Canad. Math. Bull. Vol. **18** (4), 1975

THREE REMARKS ON THE MEASUREMENT OF UNIT SPHERES

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Let X and Y be isomorphic normed linear spaces and let $\mathscr{L}$ denote the set of isomorphisms from X to Y (i.e. bounded linear mappings from X onto Y which have bounded inverses). Banach [1] suggested using

$$d(X, Y) = \inf\{\|L\|\,\|L^{-1}\| : L \in \mathscr{L}\}$$

as a measure of the distance between X and Y ($\log d$ is a pseudo-metric on each isomorphic class of Banach spaces). In recent years many writers ([2], [3], [4], [5], for example) have suggested numerical constants associated with a normed linear space. These constants are of interest in themselves, providing information about the geometry of the space in question, but they are also useful in giving lower bounds on the distance d between spaces (see, especially [5]). We have been interested in trying to explore the relationships between these constants and this note is intended to present three such connections.

For a normed linear space X, we let $\Sigma(X)$ denote the closed unit ball

$$\Sigma(X) = \{x \in X : \|x\| \leq 1\}$$

and we let $\Sigma_0(X)$ and $\partial\Sigma(X)$ denote the interior and boundary of $\Sigma(X)$ respectively.

DEFINITION 1. (See [2]). For each cardinal number α,

$$P(\alpha, X) = \sup\{r \geq 0 : \exists\ \alpha \text{ disjoint open balls of radius } r \text{ in } \Sigma(X)\}.$$

DEFINITION 2. (See [5]).

$$T(X) = \inf\{\delta \geq 0 : \forall\, \varepsilon > \delta, \exists \text{ a finite } \varepsilon \text{ net } F_\varepsilon \text{ for } \delta\Sigma(X), F_\varepsilon \subseteq \partial\Sigma(X)\}.$$

$P(\alpha, X)$ is called the *packing constant* for X and T is called the *thickness* of X.

The third definition is due to J. J. Schaffer and is as follows. If c is a rectifiable "curve" in X (for a precise definition of curve, see [3]) let $\ell(c)$ denote its length.

DEFINITION 3. (See [3]). The *inner metric* δ on $\partial\Sigma(X)$ is defined by the equation

$$\delta(p, q) = \inf\{\ell(c) : c \text{ a curve from } p \text{ to } q \text{ lying in } \partial\Sigma(X)\}$$

Received by the editors, November 13, 1974 and, in revised form, January 15, 1975.

[1] The results of this paper were contained in the first author's thesis presented in partial fulfillment of the requirements for the M.A. degree at Dalhousie. This research was supported, in part, by N.R.C. grants No. A-3999 and A-4066.

and the *girth* of X, $2m(X)$, is defined by the equation

$$m(X) = \inf\{\delta(p, -p) \colon p \in \partial\Sigma(X)\}.$$

The first theorem shows a relationship between $P(\aleph_0, X)$ and $T(X)$; the second and third show connections between $P(n, X)$ and $m(X)$ and between $m(X)$ and the "metric" d respectively.

THEOREM 1.[(2)] *Let $p=P(\aleph_0, X)$, then $T(X)\leq 2p(1-p)^{-1}$.*

Proof. Let ε be positive and let $T'=T(X)-\varepsilon$. Then there is no finite T' net for $\partial\Sigma(X)$. Therefore, if $\|x_1\|=1$, $\exists x_2 \in \partial\Sigma(X)$ such that $\|x_1-x_2\|>T'$. If $x_1, x_2, \ldots, x_n$ have been chosen with $\|x_i\|=1$ and $\|x_i-x_j\|>T'$ $i\neq j$ then this set is not a T' net for $\partial\Sigma(X)$ and so $\exists x_{n+1}$ with $\|x_{n+1}\|=1$ and $\|x_i-x_{n+1}\|>T'$ $i=1, 2, \ldots, n$. In this way we construct a sequence $\{x_n\}$ with $\|x_n\|=1$ and $\|x_n-x_m\|>T'$ for all $n, m, n\neq m$. Let $S=T'(2+T')^{-1}$ and let $y_n=(1-S)x_n$ $n=1, 2, 3, \ldots$. Then $\|y_n-y_m\|>T'(1-S)\forall n, m, n\neq m$. Then, since $(1-S)+\frac{1}{2}T'(1-S)=1$ the open balls centered at y_n and of radius $\frac{1}{2}T'(1-S)$ are disjoint and are contained in $\Sigma(X)$.

This clearly implies that $P(\aleph_0, X)\geq\frac{1}{2}T'(1-S)=T'(2+T')^{-1}$. Since ε is arbitrary we get $p\geq T(X)(2+T(X))^{-1}$ which is equivalent to $T(X)\leq 2p(1-p)^{-1}$.

REMARK. Equality holds for finite dimensional spaces and for the ℓ_p spaces. Strict inequality holds for the spaces c_0 and c. (See [2] and [5]). We remark also that Kottman showed, [2], that for any $\varepsilon>0$ there is a $(2p/(1-p)-\varepsilon)$-separated subset of $\partial\Sigma(X)$ of cardinality $\geq\alpha$.

The proof of Theorem 2 requires one further definition.

DEFINITION 4. For each natural number n and for each ρ with $0<\rho<1$, X is said to have property $J_{n,\rho}$ if $\exists x_k \in \Sigma(X)$ $k=1, 2, \ldots, n$ such that $\|\sum_{k=1}^{n} \varepsilon_k x_k\|>\rho n$ for all sequences ε_k $k=1, 2, \ldots, n$ such that $\varepsilon_k=\pm 1$ and such that every -1 (if any) precedes every $+1$ (if any).

The space X is said to have property J_n if X has property $J_{n,\rho}\forall\rho$ with $0<\rho<1$; and X has property J if X has property J_n for all n.

LEMMA (Schäffer and Sundaresan [4]). *The normed linear space X has property J if and only if $m(X)=2$.*

THEOREM 2. *Let X be a normed linear space. If $m(X)=2$ then $P(n, X)=\frac{1}{2}$ for every natural number n.*

Proof. By the lemma, X has property J. Let $n \in N$ and let ε be positive, set

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[(2)]. The authors are grateful to the referee for pointing out that this result is due to C. A. Kottman and was contained in his thesis *Packing and Reflexivity in Banach Spaces*, University of Iowa, 1969, (p. 20).

$\rho \geq 1-(\varepsilon/n)$. Then X has property $J_{n,\rho}$ and so there exists $x_k \in \Sigma(X)$, $k=1, 2, \ldots, n$ such that

$$\left\|\sum_{k=1}^{n} \varepsilon_k x_k\right\| > \rho n \geq n\left(1-\frac{\varepsilon}{n}\right) = n-\varepsilon$$

for all sequences ε_k of the prescribed type. If $j<i$ then

$$\begin{aligned}\|x_i-x_j\| &= \left\|-\sum_{k=1}^{j} x_k+\sum_{k=j+1}^{n} x_k+\sum_{k=1}^{j-1} x_k-\sum_{k=j+1}^{i-1} x_k-\sum_{k=i+1}^{n} x_k\right\| \\ &\geq \left\|-\sum_{k=1}^{j} x_k+\sum_{k=j+1}^{n} x_k\right\|-\sum_{k\neq i,j}\|x_k\| \\ &> (n-\varepsilon)-(n-2) = 2-\varepsilon.\end{aligned}$$

Now let $y_i=\frac{1}{2}x_i$. Then $\|y_i\| \leq \frac{1}{2}$, $\|y_i-y_j\|>1-(\varepsilon/2)$ and so the open balls, centered at y_i and radius $\frac{1}{2}-\frac{1}{4}\varepsilon$ are disjoint and contained in $\Sigma(X)$. Since ε is arbitrary, $P(n, X)=\frac{1}{2}$, and this holds for every natural number n.

J. J. Shäffer and K. Sundaresan have also shown that if X is non-reflexive then $m(X)=2$.

Corollary (Kottman). *If X is non-reflexive, then $P(n, X)=\frac{1}{2}$ $\forall n$.*

Theorem 3. *Let X and Y be isomorphic normed linear spaces then*

$$m(X)/m(Y) \leq d(X, Y).$$

(The left-hand side is not symmetric in X and Y but, of course, we also have $m(Y)/m(X) \leq d(X, Y)$.)

Proof. Let L be an isomorphism from X to Y, it is sufficient to show that $m(X)/m(Y) \leq \|L\|\ \|L^{-1}\|$.

Given $\varepsilon>0$, there exists $y_0 \in \partial\Sigma(Y)$ and a curve c from y_0 to $-y_0$ in $\partial\Sigma(Y)$ such that $\ell(c)<m(Y)+\varepsilon$. Let c' be the curve $\|L\|\ L^{-1}(c)$, i.e. if g_c is the canonical representation of c by arc-length parameter so that points of c are given by $g_c(t)$ $0 \leq t \leq \ell(c)$, $y_0=g_c(0)$, $-y_0=g_c(\ell(c))$. Then c' is given by $\|L\|\ L^{-1}g_c$, and, for each $y \in c$, $x=\|L\|\ L^{-1}(y)$ is a point of c'. For each such x, we have $\|x\|=\|L\|\ \|L^{-1}(y)\| \geq \|LL^{-1}(y)\|=\|y\|=1$ so that c' lies outside $\Sigma_0(X)$. Further, if $\{x_0, x_1, x_2, \ldots, x_n\}$ is a partition of c' and if $y_i=\|L\|^{-1}L(x_i)$ so that $x_i=\|L\|\ L^{-1}(y_i)$ then $\|x_i-x_{i+1}\| = \|L\|\ \|L^{-1}(y_i-y_{i+1})\| \leq \|L\|\ \|L^{-1}\|\ \|y_i-y_{i+1}\|$ and so

$$\begin{aligned}\sum_{i=0}^{n-1}\|x_i-x_{i+1}\| &\leq \|L\|\ \|L^{-1}\| \sum_{i=0}^{n-1}\|y_i-y_{i+1}\| \\ &\leq \|L\|\ \|L^{-1}\|\ \ell(c).\end{aligned}$$

Since this is true of any partition of c' we get

$$\ell(c') \leq \|L\|\ \|L^{-1}\|\ \ell(c).$$

We now appeal to a result of J. J. Schäffer [3], p. 62 which shows that to calculate the inner metric δ it is sufficient to consider curves in the complement of $\Sigma_0(X)$. Consequently, we have

$$\delta(x_0, -x_0) \leq \|L\| \; \|L^{-1}\| \; \ell(c)$$

and hence $m(X) \leq \|L\| \; \|L^{-1}\| (m(Y)+\varepsilon)$. Since ε was chosen arbitrarily, the result follows.

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