

COROLLARY 2.4. *If* $(U_t)_{t\in\mathbb{R}}$ *is a strongly continuous one-parameter group in* $\mathrm{U}(\mathcal{H})$ *with non-negative spectrum and* $\mathcal{M} \subseteq B(\mathcal{H})$ *a von Neumann algebra invariant under the automorphisms* $\alpha_t(M) := U_tMU_t^*$ *of* $B(\mathcal{H})$*, then there exists a strongly continuous unitary one-parameter group* $(V_t)_{t\in\mathbb{R}}$ *of* $\mathcal{M}$ *with non-negative spectrum and a strongly continuous unitary one-parameter group* $(W_t)_{t\in\mathbb{R}}$ *in the commutant* $\mathcal{M}'$ *such that*

$$U_t = V_tW_t \quad \textit{for} \quad t \in \mathbb{R}.$$

Proof. From Theorem 2.2 we derive the existence of (V_t) in $\mathrm{U}(\mathcal{M})$ satisfying

$$U_tMU_t^* = \alpha_t(M) = V_tMV_t^* \quad \text{for} \quad M \in \mathcal{M}, t \in \mathbb{R}.$$

Applying this relation to $V_s \in \mathcal{M}$, we see that $U_tV_s = V_sU_t$ holds for $s, t \in \mathbb{R}$. Therefore, $W_t := V_t^*U_t \in \mathrm{U}(\mathcal{M}')$ defines a one-parameter group. □

THEOREM 2.5. (Irreducibility theorem) *If* $(\pi^\sharp, \mathcal{H})$ *is an irreducible positive energy representation of* $G^\sharp = G \rtimes_\alpha \mathbb{R}$*, then its restriction* $\pi := \pi^\sharp|_G$ *is also irreducible.*

Proof. Let $\mathcal{M} := \pi(G)''$ be the von Neumann algebra generated by $\pi(G)$. Since $G \trianglelefteq G^\sharp$ is normal, $\mathcal{M}$ is invariant under conjugation with the unitary one-parameter group $U_t := \pi^\sharp(\mathbf{1}, t)$, so that Corollary 2.4 provides a factorization $U_t = V_tW_t$ with

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strongly continuous one-parameter groups (V_t) in $\mathcal{M}$ and (W_t) in $\mathcal{M}' = \pi(G)'$. It follows in particular that $U_t \in \mathcal{M} \cdot W_t$, so that

$$\pi^\sharp(G^\sharp)' = \pi(G)' \cap U'_{\mathbb{R}} = \mathcal{M}' \cap U'_{\mathbb{R}} \supseteq W_{\mathbb{R}}.$$

If $\pi^\sharp$ is irreducible, $W_t \in \pi^\sharp(G^\sharp)' = \mathbb{C}\mathbf{1}$ for $t \in \mathbb{R}$. This implies that $U_t = V_t W_t \in \mathcal{M}$, which finally leads to $\mathcal{M} = \pi^\sharp(G^\sharp)'' = B(\mathcal{H})$, i.e., $\pi(G)' = \mathcal{M}' = \mathbb{C}\mathbf{1}$, which means that π is irreducible. □

REMARK 2.6. If $(\pi, \mathcal{H})$ is an irreducible representation of G which has some extension $\pi^\sharp$ to $G^\sharp = G \rtimes_\alpha \mathbb{R}$, then Schur's Lemma implies that this extension is unique up to a unitary character of $\mathbb{R}$. For each $\mu \in \mathbb{R}$, we obtain a modified unitary representation

$$\pi^\sharp_\mu(g, t) = e^{it\mu}\pi^\sharp(g, t).$$

The set of all those $\mu \in \mathbb{R}$ for which $\pi^\sharp_\mu$ is a positive energy representation is an interval which is either empty or of the form $[\mu_0, \infty[$ for some $\mu_0 \in \mathbb{R}$. In the latter case $\mu_0 = -\inf \operatorname{Spec}(-i\mathrm{d}\pi^\sharp(0, 1))$ leads to a representation for which $\inf \operatorname{Spec}(-i\mathrm{d}\pi^\sharp_{\mu_0}(0, 1)) = 0$. We call this the *minimal positive energy extension* of π.

Since the group $G^\sharp$ has richer structure than G itself, the crucial advantage of Theorem 2.5 is that it permits us to study certain irreducible representations of G as representations of $G^\sharp$ (cf. [**14**, **17**]).

3. Some Lie group examples. In this section G denotes a Lie group modelled on a locally convex space (see [**12**] for more details). If G is a Lie group and the action α of $\mathbb{R}$ on G is smooth, the group $G^\sharp$ also is a Lie group, so that it makes sense to consider unitary representations $(\pi^\sharp, \mathcal{H})$ of $G^\sharp$ which are smooth in the sense that the subspace $\mathcal{H}^\infty$ of smooth vectors is dense in $\mathcal{H}$. The following lemma, applied with $y = (0, 1) \in \mathfrak{g}^\sharp = \mathfrak{g} \rtimes \mathbb{R}$, the Lie algebra of $G^\sharp$, shows that, in some situations the positive energy condition leads to serious restrictions on the corresponding smooth representation $(\pi, \mathcal{H})$.

LEMMA 3.1. ([**14**, Lemma 5.12]) Let $(\pi, \mathcal{H})$ be a smooth representation of a *Lie group G with exponential function* $\exp\colon \mathfrak{g} \to G$ *and* $x, y \in \mathfrak{g}$ *with* $[x, [x, y]] = 0$. *If* $-i\mathrm{d}\pi(y)$ *is bounded from below, then* $\mathrm{d}\pi([x, y]) = 0$.

With the preceding lemma, one can show in particular that:

THEOREM 3.2. *For a connected Lie group K with Lie algebra $\mathfrak{k}$, let G be the* 1*-connected covering of the connected Lie group $C^\infty(\mathbb{T}, K)_0$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, of smooth contractible loops in K, whose Lie algebra is $\mathfrak{g} = C^\infty(\mathbb{T}, \mathfrak{k})$. Writing $[t] := t + \mathbb{Z}$ for elements of $\mathbb{T}$, we obtain an action of $\mathbb{T}$ on G corresponding to the action on $\mathfrak{g}$ by*

$$(\alpha_{[t]}\xi)([s]) = \xi([s - t]).$$

Then the restriction of every smooth positive energy representations of $G^\sharp = G \rtimes_\alpha \mathbb{R}$ to G factors through a representations of an abelian quotient group of K and annihilate(s) the commutator group of G. Conversely, all such representations are positive energy representations for trivial reasons.

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Proof. (a) First we consider the case $G = C^\infty(\mathbb{T}, \mathbb{R})$. Applying Lemma 3.1 with $y = (0, 1)$ and $x \in \mathfrak{g}$, it follows that $x' = [y, x] \in \ker(\mathtt{d}\pi)$, so that

$$\left\{\xi \in \mathfrak{g} = C^\infty(\mathbb{T}, \mathbb{R}) \colon \int_{\mathbb{T}} \xi(t)\, dt = 0\right\} \subseteq \ker(\mathtt{d}\pi).$$

(b) In the general case, we obtain for each $x \in \mathfrak{k}$ a smooth homomorphism

$$\Gamma_x \colon C^\infty(\mathbb{T}, \mathbb{R}) \to C^\infty(\mathbb{T}, K)_0, \quad \Gamma_x(f)(t) := \exp(f(t)x),$$

which lifts to a homomorphism $\widetilde{\Gamma}_x \colon C^\infty(\mathbb{T}, \mathbb{R}) \to G$. As $C^\infty(\mathbb{T}, \mathbb{R}) \otimes \mathfrak{k}$ is dense in $\mathfrak{g}$, we obtain from (a) that

$$\left\{\xi \in \mathfrak{g} = C^\infty(\mathbb{T}, \mathfrak{k}) \colon \int_{\mathbb{T}} \xi(t)\, dt = 0\right\} \subseteq \ker(\mathtt{d}\pi).$$

For $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $\xi_1(t) = \cos t \cdot x$, $\xi_2(t) = \sin t \cdot x$, $\xi_3(t) = \cos t \cdot y$ and $\xi_4(t) = \sin t \cdot y$, the relation

$$[\xi_1, \xi_3] + [\xi_2, \xi_4] = [x, y]$$

now shows that $[\mathfrak{k}, \mathfrak{k}] \subseteq \ker(\mathtt{d}\pi)$, which leads to

$$[\mathfrak{g}, \mathfrak{g}] \subseteq \left\{\xi \in \mathfrak{g} = C^\infty(\mathbb{T}, \mathfrak{k}) \colon \int_{\mathbb{T}} \xi(t)\, dt \in \overline{[\mathfrak{k}, \mathfrak{k}]}\right\} \subseteq \ker(\mathtt{d}\pi).$$

Therefore, the derived representation $\mathtt{d}\pi|_{\mathfrak{g}}$ factors through a representation of the abelian quotient algebra $\mathfrak{g} \to \mathfrak{k}/\overline{[\mathfrak{k}, \mathfrak{k}]}$, $\xi \mapsto \int_{\mathbb{T}} \xi(t)\, dt + \overline{[\mathfrak{k}, \mathfrak{k}]}$. □

For the following theorem, we recall that a Lie group K is called *regular* if, for each smooth map $\xi \colon [0, 1] \to \mathfrak{k}$, the Lie algebra of K, there exists a smooth curve $\gamma_\xi \colon [0, 1] \to K$ with $\gamma_\xi(0) = \mathbf{1}$ and $\gamma_\xi'(t) = \gamma_\xi(t) \cdot \xi'(t)$ (here $\cdot$ refers to the canonical action $K \times TK \to TK$) and the map $\mathrm{evol} \colon C^\infty([0, 1], \mathfrak{k}) \to G, \xi \mapsto \gamma_\xi(1)$ is smooth. For any regular Lie group K, the parametrization of smooth curves $\gamma \colon \mathbb{R} \to K$ by their logarithmic derivatives $\xi \colon \mathbb{R} \to \mathfrak{k}$ leads to a natural Lie group structure on the group $G := C^\infty(\mathbb{R}, K)$, endowed with the pointwise multiplication (see [**15**] for details).

THEOREM 3.3. *Let K be a connected regular Lie group with Lie algebra $\mathfrak{k}$ and $G := C^\infty(\mathbb{R}, K)$. Then, $(\alpha_t f)(s) = f(s - t)$ defines a smooth action of $\mathbb{R}$ on G. All smooth positive energy representations of $G^\sharp = G \rtimes_\alpha \mathbb{R}$ are trivial on G.*

Proof. The infinitesimal generator of the derived action of $\mathbb{R}$ on $\mathfrak{g}$ is $Df := \frac{d}{dt}|_{t=0} \mathbf{L}(\alpha_t)f = -f'$. For $y = (0, 1)$ and $f(\mathbb{R}) \subseteq \mathbb{R}x$ for some $x \in \mathfrak{k}$, we derive from Lemma 3.1 that $\mathtt{d}\pi(f) = 0$ for every positive energy representation of $G^\sharp$. Therefore, $C^\infty(\mathbb{R}) \otimes \mathfrak{k} \subseteq \ker \mathtt{d}\pi$, and since this subspace of $\mathfrak{g}$ is dense, we obtain $\mathtt{d}\pi(\mathfrak{g}) = 0$. □

4. Generating the topology on projective space. In this section, we discuss the topology of the projective space $\mathbb{P}(\mathcal{H})$. Our main goal is a criterion for a subset $E \subseteq \mathcal{H}$ such that the topology on $\mathbb{P}(\mathcal{H})$ is initial with respect to the functions $h_v([w]) := |\langle v, w\rangle|$, where we represent $[w] \in \mathbb{P}(\mathcal{H})$ by a unit vector w. It suffices that the corresponding functions $\ell_v(A) := \langle Av, v\rangle$ separate the points on the space of hermitian trace class operators. Here we use the topological embedding $\mathbb{P}(\mathcal{H}) \hookrightarrow$

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$\mathrm{Herm}_1(\mathcal{H}), [w] \mapsto P_w$ of $\mathbb{P}(\mathcal{H})$ in terms of rank-one projections into the space C of positive trace class operators A with $\mathrm{tr}(A) \leq 1$. It is crucial to our argument that, if $\mathcal{H}$ is infinite-dimensional, then C can be considered as a compactification of $\mathbb{P}(\mathcal{H})$, so that compactness arguments can be used.

DEFINITION 4.1. Let $\mathcal{H}$ be a complex Hilbert space, endow its unit sphere $\mathbb{S}(\mathcal{H}) := \{v \in \mathcal{H} : \|v\| = 1\}$ with the subspace topology inherited from $\mathcal{H}$ and the projective space $\mathbb{P}(\mathcal{H}) \cong \mathbb{S}(\mathcal{H})/\mathbb{T}$ with the quotient topology. We denote its elements, the one-dimensional subspaces of $\mathcal{H}$, by $[v] = \mathbb{C}v$.

Since the sphere inherits a natural metric from $\mathcal{H}$, it is instructive to first take a closer look on the metric aspects of the topology on $\mathbb{P}(\mathcal{H})$:

LEMMA 4.2. (a) *The metric $d(x, y) = \|x - y\|$ on the sphere $\mathbb{S}(\mathcal{H})$ induces on $\mathbb{P}(\mathcal{H})$ the metric*

$$d([x], [y]) := d(\mathbb{T}x, \mathbb{T}y) = \sqrt{2(1 - |\langle x, y\rangle|)} \in [0, \sqrt{2}], \quad x, y \in \mathbb{S}(\mathcal{H}).$$

(b) *The map $\iota\colon \mathbb{P}(\mathcal{H}) \hookrightarrow \mathrm{Herm}_1(\mathcal{H}), [v] \mapsto P_v$, is a topological embedding.*

Proof. (a) follows from $d(\mathbb{T}x, \mathbb{T}y) = d(x, \mathbb{T}y) = \inf_{|t|=1} d(x, ty)$ and

$$\inf_{|t|=1} d(x, ty)^2 = \inf_{|t|=1} \|x - ty\|^2 = \inf_{|t|=1} 2(1 - \mathrm{Re}\langle x, ty\rangle) = 2(1 - |\langle x, y\rangle|).$$

(b) For $v, w \in \mathbb{S}(\mathcal{H})$ with $[v] \neq [w]$, the operator $A := P_v - P_w$ is hermitian of rank 2 with $\mathrm{tr}(A) = \|v\|^2 - \|w\|^2 = 0$. Write $w = \lambda v + \mu v'$ with $v' \in \mathbb{S}(\mathcal{H})$ orthogonal to v. Then

$$P_w = |\lambda|^2 P_v + |\mu|^2 P_{v'} + \lambda\overline{\mu} P_{v,v'} + \overline{\lambda}\mu P_{v',v}.$$

On the subspace $\mathcal{H}_0 = \mathbb{C}v + \mathbb{C}w$, we therefore have

$$\det(A|_{\mathcal{H}_0}) = (1 - |\lambda|^2)(-|\mu|^2) - |\lambda|^2|\mu|^2 = -|\mu|^2.$$

As $\mathrm{tr}(A|_{\mathcal{H}_0}) = 0$, the two eigenvalues of $A|_{\mathcal{H}_0}$ are $\pm|\mu|$, which leads to

$$\|P_v - P_w\|_1 = 2|\mu| = 2\sqrt{1 - |\lambda|^2} = 2\sqrt{1 - |\langle v, w\rangle|^2}.$$

This implies the assertion. □

Now we turn from the metric point of view to families of functions generating the topology.

DEFINITION 4.3. For $v \in \mathcal{H}$ we define the functions

$$e_v\colon \mathcal{H} \to \mathbb{C}, \quad e_v(x) := \langle v, x\rangle$$

and

$$h_v\colon \mathcal{H}/\mathbb{T} \to \mathbb{C}, \quad h_v([x]) := |\langle v, x\rangle| \quad \text{for} \quad [x] = \mathbb{T}x.$$

REMARK 4.4. (a) The topology on the sphere $\mathbb{S}(\mathcal{H})$ is the initial topology defined by the functions e_v, $\|v\| = 1$, because it coincides with the weak topology. In fact, for

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$x, y \in \mathbb{S}(\mathcal{H})$, we have

$$\|x - y\|^2 = 2(1 - \mathrm{Re}\langle x, y\rangle).$$

Let $E \subseteq \mathcal{H}$ be a total subset, i.e. span E is dense in $\mathcal{H}$. We consider on $\mathbb{S}(\mathcal{H})$ the initial topology τ_E defined by the functions $\{e_v : v \in E\}$. Then

$$\{w \in \mathcal{H} : e_w \in C(\mathbb{S}(\mathcal{H}), \tau_E)\}$$

is a closed subspace of $\mathcal{H}$ because $\|e_w\|_\infty \leq \|w\|$. Since this subspace contains E, it coincides with $\mathcal{H}$. Therefore, τ_E coincides with the metric topology on $\mathbb{S}(\mathcal{H})$.

(b) From the formula for the quotient metric in Lemma 4.2(a), it follows immediately that the topology on $\mathbb{P}(\mathcal{H})$ is the initial topology defined by the functions h_v, $\|v\| = 1$.

Remark 4.5. It is an interesting question which conditions we have to require from a subset $E \subseteq \mathcal{H}$ to ensure that $h_E := \{h_v : v \in E\}$ defines the topology on $\mathbb{P}(\mathcal{H})$. It is certainly necessary that h_E separates the points of $\mathbb{P}(\mathcal{H})$. For any proper orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, this rules out subsets $E \subseteq \mathcal{H}_1 \cup \mathcal{H}_2$ because then h_E cannot separate the elements $[v_1 + \zeta v_2] \in \mathbb{P}(\mathcal{H})$, $\zeta \in \mathbb{T}$, where $0 \neq v_j \in \mathcal{H}_j$, $j = 1, 2$. This implies in particular that (for $\dim \mathcal{H} > 1$) E needs to be total and that it cannot be decomposed into two proper mutually orthogonal subsets. However, this condition is not sufficient for h_E to separate the points of $\mathbb{P}(\mathcal{H})$. For $\mathcal{H} = \mathbb{C}^2$ and a non-orthogonal basis $v_1, v_2 \in \mathbb{C}^2$, the functions h_{v_1} and h_{v_2} do not separate the points of $\mathbb{P}(\mathcal{H})$. The level sets of both functions are families of circles on the Riemann sphere $\mathbb{P}(\mathcal{H}) \cong \mathbb{S}^2$ and two such circles can intersect in two points. Geometrically this means that a ray $\mathbb{C}v \subseteq \mathbb{C}^2$ is not determined by the two numbers $|\langle v, v_1\rangle|$ and $|\langle v, v_2\rangle|$. From the same reasoning it follows that, if $v_1, v_2, v_3 \in \mathbb{C}^2$ are such that the corresponding rays $[v_1], [v_2], [v_3] \in \mathbb{P}(\mathbb{C}^2) \cong \mathbb{S}^2$ do not lie on any great circle, then the functions h_{v_j}, $j = 1, 2, 3$, separate the points of $\mathbb{P}(\mathbb{C}^2)$.

Problem 4.6. Suppose that h_E separates the points of $\mathbb{P}(\mathcal{H})$. Is the topology on $\mathbb{P}(\mathcal{H})$ the initial topology with respect to the set h_E?

Below, we prove a slightly weaker statement, which requires h_E to separate the functions on a slightly larger set that we introduce below.

Definition 4.7. Let

$$\overline{\mathbb{S}}(\mathcal{H}) := \{v \in \mathcal{H} : \|v\| \leq 1\}$$

denote the closed unit ball in $\mathcal{H}$, endowed with the weak topology, with respect to which it is a compact space. Since the scalar multiplication action $\mathbb{T} \times \overline{\mathbb{S}}(\mathcal{H}) \to \overline{\mathbb{S}}(\mathcal{H})$ is continuous and $\mathbb{T}$ is compact, we obtain on the quotient space

$$\overline{\mathbb{P}}(\mathcal{H}) := \overline{\mathbb{S}}(\mathcal{H})/\mathbb{T}$$

a compact Hausdorff topology and a topological embedding $\mathbb{P}(\mathcal{H}) \hookrightarrow \overline{\mathbb{P}}(\mathcal{H})$.

Remark 4.8. (a) If $\mathcal{H}$ is infinite-dimensional, then $\mathbb{S}(\mathcal{H})$ is dense in $\overline{\mathbb{S}}(\mathcal{H})$, so that we may consider the ball as a compactification of the unit sphere. Likewise, $\overline{\mathbb{P}}(\mathcal{H})$ is a compactification of $\mathbb{P}(\mathcal{H})$. If $\mathcal{H}$ is finite-dimensional, the unit sphere and the projective space are compact.

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(b) If X is a compact (Hausdorff) space and $\mathcal{E} \subseteq C(X, \mathbb{C})$ is a point separating set of continuous functions, then the topology on X coincides with the initial topology defined by $\mathcal{E}$ because the map $X \to \mathbb{C}^{\mathcal{E}}, x \mapsto (f(x))_{f\in\mathcal{E}}$ is a topological embedding.

PROPOSITION 4.9. *If $E \subseteq \mathcal{H}$ is a subset for which the functions $h_v([x]) := |\langle v, x\rangle|$, $v \in E$, separate the points of $\overline{\mathbb{P}}(\mathcal{H})$, then the topology on $\mathbb{P}(\mathcal{H})$ is the initial topology with respect to the family $(h_v)_{v\in E}$. This is in particular the case for $E = \mathcal{H}$.*

Proof. (a) Since the functions $x \mapsto |\langle x, v\rangle|$ on $\overline{\mathbb{S}}(\mathcal{H})$ are continuous, the function h_v defines a continuous function on $\overline{\mathbb{P}}(\mathcal{H})$. In view of Remark 4.8(b), the topology on the compact space $\overline{\mathbb{P}}(\mathcal{H})$ is initial with respect to the functions $(h_v)_{v\in E}$. Now the first assertion follows from the fact that $\mathbb{P}(\mathcal{H})$ is a topological subspace of $\overline{\mathbb{P}}(\mathcal{H})$.

(b) To verify the second assertion, we show that, for $E = \mathbb{S}(\mathcal{H})$, the functions $(h_v)_{v\in E}$ separate the points of $\overline{\mathbb{P}}(\mathcal{H})$. In fact, they obviously separate 0 from the non-zero elements in $\overline{\mathbb{S}}(\mathcal{H})$. Moreover, $\|x\| = \sup_{\|v\|=1} h_v(x)$ implies that they determine the norm of an element. If $h_v(x) = h_v(y)$ holds for two non-zero elements $x, y \in \overline{\mathbb{S}}(\mathcal{H})$ and all $v \in \mathbb{S}(\mathcal{H})$, we obtain for $v = x/\|x\|$ the relation

$$\|x\| = h_v(x) = h_v(y) = \frac{|\langle x, y\rangle|}{\|x\|} \quad \text{and likewise} \quad \|y\| = \frac{|\langle x, y\rangle|}{\|y\|},$$

so that

$$|\langle x, y\rangle| = \|x\| \cdot \|y\|.$$

We conclude that $y \in \mathbb{C}x$, and since the preceding discussion also implies that $\|y\| = \|x\|$, it follows that $y \in \mathbb{T}x$, i.e., $[x] = [y]$ in $\overline{\mathbb{P}}(\mathcal{H})$. □

EXAMPLE 4.10. Not every family $(h_v)_{v\in E}$ which separates the points of $\mathbb{P}(\mathcal{H})$ also separates the points of $\overline{\mathbb{P}}(\mathcal{H})$. A simple example arises for $\dim \mathcal{H} = 1$, where $E = \emptyset$ suffices to separate the points of the one point set $\mathbb{P}(\mathcal{H})$, but this is not enough for the interval $\overline{\mathbb{P}}(\mathcal{H})$.

For $\mathcal{H} = \mathbb{C}^2$, we have seen in Remark 4.5 that, if v_1, v_2, v_3 are three unit vectors for which the corresponding rays $[v_1], [v_2], [v_3] \in \mathbb{P}(\mathbb{C}^2) \cong \mathbb{S}^2$ do not lie on a great circle, the functions $h_{v_j}, j = 1, 2, 3$, separate the points of $\mathbb{P}(\mathbb{C}^2)$. Since the space $\mathrm{Herm}_2(\mathbb{C})$ of hermitian 2×2-matrices is four-dimensional, there exists a non-zero matrix $A \in \mathrm{Herm}_2(\mathbb{C})$ with

$$\langle Av_j, v_j\rangle = 0 \quad \text{for} \quad j = 1, 2, 3.$$

If A is positive or negative semidefinite, then these relations imply $Av_j = 0$ for $j = 1, 2, 3$, and since v_j are linearly independent, this contradicts $A \neq 0$. Therefore, A has eigenvalues $\lambda_1 < 0 < \lambda_2$. We assume without loss of generality that $|\lambda_j| \leq 1$ for $j = 1, 2$ and write $u_1, u_2 \in \mathbb{C}^2$ for corresponding unit eigenvectors of A with $Au_j = \lambda_j u_j$. From $A = \lambda_1 P_{u_1} + \lambda_2 P_{u_2}$ we then obtain for $w_j := \sqrt{|\lambda_j|}u_j$ the relation $A = P_{w_2} - P_{w_1}$, and thus

$$0 = \langle Av_j, v_j\rangle = |\langle w_2, v_j\rangle|^2 - |\langle w_1, v_j\rangle|^2$$

implies that the functions h_{v_j} do not separate the two elements $[w_1]$ and $[w_2]$ in $\overline{\mathbb{P}}(\mathbb{C}^2)$.

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We have already seen in Lemma 4.2(b) that we have a topological embedding $\eta\colon \mathbb{P}(\mathcal{H}) \hookrightarrow \mathrm{Herm}_1(\mathcal{H}), [v] \mapsto P_v$. The subset

$$C := \{A = A^* \in \mathrm{Herm}_1(\mathcal{H})\colon 0 \leq A, \mathrm{tr}\, A \leq 1\}$$

of $\mathrm{Herm}_1(\mathcal{H})$ is convex, bounded and weak-$*$-closed if we consider $B_1(\mathcal{H})$ via the trace pairing as the dual space of the space $K(\mathcal{H})$ of compact operators on $\mathcal{H}$. We conclude that C is a weak-$*$-compact subset. Next, we observe that η extends to a map

$$\eta\colon \overline{\mathbb{P}}(\mathcal{H}) \hookrightarrow C, \quad [v] \mapsto P_v.$$

To see that this map is continuous, we first recall that the subset $\{P_v\colon v \in \mathcal{H}\} \subseteq K(\mathcal{H})$ spans a dense subspace, which implies that the topology on C is the initial topology with respect to the functions

$$\ell_v\colon C \to \mathbb{C}, \quad A \mapsto \mathrm{tr}(AP_v) = \mathrm{tr}(P_{Av,v}) = \langle Av, v\rangle \tag{2}$$

(Remark 4.8(b)). Therefore

$$\ell_v(P_w) = \langle P_w v, v\rangle = \langle v, w\rangle\langle w, v\rangle = |\langle v, w\rangle|^2$$

shows that all function $\ell_v \circ \eta$ are continuous (cf. Definition 4.3), so that $\eta\colon \overline{\mathbb{P}}(\mathcal{H}) \to C$ is continuous. From $\|P_v\| = \|v\|^2$ it further follows that η is injective, hence a topological embedding of $\overline{\mathbb{P}}(\mathcal{H})$ onto a weak-$*$-compact subset of C.

This leads to the following criterion.

PROPOSITION 4.11. *If $E \subseteq \mathcal{H}$ is such that the functions $(\ell_v)_{v\in E}$ separate the points of* $\mathrm{Herm}_1(\mathcal{H})$*, then $(h_v)_{v\in E}$ separates the points of $\overline{\mathbb{P}}(\mathcal{H})$. In particular, the topology on $\mathbb{P}(\mathcal{H})$ is the initial topology with respect to $(h_v)_{v\in E}$.*

LEMMA 4.12. *For a subset $E \subseteq \mathcal{H}$, the functions $(\ell_v)_{v\in E}$ separate the points of C if and only if $\{P_v\colon v \in E\}$ spans a dense subspace of $K(\mathcal{H})$.*

Proof. The set $\{P_v\colon v \in E\}$ is not total in $K(\mathcal{H})$, i.e., its closed span is a proper subspace, if and only if there exists a hermitian trace class operator $0 \neq A \in \mathrm{Herm}_1(\mathcal{H}) \cong K(\mathcal{H})'$ with $\langle Av, v\rangle = 0$ for every $v \in E$.

Writing $A = A_+ - A_-$ with positive operators $A_\pm$, we find a $\lambda > 0$ such that $\mathrm{tr}(\lambda A_\pm) < 1$. Then $\lambda A_\pm \in C$ satisfy $\langle A_+ v, v\rangle = \langle A_- v, v\rangle$ for every $v \in E$. Therefore, $(\ell_v)_{v\in E}$ does not separate the points of C.

If, conversely, $A_\pm \in C$ are two different operators not separated by the functions $(\ell_v)_{v\in E}$, then $A := A_+ - A_- \in \mathrm{Herm}_1(\mathcal{H})$ is non-zero with $\mathrm{tr}(AP_v) = 0$ for every $v \in E$. Therefore, $\{P_v\colon v \in E\}$ is not total. □

REMARK 4.13. If $\dim \mathcal{H} = 2$, then $\dim \mathrm{Herm}_1(\mathcal{H}) = \dim \mathrm{Herm}_2(\mathbb{C}) = 4$, so that any subset $E \subseteq \mathcal{H}$ for which $(\ell_v)_{v\in E}$ separate the points of $\mathrm{Herm}_1(\mathcal{H})$ has to contain at least 4 elements (cf. Example 4.10).

The following criterion is sometimes useful to verify the condition in Lemma 4.12.

PROPOSITION 4.14. *If M is a connected complex manifold and $F\colon M \to \mathcal{H}$ a holomorphic map with total range, then $(\ell_{F(m)})_{m\in M}$ separates the points of* $\mathrm{Herm}_1(\mathcal{H})$.

Proof. Let $\overline{M}$ denote the real manifold M, endowed with the opposite complex structure. Then, for each $A \in \mathrm{Herm}_1(\mathcal{H})$, the function $\alpha_A(m, n) := \langle AF(m), F(n)\rangle$ on

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$M \times \overline{M}$ is holomorphic. If $\ell_{F(m)}(A) = \langle AF(m), F(m)\rangle = 0$ for every $m \in M$, then the holomorphic function α_A vanishes on the totally real submanifold $\Delta_M = \{(m, m): m \in M\}$ of $M \times \overline{M}$, which implies that $\alpha_A = 0$ (cf. [**11**, Proposition A.III.7]). We conclude that for each $m \in M$, $AF(m) \in \mathrm{im}(F)^\perp = \{0\}$, and since $\mathrm{im}(F)$ is total, it follows that $A = 0$. □

The preceding proposition applies in particular to all reproducing kernel Hilbert spaces of holomorphic functions or holomorphic sections (cf. [**17**, **13**]).

5. Continuity of projective unitary representations. It is well known that a unitary representation $\pi : G \to \mathrm{U}(\mathcal{H})$ of a topological group is continuous if and only if, for each v in a total subset $E \subseteq \mathcal{H}$, the function $\pi^{v,v}(g) := \langle \pi(g)v, v\rangle$ is continuous (cf. [**11**, Lemma VI.1.3]). In this section, we discuss a similar continuity criterion for projective unitary representations $\pi : G \to \mathrm{PU}(\mathcal{H})$.

PROPOSITION 5.1. (a) *The topology on* $\mathrm{PU}(\mathcal{H})$ *is the coarsest topology for which all functions*

$$h_{v,w} \colon \mathrm{PU}(\mathcal{H}) \to \mathbb{R}, \quad [g] \mapsto |\langle gv, w\rangle|, \quad v, w \in \mathcal{H},$$

are continuous.

(b) *The quotient map* $q\colon \mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H})$ *has continuous local sections, i.e. each* $[g] \in \mathrm{PU}(\mathcal{H})$ *has an open neighbourhood* U *on which there exists a continuous section* $\sigma : U \to \mathrm{U}(\mathcal{H})$ *of* q.

Proof. (a) Let $q\colon \mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H})$ denote the quotient map. Then all functions $f_{v,w} := h_{v,w} \circ q$ are continuous on $\mathrm{U}(\mathcal{H})$, which implies that the functions $h_{v,w}$ are continuous on $\mathrm{PU}(\mathcal{H})$.

Let τ denote the coarsest topology on $\mathrm{PU}(\mathcal{H})$ for which all functions $h_{v,w}$ are continuous. We know already that this topology is coarser than the quotient topology. Next we observe that the relations

$$h_{v,w}([g][g']) = h_{g'v,w}([g]) = h_{v,g^{-1}w}([g'])$$

imply that left and right multiplications are continuous in τ. To see that τ coincides with the quotient topology, it therefore remains to see that $[g_i] \to \mathbf{1}$ in τ implies that $[g_i] \to \mathbf{1}$ in the quotient topology.

For a net $([g_i])_{i\in I}$ in $\mathrm{PU}(\mathcal{H})$ we consider a lift $(g_i)_{i\in I}$ in $\mathrm{U}(\mathcal{H})$. Since the closed operator ball $\mathcal{B} := \{A \in B(\mathcal{H}): \|A\| \leq 1\}$ is compact in the weak operator topology, there exists a convergent subnet $g_{\alpha(j)} \to g_0 \in \mathcal{B}$. For $v, w \in \mathcal{H}$, we then have

$$h_{v,w}([g_{\alpha(j)}]) \to h_{v,w}(\mathbf{1}) = |\langle v, w\rangle|$$

and also

$$h_{v,w}([g_{\alpha(j)}]) = |\langle g_{\alpha(j)}v, w\rangle| \to |\langle g_0 v, w\rangle|,$$

hence $|\langle g_0 v, w\rangle| = |\langle v, w\rangle|$. This implies, in particular, that for each non-zero vector v, we have

$$g_0 v \in (v^\perp)^\perp = \mathbb{C}v,$$

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so that each vector is an eigenvector, and this implies that $g_0 = t\mathbf{1}$ for some $t \in \mathbb{C}$. If $v = w$ is a unit vector, we obtain $|t| = |\langle g_0 v, v\rangle| = 1$. Therefore, we have $g_{\alpha(j)} \to t\mathbf{1}$ in $\mathrm{U}(\mathcal{H})$, and this implies that $[g_{\alpha(j)}] \to [\mathbf{1}]$ in $\mathrm{PU}(\mathcal{H})$.

If the net $(g_i)_{i\in I}$ does not converge to $\mathbf{1}$ in $\mathrm{PU}(\mathcal{H})$, then there exists an open $\mathbf{1}$-neighbourhood U for which the set $I_U := \{i \in I : g_i \notin U\}$ is cofinal, which leads to a subnet $(g_i)_{i\in I_U}$ converging to $\mathbf{1}$ in τ and contained in the closed subset U^c. Applying the preceding argument to this subnet now leads to a contradiction since it cannot have any subnet converging to $\mathbf{1}$ because U^c is closed.

(b) Since we can move sections with left multiplication maps, it suffices to assume that $g = \mathbf{1}$. Pick $0 \neq v_0 \in \mathcal{H}$. Then

$$\Omega := \{g \in \mathrm{U}(\mathcal{H}) : \langle gv_0, v_0\rangle \neq 0\}$$

is an open $\mathbf{1}$-neighbourhood in $\mathrm{U}(\mathcal{H})_s$ with $\Omega\mathbb{T} = \Omega$. Therefore, $\widetilde{\Omega} := \{[g] : g \in \Omega\}$ is an open $\mathbf{1}$-neighbourhood of $\mathrm{PU}(\mathcal{H})$. For each $g \in \Omega$ there exists a unique $t \in \mathbb{T}$ with

$$tg \in \Omega_+ := \{g \in \mathrm{U}(\mathcal{H}) : \langle gv_0, v_0\rangle > 0\}.$$

We now define a map

$$\sigma : \widetilde{\Omega} \to \Omega, \quad [g] \mapsto g \quad \text{for} \quad g \in \Omega_+.$$

To see that σ is continuous, it suffices to observe that the map

$$\Omega \to \Omega_+, \quad g \mapsto \frac{|\langle gv_0, v_0\rangle|}{\langle gv_0, v_0\rangle} g$$

is continuous and constant on the cosets of $\mathbb{T}$. Hence, it factors through a continuous map $\widetilde{\Omega} \to \Omega_+$ which is σ. Therefore, the quotient map

$$q : \mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H}), \quad g \mapsto [g]$$

has a continuous section in the $\mathbf{1}$-neighbourhood $\widetilde{\Omega}$ of $\mathrm{PU}(\mathcal{H})$. □

COROLLARY 5.2. *Let G be a topological group and $\pi : G \to \mathrm{PU}(\mathcal{H})$ be a group homomorphism. Then the following are equivalent:*

(i) *π is continuous.*
(ii) *For all $v, w \in \mathcal{H}$, the function $G \to \mathbb{R}, \mapsto |\langle \pi(g)v, w\rangle|$ is continuous.*
(iii) *For each $[v] \in \mathbb{P}(\mathcal{H})$, the orbit map $G \to \mathbb{P}(\mathcal{H}), g \mapsto \pi(g)[v]$ is continuous.*

Proof. The equivalence of (i) and (ii) is an immediate consequence of Proposition 5.1. Clearly, (ii) is equivalent to the requirement that, for any $0 \neq v \in \mathcal{H}$, the functions $g \mapsto h_w(\pi(g)[v])$, $w \in \mathcal{H}$, are continuous (cf. Definition 4.3). As the topology on $\mathbb{P}(\mathcal{H})$ is initial with respect to the functions $(h_w)_{w\in\mathcal{H}}$ (Proposition 4.9), the corollary follows. □

The equivalence of (i) and (iii) in the preceding corollary can already be found in [**18**, Lemma 8.1] (see also [**7**, Proposition 6]). The characterization of the continuity of a projective representation in Corollary 5.2 involves all elements $v \in \mathcal{H}$. This makes it inconvenient to use in practice. Below we develop a criterion which makes it much easier to check continuity.

DEFINITION 5.3. Let G be a topological group. For a homomorphism $\pi\colon G \to \mathrm{PU}(\mathcal{H})$, we write $\mathbb{P}(\mathcal{H})_c \subseteq \mathbb{P}(\mathcal{H})$ for the set of all elements $[v]$ for which the G-orbit map is continuous.

We now take a closer look at the structure of the set $\mathbb{P}(\mathcal{H})_c$. We start with an elementary observation on isometric actions of topological groups.

LEMMA 5.4. *If the topological group G acts isometrically by*

$$\sigma\colon G \times X \to X, \quad (g, x) \mapsto g.x$$

on the metric space (X, d), then the set X_c of all points with continuous orbit maps is closed and the G-action on this set is continuous.

Proof. If $x_n \to x$ in X, then the orbits maps $\sigma^{x_n}\colon G \to X$ converge uniformly to the orbit map σ^x. This proves that X_c is closed. The second assertion follows from

$$d(g_0.x_0, g_1.x_1) = d\left(g_1^{-1}g_0.x_0, x_1\right) \leq d\left(g_1^{-1}g_0.x_0, x_0\right) + d(x_0, x_1). \qquad \square$$

REMARK 5.5. For a unitary representation $\pi\colon G \to \mathrm{U}(\mathcal{H})$, one can also consider the subset $\mathcal{H}_c \subseteq \mathcal{H}$ of those elements with a continuous orbit map. This is obviously a subspace which is closed by Lemma 5.4. Now $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_c^\perp$ provides a decomposition of the unitary representation as a direct sum of a continuous representation and one without non-zero continuous orbit maps. We shall see below that the situation is more complicated in the projective case.

DEFINITION 5.6. On $\mathbb{P}(\mathcal{H})$ we now consider the Riemannian metric given by

$$d_R([x], [y]) := \arccos|\langle x, y\rangle| \in [0, \pi/2] \quad \text{for} \quad x, y \in \mathbb{S}(\mathcal{H}).$$

We write $[x]\perp[y]$ and say that $[x]$ and $[y]$ are *orthogonal* if $x\perp y$, which is equivalent to $d_R([x], [y]) = \pi/2$. If $[x]$ and $[y]$ are not orthogonal, then we write $[x]\sharp[y]$ for the unique metric midpoint of $[x]$ and $[y]$.

LEMMA 5.7. *The midpoint operation is a continuous map*

$$\sharp\colon \{([x], [y]) \in \mathbb{P}(\mathcal{H})^2 \colon \langle x, y\rangle \neq 0\} \to \mathbb{P}(\mathcal{H}).$$

Proof. In the following argument, we represent elements of $\mathbb{P}(\mathcal{H})$ by non-zero vectors not necessarily normalized to unit length. Recall that $\mathbb{P}(\mathcal{H})$ is a symmetric space with the point reflections given by

$$r_{[x]}([y]) := \left[-y + 2\frac{\langle y, x\rangle}{\|x\|^2}x\right].$$

For the midpoint $[z] = [x]\sharp[y]$ of two non-orthogonal rays $[x]$ and $[y]$ we then have

$$[y] = r_{[z]}([x]) = \left[-x + 2\frac{\langle x, z\rangle}{\|z\|^2}z\right].$$

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For $y = x + v$ and $z = x + w$ with $v, w \in x^\perp$ (here we normalize by $\langle y, x\rangle = \langle z, x\rangle = 1$), we then have

$$[x+v] = [y] = \left[-x + \frac{2}{\|x+w\|^2}(x+w)\right] = \left[-x + \frac{2}{1+\|w\|^2}(x+w)\right]$$
$$= \left[\frac{1-\|w\|^2}{1+\|w\|^2}x + \frac{2}{1+\|w\|^2}w\right] = \left[x + \frac{2}{1-\|w\|^2}w\right].$$

Note that this calculation actually shows that $\|w\| \neq 1$ because $[y] \neq [w]$. We further obtain

$$v = \frac{2}{1-\|w\|^2}w \quad \text{and} \quad \|v\| = \frac{2\|w\|}{1-\|w\|^2},$$

which in turn yields

$$\|w\| = \frac{\sqrt{1+\|v\|^2}-1}{\|v\|}$$

and hence

$$w = \frac{\sqrt{1+\|v\|^2}-1}{\|v\|^2}v.$$

This argument also implies that w is uniquely determined by x and y and that the midpoint operation is continuous in each argument separately.

We now show that it is continuous. Pick $[x_0] \in \mathbb{P}(\mathcal{H})$ and let U be an open neighbourhood of $[x_0]$ for which there exists a continuous map $\sigma: U \to \mathrm{U}(\mathcal{H})$ with $\sigma_u[x_0] = u$ for $u \in U$. For $\langle x_0, y_0\rangle \neq 0$ and $x_i \to x_0$, $y_i \to y_0$ we may without loss of generality assume that $\langle x_i, y_i\rangle \neq 0$ for every i. We then have for $[x_i] \in U$ the relation

$$[x_i]\sharp[y_i] = \sigma_{[x_i]}\left([x_0]\sharp\sigma_{[x_i]}^{-1}[y_i]\right) \to [x_0]\sharp[y_0]$$

because the action of $\mathrm{U}(\mathcal{H})$ on $\mathbb{P}(\mathcal{H})$ is continuous. □

LEMMA 5.8. *Let $\pi: G \to \mathrm{PU}(\mathcal{H})$ be a homomorphism. Then the set $\mathbb{P}(\mathcal{H})_c$ has the following properties:*

(i) *$\mathbb{P}(\mathcal{H})_c$ is closed.*

(ii) *If $[x], [y] \in \mathbb{P}(\mathcal{H})_c$ are not orthogonal, then $[x, y] := \mathbb{P}(\mathbb{C}x + \mathbb{C}y) \subseteq \mathbb{P}(\mathcal{H})_c$.*

(iii) *If $[x_0], \ldots, [x_n] \in \mathbb{P}(\mathcal{H})_c$ are such that $\langle x_j, x_{j+1}\rangle \neq 0$ for $j = 0, \ldots, n-1$, then $[x_0, \ldots, x_n] := \mathbb{P}(\mathrm{span}\{x_0, \ldots, x_n\}) \subseteq \mathbb{P}(\mathcal{H})_c$.*

Proof. (i) follows immediately from Lemma 5.4.

(ii) We obtain from Lemma 5.7 and the equivariance of the midpoint operation under $\mathrm{PU}(\mathcal{H})$ that, for two non-orthogonal elements $[x], [y] \in \mathbb{P}(\mathcal{H})_c$, we also have $[x]\sharp[y] \in \mathbb{P}(\mathcal{H})_c$.

Let

$$\mathrm{Exp}: T(\mathbb{P}(\mathcal{H})) \to \mathbb{P}(\mathcal{H})$$

denote the exponential map of the symmetric space $\mathbb{P}(\mathcal{H})$. For $[y] = \mathrm{Exp}(v)$ and $v \in T_{[x]}(\mathbb{P}(\mathcal{H}))$ with $\|v\| < \pi/2$, the whole geodesic arc $[y_t] = \mathrm{Exp}(tv)$, $0 \leq t \leq 1$, consists of

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elements not orthogonal to $[x]$. By successive dyadic division, we can generate a dense subset of this arc by the midpoint operation from $[x]$ and $[y]$. Therefore, $[x], [y] \in \mathbb{P}(\mathcal{H})_c$ implies that $[y_t] \in \mathbb{P}(\mathcal{H})_c$ for $0 \leq t \leq 1$.

Since Exp is $U(\mathcal{H})$-equivariant, we conclude that, for the action of G on the tangent bundle $T(\mathbb{P}(\mathcal{H}))$, the set $T(\mathbb{P}(\mathcal{H}))_c$ of G-continuous elements has the property that, if $v \in T(\mathbb{P}(\mathcal{H}))_c$ with $\|v\| < \pi/2$, then $[0, 1]v \subseteq T(\mathbb{P}(\mathcal{H}))_c$. Since G acts on $T(\mathbb{P}(\mathcal{H}))$ by bundle automorphisms, it also follows that, for each $[x] \in \mathbb{P}(\mathcal{H})_c$, the set $T_{[x]}(\mathbb{P}(\mathcal{H}))_c$ is a closed complex linear subspace. This implies that, for two non-orthogonal elements $[x], [y] \in \mathbb{P}(\mathcal{H})_c$, the whole projective plane $[x, y] \subseteq \mathbb{P}(\mathcal{H})$ consists of G-continuous vectors.

(iii) We argue by induction and observe that the case $n = 1$ follows from (ii). Assume that $n > 1$. Then the induction hypothesis implies that $[x_0, \dots, x_{n-1}] \subseteq \mathbb{P}(\mathcal{H})_c$. Since x_n is not orthogonal to this space, the set of all elements $[y] \in [x_0, \dots, x_{n-1}]$ with $\langle y, x_n \rangle \neq 0$ is open dense. Hence, the closedness of $\mathbb{P}(\mathcal{H})_c$ implies that $[x_0, \dots, x_n] \subseteq \mathbb{P}(\mathcal{H})_c$. □

Definition 5.9. We call a subset $E \subseteq \mathbb{P}(\mathcal{H})$ *indecomposable* if

$$E = E_1 \dot{\cup} E_2, \quad E_1 \perp E_2, \quad E_1 \neq \emptyset \quad \Rightarrow \quad E_2 = \emptyset,$$

i.e. E cannot be decomposed into two proper mutually orthogonal subsets.

Remark 5.10. On every subset $E \subseteq \mathbb{P}(\mathcal{H})$ we obtain an equivalence relation by $[x] \sim [y]$ if there exists a sequence $[x_0], \dots, [x_n] \in E$ with $[x_0] = [x]$, $[x_n] = y$ and $\langle x_j, x_{j+1} \rangle \neq 0$ for $j = 0, \dots, n-1$. The corresponding equivalence classes are the maximal indecomposable subsets of E. We call them the *indecomposable components of* E.

Theorem 5.11. (Structure theorem for $\mathbb{P}(\mathcal{H})_c$) *The indecomposable components of* $\mathbb{P}(\mathcal{H})_c$ *are of the form* $\mathbb{P}(\mathcal{H}_j)$, $j \in J$, *where* $(\mathcal{H}_j)_{j \in J}$ *is a family of mutually orthogonal closed subspaces of* $\mathcal{H}$. *In particular,*

$$\mathbb{P}(\mathcal{H})_c = \bigcup_{j \in J} \mathbb{P}(\mathcal{H}_j),$$

where each $\mathbb{P}(\mathcal{H}_j)$ *is closed and open in* $\mathbb{P}(\mathcal{H})_c$.

Proof. Let $C \subseteq \mathbb{P}(\mathcal{H})_c$ be an indecomposable component. Then, $\mathbb{P}(\mathcal{H})_c \subseteq C \dot{\cup} C^{\perp}$ and the closedness of $C^{\perp}$ already implies that C is relatively open in $\mathbb{P}(\mathcal{H})_c$. Since this is also true for the other indecomposable components, C is also relatively closed, hence closed in $\mathbb{P}(\mathcal{H})$ because $\mathbb{P}(\mathcal{H})_c$ is closed.

For $[x], [y] \in C$ we find $[x_0], \dots, [x_n] \in C$ with $[x] = [x_0], [y] = [x_n]$ and $\langle x_j, x_{j+1} \rangle \neq 0$ for $j = 0, \dots, n-1$. Therefore, Lemma 5.8(iii) implies that

$$[x, y] \subseteq [x_0, \dots, x_n] \subseteq \mathbb{P}(\mathcal{H})_c,$$

and since $[x, y]$ is indecomposable, $[x, y] \subseteq C$. This proves that $C = \mathbb{P}(\mathcal{K})$ for a linear subspace $\mathcal{K} \subseteq \mathcal{H}$. The closedness of C now implies that $\mathcal{K}$ is closed in $\mathcal{H}$.

If $\mathbb{P}(\mathcal{K}_1)$ and $\mathbb{P}(\mathcal{K}_2)$ are different indecomposable components of $\mathbb{P}(\mathcal{H})_c$, then obviously $\mathcal{K}_1 \perp \mathcal{K}_2$. This completes the proof. □

Corollary 5.12. *If* $\mathbb{P}(\mathcal{H})_c$ *is total and indecomposable, then* $\mathbb{P}(\mathcal{H})_c = \mathbb{P}(\mathcal{H})$.

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COROLLARY 5.13. *If $\mathbb{P}(\mathcal{K}) \subseteq \mathbb{P}(\mathcal{H})_c$ is an indecomposable component, then*

$$G_{\mathcal{K}} := \{g \in G : \pi(g)\mathbb{P}(\mathcal{K}) \subseteq \mathbb{P}(\mathcal{K})\}$$

is an open subgroup of G and the induced homomorphism $G_{\mathcal{K}} \to \mathrm{PU}(\mathcal{K})$ is continuous.

Proof. Let $[v] \in \mathbb{P}(\mathcal{K})$. Since $\mathbb{P}(\mathcal{K})$ is an open subset of the G-invariant subset $\mathbb{P}(\mathcal{H})_c$, the subset $U := \{g \in G : \pi(g)[v] \in \mathbb{P}(\mathcal{K})\}$ is open. Since G also permutes the indecomposable components of $\mathbb{P}(\mathcal{H})_c$, any $g \in U$ preserves $\mathbb{P}(\mathcal{K})$, so that $U = G_{\mathcal{K}}$. The continuity of the projective representation of $G_{\mathcal{K}}$ on $\mathcal{K}$ follows from Corollary 5.2. □

EXAMPLE 5.14. Let $\chi : \mathbb{R} \to \mathbb{T}$ be a discontinuous character and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ be a direct sum of two Hilbert spaces. Then, $\pi(t)(v_1 + v_2) := v_1 + \chi(t)v_2$ for $v_j \in \mathcal{H}_j$ defines a unitary representation on $\mathcal{H}$ with $\mathcal{H}_c = \mathcal{H}_1$, but for the corresponding projective representation we have $\mathbb{P}(\mathcal{H})_c = \mathbb{P}(\mathcal{H}_1)\dot\cup\mathbb{P}(\mathcal{H}_2)$.

THEOREM 5.15. (Continuity criterion for projective representations) *Let G be a connected topological group and $\pi : G \to \mathrm{PU}(\mathcal{H})$ be a projective unitary representation with cyclic ray $[v] \in \mathbb{P}(\mathcal{H})_c$. Then π is continuous.*

Proof. Since open subgroups of topological groups are also closed, the connectedness of G implies that it preserves all indecomposable components of $\mathbb{P}(\mathcal{H})_c$ (Corollary 5.13). Hence, $\pi(G)[v]$ lies in a single indecomposable component $\mathbb{P}(\mathcal{K})$ for a closed subspace $\mathcal{K} \subseteq \mathcal{H}$. As $[v]$ is cyclic, we find that $\mathcal{K} = \mathcal{H}$, hence $\mathbb{P}(\mathcal{H})_c = \mathbb{P}(\mathcal{H})$, so that π is continuous (Corollary 5.2). □

COROLLARY 5.16. *Let G be a connected topological group and $\pi : G \to \mathrm{PU}(\mathcal{H})$ be an irreducible projective unitary representation with $\mathbb{P}(\mathcal{H})_c \neq \emptyset$. Then π is continuous.*

THEOREM 5.17. *Let G and T be topological groups and $\alpha : T \to \mathrm{Aut}(G)$ be a homomorphism defining a continuous action of T on G. Suppose that G is connected and that $(\pi, \mathcal{H})$ is an irreducible continuous unitary representation of G for which $\pi \circ \alpha_t$ is equivalent to π for every $t \in T$. We then obtain a well-defined projective unitary representation $\overline{U} : T \to \mathrm{PU}(\mathcal{H})$, defined by*

$$U_t\pi(g)U_t^* = \pi(\alpha_t(g)) \quad \textit{for} \quad g \in G, t \in T,$$

where $U_t \in \mathrm{U}(\mathcal{H})$ is a unitary lift of $\overline{U}_t \in \mathrm{PU}(\mathcal{H})$. Then the following assertions hold:

(i) *$\overline{U}$ is continuous if there exists a ray $[v] \in \mathrm{PU}(\mathcal{H})$ with a continuous orbit map.*

(ii) *If $\{\ell_{\pi(g)v} : g \in G\}$ separates the points of $\mathrm{Herm}_1(\mathcal{H})$, then $[v]$ has a continuous orbit map under $\overline{U}$ if and only if all functions*

$$T \to \mathbb{R}, \quad t \mapsto |\langle U_t v, \pi(g)v\rangle|, \quad g \in G,$$

are continuous.

(iii) *If $\overline{U}$ is continuous, then*

$$\widetilde{T} := \{(t, U) \in T \times \mathrm{U}(\mathcal{H})_s : \overline{U}_t = \overline{U}\}, \quad p : \widetilde{T} \to T, (t, U) \mapsto t,$$

is a central $\mathbb{T}$-extension of T. By $\widetilde{\alpha}(t, U) := \alpha(t)$ we then obtain a topological group $\widetilde{G} := G \rtimes_{\widetilde{\alpha}} \widetilde{T}$ and $\widetilde{\pi}(g, (t, U)) := \pi(g)U$ defines a continuous0 unitary representation of $\widetilde{G}$ on $\mathcal{H}$ extending π.

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(iv) *If, in addition, the central extension* $p\colon \widetilde{T} \to T$ *splits, then* π *extends to a continuous unitary representation of* $G^\sharp = G \rtimes_\alpha T$*. This is always the case for* $T = \mathbb{R}$.

Proof. (i) Let $\mathbb{P}(\mathcal{H})_c$ denote the set of continuous rays for the projective representation $\overline{U}$. Since T acts continuously on G and π is continuous, the set $\mathbb{P}(\mathcal{H})_c$ is G-invariant because

$$\overline{U}_t[\pi(g)v] = (U_t\pi(g)U_t^*)\overline{U}_t[v] = \pi(\alpha_t(g))\overline{U}_t[v].$$

Suppose that $[v] \in \mathbb{P}(\mathcal{H})_c$. Then $\pi(G)[v] \subseteq \mathbb{P}(\mathcal{H})_c$ is total and indecomposable (because G is connected), so that Corollary 5.12 implies that $\mathbb{P}(\mathcal{H})_c = \mathbb{P}(\mathcal{H})$, i.e., that $\overline{U}$ is continuous.

(ii) follows from Proposition 4.11. Note that $|\langle U_t v, \pi(g)v\rangle|$ depends only on $\overline{U}_t$ and not on the choice of the unitary lift U_t.

(iii) Since $\widetilde{T}$ is the pullback $\overline{U}^*\,\mathrm{U}(\mathcal{H})$ of the central $\mathbb{T}$-extension $\mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H})$, it also is a central extension of topological groups, i.e., $p\colon \widetilde{T} \to T$ is a quotient map with continuous local sections. The rest of (iii) is clear.

(iv) If $\sigma : T \to \widetilde{T}$, $t \mapsto (t, U_t)$ is a continuous splitting of $\widetilde{T}$, then $\pi^\sharp(g, t) := \pi(g)U_t$ is a continuous extension of π to $G^\sharp$.

For $T = \mathbb{R}$, the discussion in [**10**, Chapter 9] implies that $\widetilde{T}$ actually is a Lie group and since one-parameter groups of quotients of Lie groups lift, there exists a continuous splitting $\sigma : T \to \widetilde{T}$. □

6. Extending representations to semidirect products. Let G be a topological group. Then there exists a W^*-algebra $W^*(G)$, together with a homomorphism $\eta\colon G \to \mathrm{U}(W^*(G))$ which is continuous with respect to the weak topology on $W^*(G)$, and which has the universal property that, for every continuous unitary representation $(\pi, \mathcal{H})$ of G, there exists a unique normal representation $\widetilde{\pi}\colon W^*(G) \to B(\mathcal{H})$ with $\widetilde{\pi} \circ \eta = \pi$ (cf. [**11**, Rem. IV.1.2], [**9**]). In the appendix of [**9**] it is shown that the predual $W^*(G)_*$ of this W^*-algebra can be identified with the subspace $B(G) \subseteq C_b(G)$ of bounded continuous functions on G, spanned by the convex cone $\mathcal{P}_c(G)$ of continuous positive definite functions on G. The natural map

$$\eta_*\colon W^*(G)_* \to B(G), \quad \eta_*(\varphi) := \varphi \circ \eta$$

is a linear bijection which is continuous with respect to $\|\cdot\|_\infty$ on $B(G)$.

Remark 6.1. In general the subspace $B(G)$ is not closed in $(C_b(G), \|\cdot\|_\infty)$, so that η_* is not an open map with respect to $\|\cdot\|_\infty$ on $B(G)$.

In fact, if G is locally compact abelian, Bochner's Theorem implies that $B(G)$ is the range of the Fourier transform $M(\widehat{G}) \to C(G)$ from the convolution algebra $M(\widehat{G})$ of finite complex Radon measures on $\widehat{G}$ to $C(G)$. For $G = \mathbb{T}$ and $\widehat{G} = \mathbb{Z}$, we have $M(\widehat{G}) \cong \ell^1(\mathbb{Z})$, so that $B(\mathbb{T})$ is the Wiener algebra of all continuous functions with absolutely convergent Fourier series. This algebra is dense in $C(\mathbb{T})$ and a proper subspace, hence not closed.

We endow $B(G)$ with the norm $\|\cdot\|$ that turns η_* into an isometry. According to [**9**, Prop. A.3], we then have $\|\varphi\| = \|\varphi\|_\infty = \varphi(\mathbf{1})$ for $\varphi \in \mathcal{P}_c(G)$, but Remark 6.1 implies that, in general, $\|\cdot\|$ is not equivalent to $\|\cdot\|_\infty$ because $B(G)$ may be incomplete w.r.t. $\|\cdot\|_\infty$.

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Let T be a topological group and $\alpha\colon T \to \mathrm{Aut}(G)$, $t \mapsto \alpha_t$ be a homomorphism defining a continuous action of T on G. We write $G^\sharp := G \rtimes_\alpha T$ for the corresponding semidirect product group. From the universal property of $W^*(G)$ we obtain a homomorphism $\widetilde{\alpha}\colon T \to \mathrm{Aut}(W^*(G))$ which is uniquely determined by $\widetilde{\alpha}_t \circ \eta = \eta \circ \alpha_t$ for every $t \in T$. This defines a W^*-*dynamical system* in the sense of Borchers (where no continuity of $\widetilde{\alpha}$ is required; cf. [**3**]). Let $B(G)_c \subseteq B(G)$ denote the set of all functions φ for which the map $T \to B(G)$, $t \mapsto \alpha_t^*\varphi = \varphi \circ \alpha_t$ is continuous. From [**3**, Proposition II.5] it follows that $B(G)_c$ is a closed subspace invariant under T which is generated by the convex cone $B(G) \cap \mathcal{P}_c(G)$.

THEOREM 6.2. (Characterization theorem) *Let $(\pi, \mathcal{H})$ be a continuous unitary representation of G and $F_\pi \subseteq B(G)$ the corresponding folium, i.e., the set of all functions of the form $\varphi_S(g) := \mathrm{tr}(\pi(g)S)$, where $S \in \mathrm{Herm}_1(\mathcal{H})$ is non-negative with $\mathrm{tr}(S) = 1$. Then the following are equivalent:*

(i) *π is quasi-equivalent to a representation $(\pi', \mathcal{H}')$ that extends to a continuous unitary representation $(\pi^\sharp, \mathcal{H}')$ of $G^\sharp = G \rtimes_\alpha T$.*

(ii) *F_π is T-invariant and contained in $B(G)_c$ (which implies that T acts continuously on F_π).*

Proof. This follows immediately from [**3**, Theorem III.2], applied to the W^*-dynamical system defined by $(\widetilde{\alpha}_t)_{t\in T}$ and the one-to-one correspondence between normal representations of $W^*(G)$ and continuous unitary representations of G. □

COROLLARY 6.3. (Extendability criterion) *For a continuous positive definite function $\varphi \in \mathcal{P}_c(G)$, the following are equivalent:*

(i) *φ extends to a continuous positive definite function of $G^\sharp = G \rtimes_\alpha T$.*

(ii) *$\varphi \in B(G)_c$, i.e., the T-orbit map of φ is continuous with respect to the norm $\|\cdot\|$ on $B(G) \cong W^*(G)_*$.*

Proof. (i) $\Rightarrow$ (ii): If $\varphi^\sharp\colon G^\sharp \to \mathbb{C}$ is a continuous positive definite extension of φ, then the corresponding GNS representation of $G^\sharp$ is continuous, so that (ii) follows from Theorem 6.2.

(ii) $\Rightarrow$ (i): According to [**4**], the subspace

$$\mathcal{M} := \{A \in W^*(G)\colon AB(G)_c \cup B(G)_c A \subseteq B(G)_c\}$$

is a W^*-subalgebra, $\mathcal{N} := B(G)_c^\perp \cap \mathcal{M}$ is a W^*-ideal, and $B(G)_c$ is the predual of the W^*-algebra $\mathcal{M}/\mathcal{N}$.

Since the left and right multiplications with elements of $\eta(G) \subseteq W^*(G)$ define isometries of $B(G) \cong W^*(G)_*$, and Theorem 6.2 implies, in particular, that for each $\varphi \in B(G)$, the maps

$$G \to B(G), \quad g \mapsto \eta(g)\varphi, \quad g \mapsto \varphi\eta(g)$$

are continuous, Lemma 5.4 shows that the left and right multiplication actions of G on $B(G)$ are continuous. From

$$\alpha_t^*(\eta(g)\varphi) = \eta(\alpha_{t^{-1}}g)\alpha_t^*\varphi$$

we now conclude that $\eta(G) \subseteq \mathcal{M}$. Since $W^*(G)$ is generated by $\eta(G)$ as a W^*-algebra, it follows that $\mathcal{M} = W^*(G)$. Any faithful normal representation of $\mathcal{M}/\mathcal{N}$ now yields

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a continuous unitary representation $(\pi, \mathcal{H})$ of G for which

$$F_\pi = B(G)_c \cap \{\varphi \in \mathcal{P}_c(G) : \varphi(\mathbf{1}) = 1\}$$

is the corresponding folium. Since it is T-invariant, Theorem 6.2 implies the existence of a quasi-equivalent representation $(\pi', \mathcal{H}')$ of G that extends to $G^\sharp$. As $F_\pi = F_{\pi'}$ by quasi-equivalence, (i) follows. □

REMARK 6.4. (a) Unfortunately, the characterization of the extendable positive definite functions on G is given in terms of the norm $\|\cdot\|$ on $B(G)$ which is not very accessible. Since the inclusion $(B(G), \|\cdot\|) \to (C_b(G), \|\cdot\|_\infty)$ is continuous, Corollary 6.3 shows that the continuity of the map

$$T \to C_b(G), \quad t \mapsto \varphi \circ \alpha_t$$

is necessary for the existence of a continuous positive definite extension to $G^\sharp$.

Since this condition is much easier to check for concrete cases, it would be interesting to know whether it is also sufficient.

(b) Assume that $T = \mathbb{R}$. Although, in general, the group $G^\sharp = G \rtimes_\alpha \mathbb{R}$ is not locally compact, the existence of an invariant measure on the quotient $G^\sharp/G \cong \mathbb{R}$ implies that unitary induction makes sense as a passage from continuous unitary representations of G to continuous unitary representations of $G^\sharp$.

Starting with a unitary representation $(\pi, \mathcal{H})$ of G, we consider the space $\mathcal{H}^\sharp := L^2(\mathbb{R}, \mathcal{H})$, endowed with the continuous unitary representation $\pi^\sharp := \mathrm{Int}_G^{G^\sharp}(\pi)$, given by

$$(\pi^\sharp(g, t)f)(s) = \pi(\alpha_{-s}(g))f(s - t) \quad \text{for} \quad g \in G, s, t \in \mathbb{R}.$$

Then $(\pi^\sharp(g, 0)f)(s) = \pi(\alpha_{-s}(g))f(s)$ shows that G acts by multiplication operators and $(U_t f)(s) := (\pi^\sharp(\mathbf{1}, t)f)(s) = f(s - t)$ shows that $\mathbb{R}$ acts by translations. It follows in particular that $\mathrm{Spec}(U) = \mathbb{R}$, so that $\pi^\sharp$ never is a positive energy representation.

Let $v \in \mathcal{H}$ be such that the positive definite function $\varphi(g) := \langle \pi(g)v, v\rangle$ has the property that $\mathbb{R} \to C_b(G)$, $t \mapsto \varphi \circ \alpha_t$ is continuous. For $h \in C_c(\mathbb{R})$, we consider the element $f(t) := h(t)v$ of $\mathcal{H}^\sharp$. In the representation $(\pi^\sharp, \mathcal{H}^\sharp)$ we then obtain the matrix coefficient

$$\langle \pi^\sharp(g, t)f, f\rangle = \int_{\mathbb{R}} \langle \pi(\alpha_{-s}(g))v, v\rangle h(s - t)\overline{h(s)}\, ds = \int_{\mathbb{R}} \varphi(\alpha_{-s}(g))h(s - t)\overline{h(s)}\, ds.$$

This is a continuous positive definite function whose restriction to G is given by

$$\langle \pi^\sharp(g, 0)f, f\rangle = \int_{\mathbb{R}} \varphi(\alpha_{-s}(g))|h(s)|^2\, ds.$$

If the functions $|h_n|_2^2$, $n \in \mathbb{N}$, form an approximate identity on $\mathbb{R}$, we thus obtain a sequence $(\varphi_n^\sharp)_{n\in\mathbb{N}}$ of continuous positive definite functions on $G^\sharp$ such that $\varphi_n := \varphi_n^\sharp|_G$ converges in $C_b(G)$ to φ (cf. [**2**, Section III]).

If we apply the same construction to the discrete group $\mathbb{R}_d$ instead, we have $\mathcal{H}^\sharp = \ell^2(\mathbb{R}, \mathcal{H})$ and for $f = \delta_0 v$, we obtain the positive definite function

$$\varphi^\sharp(g, t) = \delta_{0,t}\varphi(g)$$

which is not continuous on $G^\sharp$ if φ is not constant.

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Applications to C^*-dynamical systems. We conclude this section with a brief discussion of the link to Borchers' criterion for the existence of covariant representations in the context of C^*-algebras.

Let $\mathcal{A}$ be a C^*-algebra, T be a topological group and $\alpha\colon T \to \mathrm{Aut}(\mathcal{A})$ be a group homomorphism (not necessarily strongly continuous). A covariant representation of $(\mathcal{A}, \alpha)$ is a triple $(\pi, U, \mathcal{H})$, where $\pi : \mathcal{A} \to B(\mathcal{H})$ is a non-degenerate representation of $\mathcal{A}$ and $U\colon T \to \mathrm{U}(\mathcal{H})$ is a continuous representation satisfying

$$U_t \pi(A) U_t^* = \pi(\alpha_t(A)) \quad \text{for} \quad A \in \mathcal{A}, t \in T.$$

If $(\pi, U, \mathcal{H})$ is a continuous unitary representation and $v \in \mathcal{H}$ a unit vector, then $\varphi(A) := \langle \pi(A)v, v\rangle$ is a normalized state $\varphi \in \mathcal{A}^*$ for which the α-orbit map $T \to \mathcal{A}^*, t \mapsto \alpha_t^*\varphi := \varphi \circ \alpha_t$ is norm continuous. This follows immediately from

$$\left(\alpha_t^*\varphi\right)(A) = \langle \pi(\alpha_t A)v, v\rangle = \left\langle U_t\pi(A)U_t^* v, v\right\rangle = \left\langle \pi(A)U_t^*v, U_t^*v\right\rangle = \mathrm{tr}(\pi(A)P_{U_t^*v}). \quad (3)$$

There is an interesting converse result, namely that every state $\varphi \in \mathcal{A}^*$ with a norm-continuous α-orbit map comes from a vector in a covariant representation (cf. [**3**]). Since Borchers' proof of this result is quite involved, one would like to have a more direct argument based on the criteria from above.

REMARK 6.5. (cf. [**1**]) Let $\pi : \mathcal{A} \to B(\mathcal{H})$ be a representation of the C^*-algebra and

$$\pi_*\colon B_1(\mathcal{H}) \to \mathcal{A}^*, \quad \pi_*(X)(A) := \mathrm{tr}(\pi(A)X).$$

Then π_* is a continuous linear map whose adjoint $\pi_*^*\colon \mathcal{A}^{**} \to B_1(\mathcal{H})^* \cong B(\mathcal{H})$ is a normal representation of the W^*-algebra $\mathcal{A}^{**}$ (cf. [**19**]). In particular, its range is closed, and therefore the range of π_* is closed as well. Therefore, π_* induces a topological embedding $B_1(\mathcal{H})/\ker \pi_* = B_1(\mathcal{H})/\pi(\mathcal{A})^\perp \hookrightarrow \mathcal{A}^*$ of Banach spaces.

The map π_* is injective if and only if $\pi(\mathcal{A})$ is weakly dense in $B(\mathcal{H})$, which is equivalent to $\pi(\mathcal{A})'' = \pi_*^*(\mathcal{A}^{**}) = B(\mathcal{H})$. This means that π is irreducible and then $\pi_*\colon B_1(\mathcal{H}) \to \mathcal{A}^*$ is a topological embedding.

THEOREM 6.6. *Assume that* $\alpha\colon T \to \mathrm{Aut}(\mathcal{A})$ *defines a strongly continuous automorphic T-action on $\mathcal{A}$. Let $(\pi, \mathcal{H})$ be an irreducible representation of $\mathcal{A}$ for which $\pi \circ \alpha_t$ is equivalent to π for every $t \in T$ and let $v \in \mathcal{H}$ be a unit vector. Then the following are equivalent:*

(i) *The state $\varphi(A) := \langle \pi(A)v, v\rangle$ of $\mathcal{A}$ has a norm-continuous orbit map $T \to \mathcal{A}^*, t \mapsto \varphi \circ \alpha_t$.*

(ii) *There exists a central $\mathbb{T}$-extension $p\colon \widetilde{T} \to T$ and a continuous unitary representation $\widetilde{U}\colon \widetilde{T} \to \mathrm{U}(\mathcal{H})$ such that $(\pi, \widetilde{U}, \mathcal{H})$ is a covariant representation with respect to $\widetilde{\alpha} := \alpha \circ p\colon \widetilde{T} \to \mathrm{Aut}(\mathcal{A})$.*

Proof. In view of (3), (ii) implies (i). So we assume (i). Then there exists a family $(U_t)_{t\in T}$ of unitary operators with

$$U_t\pi(A)U_t^* = \pi(\alpha_t(A)) \quad \text{for} \quad t \in T, A \in \mathcal{A}. \quad (4)$$

From (3) we derive the relation $\alpha_t^*\varphi = \pi_*(P_{U_{t^{-1}}v})$. Since π_* is a topological embedding by Remark 6.5, the map $T \to \mathrm{Herm}_1(\mathcal{H}), t \mapsto P_{U_tv}$ is also continuous. Therefore, Lemma 4.2 shows that $[v] \in \mathbb{P}(\mathcal{H})_c$ is a continuous ray for the corresponding homomorphism $\overline{U}\colon T \to \mathrm{PU}(\mathcal{H})$. As in the proof of Theorem 5.17, we see that

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$\pi(\mathrm{U}(\mathcal{A}))[v] \subseteq \mathbb{P}(\mathcal{H})_c$ consists of continuous rays for $\overline{U}$. As $\pi(\mathrm{U}(\mathcal{A}))$ acts transitive on $\mathrm{U}(\mathcal{H})$ ([**8**, Theorem 2.8.3(iii)]), we obtain $\mathbb{P}(\mathcal{H}) = \mathbb{P}(\mathcal{H})_c$, so that Corollary 5.2 implies that the T-action on $\mathbb{P}(\mathcal{H})$ is continuous and now one proceeds as in the proof of Theorem 5.17. □

REMARK 6.7. If π is not irreducible, then the situation is more complicated. Then $\pi(\mathcal{A})' \neq \mathbb{C}\mathbf{1}$, so that the ambiguity in the choice of U is rather large and the requirement that $\pi \circ \alpha_t \sim \pi$ for every t does not lead to a canonical projective unitary representation.

EXAMPLE 6.8. Let $\mathcal{A}$ be a C^*-algebra and $\alpha\colon T \to \mathrm{Aut}(\mathcal{A})$ a homomorphism defining a continuous action of T on $\mathcal{A}$. Then α defines in particular a continuous T-action on the unitary group $G := \mathrm{U}(\mathcal{A})$, endowed with the norm topology. Let $\varphi\colon G \to \mathbb{C}$ be a positive definite function defined by a linear positive functional $\psi \in \mathcal{A}^*$. We claim that, in this case, the requirement that the curve φ_t in $C_b(G)$ is continuous implies that the curve $\psi_t := \psi \circ \alpha_t$ in $\mathcal{A}^*$ is norm continuous (cf. Remark 6.4(a)).

This claim follows from the fact that the absolute convex hull of $\mathrm{U}(\mathcal{A})$ is a 0-neighbourhood in $\mathcal{A}$. In fact, if $A = A^*$ with $\|A\| < 1$, then $U := A + i\sqrt{\mathbf{1} - A^2}$ is unitary and $A = \frac{1}{2}(U + U^*)$. Therefore, the absolute convex hull of $\mathrm{U}(\mathcal{A})$ contains all hermitian operators A with $\|A\| < 1$, hence all operators C with $\|C\| < \frac{1}{2}$ because $C = \frac{1}{2}(C + C^*) - i\frac{1}{2}(iC - iC^*)$.

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