1 Phases of Gauge Theories

Spontaneous breaking of global and local symmetries. — The Higgs regime. — The Coulomb and infrared free phases. — Color confinement (closed and open strings). Does confinement imply chiral symmetry breaking? — Conformal regime. — Conformal window.

Illustration by Olga Kulakova: *Open string in nonperturbative regime*

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1.1 Spontaneous Symmetry Breaking

1.1.1 Introduction

We will begin with a general survey of various patterns of spontaneous symmetry breaking in field theory. Our first task is to get acquainted with the breaking of global symmetries – at first discrete, then continuous. After that we will familiarize ourselves with the manifestations of spontaneous symmetry breaking.

Spontaneous symmetry breakdown: what does that mean?

Assume that a dynamical system under consideration is described by a Lagrangian $\mathcal{L}$ possessing a certain global symmetry $\mathcal{G}$. Assume that the ground state of this system is known. Generally speaking, there is no reason why the ground state should be symmetric under $\mathcal{G}$. Examples of such situations are well known. For instance, although spin interactions in magnetic materials are rotationally symmetric, spontaneous magnetization does occur: spins in the ground state are predominantly aligned along a certain direction, as well as the magnetic field they induce. Even though the Hamiltonian is rotationally invariant, the ground state is not. If this is the case then, in fact, we are dealing with infinitely many ground states, since all alignment directions are equivalent (strictly speaking, they are equivalent for an infinitely large ferromagnet in which the impact of the boundary is negligible).

This situation is usually referred to as *spontaneous symmetry breaking*. This terminology is rather deceptive, however, since the symmetry has not disappeared but, is realized in a special manner. The reason why people say that the symmetry is broken is, probably, as follows. Assume that a set of small detectors is placed inside a given ferromagnet far from the boundaries. Experiments with these detectors will not reveal the rotational invariance of the fundamental interactions because there is a preferred direction, that of the background magnetic field in the ferromagnet. For the uninitiated, inside-the-sample measurements give no direct hint that there are infinitely many degenerate ferromagnets, which, taken together, form a rotationally invariant family. Indeed, one can change the direction of only a finite number of spins at a time by tuning one's apparatus. To obtain a ferromagnet with a different direction of spontaneous magnetization, one will need to make an infinite number of steps.

A learned theoretician will be able to guess that the fundamental interaction is rotationally invariant from the presence of Goldstone bosons.

Thus, the rotational symmetry of the Hamiltonian, as observed from "inside," is hidden. Of course, it becomes perfectly obvious if we make observations from "outside." However, in many problems in solid state physics and in all problems in high-energy physics, the spatial extension is infinite for all practical purposes. An observer living inside such a world, will have to use guesswork to uncover the genuine symmetry of the fundamental interactions.

Since the terminology "spontaneous symmetry breaking" is common, we will use it too, at least with regard to the breaking of global symmetries. Now we will discuss discrete symmetries; the simplest example is Z_2.

1.1.2 Real Scalar Field with Z_2-Invariant Interactions

Let us consider a system with one real field $\phi(x)$ with action

$$S = \int d^D x \left[\frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - U(\phi)\right], \tag{1.1}$$

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where $U(\phi)$ is the self-interaction (or potential energy) and D is the number of dimensions. In field theory one can consider three distinct cases, $D = 2$, $D = 3$, and $D = 4$. The first two cases may be relevant for both solid state and high-energy physics, while the third case refers only to high-energy physics.

The potential energy may be chosen in many different ways. In this subsection we will limit ourselves to the simplest choice, a quartic polynomial of the form

$$U(\phi) = \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{4}g^2\phi^4, \tag{1.2}$$

where m^2 and g^2 are constants. We will assume that g^2 is small, so that a quasiclassical treatment applies.

It is obvious that the system described by Eqs. (1.1), (1.2) possesses a discrete Z_2 symmetry:

The symmetry Z_2 as an example of the discrete global symmetry

$$\phi(x) \longrightarrow -\phi(x). \tag{1.3}$$

Indeed, only even powers of ϕ enter the action. This is a global symmetry since the transformation (1.3) must be performed for all x simultaneously.

For the time being we will treat our theory purely classically but will use quantum-mechanical language. We will refer to the lowest energy state (the ground state) as the *vacuum*. To determine the vacuum states one should examine the Hamiltonian of the system,

$$\mathcal{H} = \int d^{D-1}x \left[\tfrac{1}{2}(\partial_0\phi)(\partial_0\phi) + \tfrac{1}{2}\left(\vec{\partial}\phi\right)\left(\vec{\partial}\phi\right) + U(\phi)\right]. \tag{1.4}$$

Since the kinetic term is positive definite, it is clear that the state of lowest energy is that for which the value of the field ϕ is constant, i.e., independent of the spatial and time coordinates. For a constant-field configuration the minimal energy is determined by the minimization of $U(\phi)$. We will refer to the corresponding value of ϕ as the vacuum value.

Within the given class of theories with the potential energy (1.2) we can find both dynamical scenarios: manifest Z_2 symmetry or spontaneously broken Z_2 symmetry, depending on the sign of the parameter m^2.

1.1.3 Symmetric Vacuum

Let us start from the case of positive m^2; see Fig. 1.1. The vacuum is achieved at

$$\phi = 0. \tag{1.5}$$

This solution is obviously invariant under the transformation (1.3). Thus the ground state of the system has the same Z_2 symmetry as the Hamiltonian. In this case we will say that *the vacuum does not break the symmetry spontaneously*. One can make one step further and consider small oscillations around the vacuum. Since the vacuum is at zero, small oscillations coincide with the field ϕ itself. In the quadratic approximation the action becomes

$$S_2 = \int d^Dx \left[\tfrac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - \tfrac{1}{2}m^2\phi^2\right]. \tag{1.6}$$

We immediately recognize m as the mass of the ϕ particle. Moreover, from the quartic term $g^2\phi^4$ one can readily extract the interaction vertex and develop the

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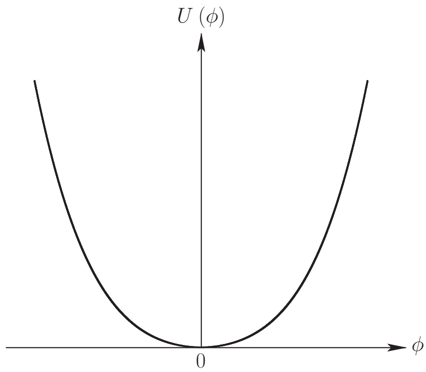

Fig. 1.1 The potential energy (1.2) at positive m^2.

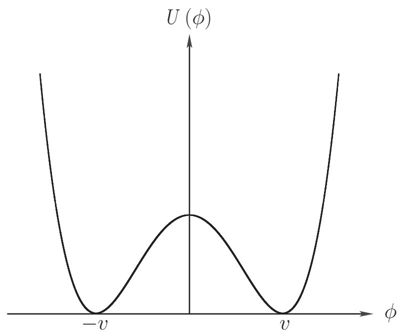

Fig. 1.2 The potential energy at negative m^2.

corresponding Feynman graph technique. The Z_2 symmetry of the interactions is apparent. Because of the invariance under (1.3), if in any scattering process the initial state has an odd number of particles then, so does the final state. Starting with any even number of particles in the initial state one can obtain only an even number of particles in the final state. Thus, a smart experimentalist, colliding two particles and never finding three, five, seven, and so on particles in his detectors, will deduce the Z_2 invariant nature of the theory.

1.1.4 Nonsymmetric Vacuum

Let us pass now to another case, that of negative m^2. To ease the notation we will introduce a positive parameter, $\mu^2 \equiv -m^2$. The new potential is shown in Fig. 1.2. Strictly speaking, I am cheating a little bit here; in fact, what is shown in Fig. 1.2 is *not* the potential (1.2). Rather, I have added a constant to this potential, $\Delta U = \mu^4/(4g^2)$, chosen in such a way as to adjust to zero the value of U at the minima. As you know, numerical additive constants in the Lagrangian are unobservable – they have no impact on the dynamics of the system.

The symmetric solution $\phi = 0$ is now at a maximum of the potential rather than a minimum. Small oscillations near this solution would be unstable; in fact, they would represent tachyonic objects rather than normal particles.

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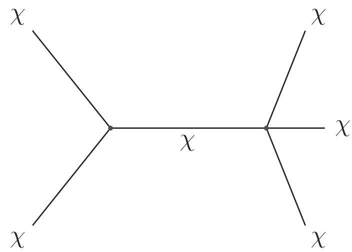

Fig. 1.3 The Feynman graph for the transition of two χ quanta into three in an asymmetric vacuum.

The true ground states are asymmetric with respect to (1.3),

$$\phi = \pm v, \qquad v = \frac{\mu}{g}. \tag{1.7}$$

The two-fold degeneracy of the vacuum follows from the Z_2 symmetry of the Lagrangian in (1.6). Indeed, under the action of (1.3) the positive vacuum goes into the negative vacuum, and vice versa.

In terms of v the potential takes the form

$$U(\phi) = \tfrac{1}{4} g^2 \left(\phi^2 - v^2\right)^2. \tag{1.8}$$

To investigate the physics near one of the two asymmetric vacua, let us define a new "shifted" field χ,

$$\phi = v + \chi, \tag{1.9}$$

which represents small oscillations, i.e., the particles of the theory. First let us examine the particle mass. To this end we substitute the decomposition (1.9) into the Lagrangian with a potential term given by Eq. (1.8). In this way we get

$$\mathcal{L} = \tfrac{1}{2} \left(\partial_\mu \chi\right) \left(\partial^\mu \chi\right) - \left(\mu^2 \chi^2 + \mu g \chi^3 + \tfrac{1}{4} g^2 \chi^4\right), \tag{1.10}$$

using Eq. (1.7) for v. By comparing the kinetic term with the term $\mu^2 \chi^2$ within the large parentheses we immediately conclude, for the mass of the χ quantum, that

$$m_\chi = \sqrt{2}\mu. \tag{1.11}$$

In the unbroken case of positive m^2 the particle's mass was m (see Eq. (1.6)). We see that changing the sign of m^2 leads to a factor of $\sqrt{2}$ difference in the particle mass.

The occurrence of the term cubic in χ in (1.10) is even more dramatic. Indeed this term, in conjunction with the quartic term, will generate amplitudes with an arbitrary number of quanta. For instance, the scattering amplitude for two quanta into three quanta is displayed in Fig. 1.3.[1]

The selection rule prohibiting the transition of an even number of particles into an odd number, as was the case for positive m^2 (a symmetric vacuum), is gone. Even for a smart physicist, doing scattering experiments, it would be rather hard now to discover the Z_2 symmetry of the original theory.

[1] Let us note parenthetically that there is an easy heuristic way to generate Feynman graphs in the asymmetric-vacuum theory from those of the symmetric theory. In the symmetric-vacuum theory, where all vertices are quartic, one starts for instance from the graph of Fig. 1.4a and replaces one external line by the vacuum expectation value of ϕ (Fig. 1.4b). Since $\phi_{\rm vac}$ is just a number, one immediately arrives at the graph of Fig. 1.3.

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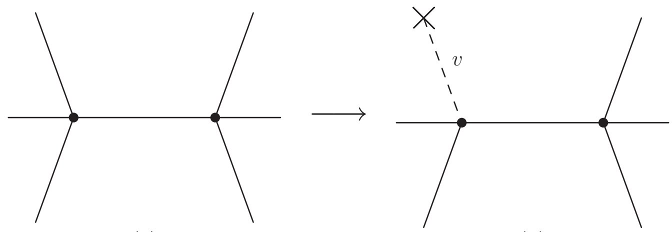

Fig. 1.4 Converting Feynman graphs in the symmetric theory (a) into those of the theory with asymmetric vacua (b). The cross on the broken line means that this line is replaced by the vacuum value of the field ϕ.

A trace of this symmetry remains in the broken phase, namely a relation between the cubic coupling constant in the Lagrangian ($-\mu g$), the quartic constant ($-g^2/4$), and the particle mass squared ($2\mu^2$):

This relation does not hold for generic cubic and quartic interaction vertices in (1.2).

$$\text{quartic constant} = -\frac{(\text{cubic constant})^2}{2m_\chi^2}. \tag{1.12}$$

A qualitative signature of the underlying spontaneously broken Z_2 symmetry is the existence of domain walls.

1.1.5 Equivalence of Asymmetric Vacua

Two questions remain to be discussed. Let us start with the simpler. What would happen if, instead of the vacuum at $\phi = v$, we (or, rather, nature) chose the second vacuum, at $\phi = -v$? The decomposition (1.9) would obviously be replaced by $\phi = -v+\chi$. This would change the sign of the cubic term in the Lagrangian, which, in turn, would entail the change in sign of all amplitudes with an odd number of external lines. We should remember, however, that it is not amplitudes but probabilities that are measurable. Since there is no interference between amplitudes with odd and even numbers of external lines, the sign is unobservable. *The physics in the two vacua is perfectly equivalent!*

This brings us to the second question: is there a *direct* manifestation of the fact that the underlying theory is Z_2 symmetric and the Z_2 symmetry is spontaneously broken by the choice of vacuum state? The answer is yes, at least in theory. We will discuss this phenomenon at length later (see Chapter 2).

1.1.6 Spontaneous Breaking of the Continuous Symmetry

To begin with, we will consider the simplest continuous symmetry, U(1). Consider a *complex* field $\phi(x)$ with action

$$S = \int d^D x \left[\left(\partial_\mu \phi\right)^* \left(\partial^\mu \phi\right) - U(\phi) \right], \tag{1.13}$$

where the potential energy $U(\phi)$ in fact depends only on $|\phi|$, for instance,

$$U(\phi) = m^2|\phi|^2 + \tfrac{1}{2}g^2|\phi|^4. \tag{1.14}$$

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In this case the Lagrangian is invariant under a (global) phase rotation of the field ϕ:

$$\phi \to e^{i\alpha}\phi, \qquad \phi^* \to e^{-i\alpha}\phi^*. \tag{1.15}$$

If the mass parameter m^2 is positive, the minimum of the potential energy is achieved at $\phi = 0$. This is the unbroken phase. The vacuum is unique. There are two particles, that is, two elementary excitations, corresponding to Re ϕ and Im ϕ. The mass of both these elementary excitations is m.

Changing the sign of m^2 from positive to negative drives one into the broken phase. The potential energy can be rewritten (after addition of an irrelevant constant) as

$$U(\phi) = \tfrac{1}{2}g^2\left(|\phi|^2 - v^2\right)^2, \tag{1.16}$$

where

$$v^2 = \frac{\mu^2}{g^2} \equiv -\frac{m^2}{g^2}; \tag{1.17}$$

$U(\phi)$ has the form of a "Mexican hat," see Fig. 1.5. The degenerate minima in the potential energy are indicated by the black circle. An arbitrary point on this circle is a valid vacuum. Thus there is a continuous set of vacuum states, called the *vacuum manifold*. All these vacua are physically equivalent.

As an example let us consider the vacuum state at $\phi = v$. Near this vacuum the field ϕ can be represented as

$$\phi(x) = v + \frac{1}{\sqrt{2}}\varphi(x) + \frac{i}{\sqrt{2}}\chi(x), \tag{1.18}$$

where φ and χ are real fields. Then in terms of these fields

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}\left[(\partial_\mu\varphi)^2 + (\partial_\mu\chi)^2\right] \\ &\quad - \left[g^2v^2\varphi^2 + \frac{g^2 v}{\sqrt{2}}\varphi(\varphi^2+\chi^2) + \frac{g^2}{8}(\varphi^2+\chi^2)^2\right].\end{aligned} \tag{1.19}$$

The mass of an elementary excitation of the φ field is $m_\varphi = \sqrt{2}gv = \sqrt{2}\mu$. A remarkable feature is that the mass of the χ quantum vanishes: the potential energy has no terms quadratic in χ in (1.19).

The Goldstone theorem, Section 6.5.1

This is a general situation: the spontaneous breaking of continuous symmetries entails the occurrence of massless particles, which are referred to as Goldstone particles, or Goldstones for short.[2] In solid state physics they are also known as gapless excitations. For instance, in the example of the ferromagnet discussed at the beginning of this section such gapless excitations exist too; they are called magnons. Detecting magnons within the ferromagnet sample gives a clue that in fact one is dealing with an underlying symmetry that has been spontaneously broken.

In the problem at hand, that of a single complex field, the spontaneously broken symmetry is U(1). It has a single generator; hence the Goldstone boson, the phase of the order parameter, is unique.

[2] Sometimes the Goldstone bosons are referred to as the Nambu–Goldstone bosons. They were discussed first by Nambu in the context of the Bardeen–Cooper–Schrieffer superconductivity and independently by Vaks and Larkin who constructed a model now known as the Nambu–Jona-Lasinio model. [1]. In the context of high-energy physics they were discovered by Goldstone [2].

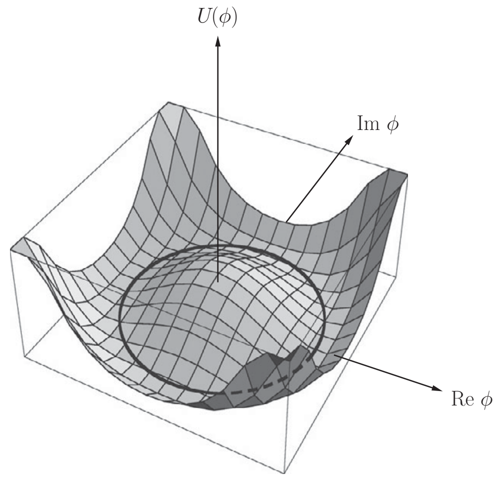

Fig. 1.5 The potential energy (1.16). The black circle marks the minimum of the potential energy, the vacuum manifold.

To conclude this section we will consider another example, with a slightly more sophisticated pattern of symmetry breaking, which we will need in our study of monopoles (Section 4.1).

The model for analysis is a triplet of real fields ϕ_a $(a = 1, 2, 3)$ with the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \vec{\phi})^2 - \left[-\tfrac{1}{2}\mu^2\vec{\phi}^2 + \tfrac{1}{4}g^2 4(\vec{\phi}^2)^2\right], \tag{1.20}$$

where $\vec{\phi} = \{\phi_1, \phi_2, \phi_3\}$ and $\mu^2 > 0$. It is obvious that this Lagrangian is O(3)-symmetric while the vacuum state is not. The minimum of the potential energy is achieved at $\vec{\phi}^2 = \mu^2/g^2$; thus $|\phi_{\rm vac}| = \mu/g \equiv v$. The angular orientation of the vector of the vacuum field in the O(3) space ("isospace") is arbitrary. The vacuum manifold is a two-dimensional sphere of radius v. All points on this manifold are physically equivalent.

Suppose that we choose $\vec{\phi}_{\rm vac} = \{0, 0, v\}$, i.e., we align the vacuum value of the field along the third axis in isospace. The original symmetry is broken down to U(1). The fact that there is a residual U(1) is quite transparent. Indeed, rotations in the isospace around the third axis do not change $\phi_{\rm vac}$. Thus, in this problem we are dealing with the following pattern of symmetry breaking:

$$\mathrm{O}(3) \to \mathrm{U}(1). \tag{1.21}$$

Two out of three generators are broken; hence, we expect two Goldstone bosons. Let us see whether this expectation comes true.

Parametrizing the field $\vec{\phi}$ near this vacuum as $\vec{\phi}(x) = \{\varphi(x), \chi(x), v + \eta(x)\}$ and calculating $U(\varphi, \chi, \eta)$, it is easy to see that only one field, η, has a mass term, $m_\eta = \sqrt{2}\mu$, while the fields φ and χ have only cubic and quartic interactions and remain massless. The fields φ and χ present two Goldstone bosons in the problem at hand. The interaction depends on the combination $\varphi^2 + \chi^2$ and is invariant under the U(1) rotations

$$\varphi \to \varphi \cos\alpha + \chi \sin\alpha, \qquad \chi \to -\varphi \sin\alpha + \chi \cos\alpha, \tag{1.22}$$

in full agreement with the existence of an unbroken U(1) symmetry.

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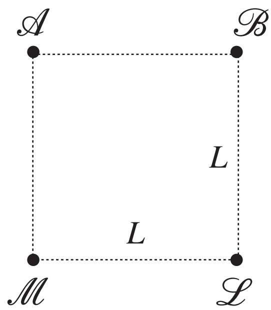

Fig. 1.6 Four towns on the map form a perfect square. Dotted lines are for orientation.

The signature of discrete symmetry breaking is the occurrence of domain walls (kinks).

Summarizing, if continuous (global) symmetries are spontaneously broken then massless Goldstone bosons emerge, one such boson for each broken generator. The occurrence of Goldstones (gapless excitations) is the signature of spontaneous continuous symmetry breaking. A reservation must be added immediately: Goldstone bosons do not appear in $D = 1 + 1$ theories unless they are sterile. We will discuss this subtle aspect in more detail later (see Section 6.5.2).

The interactions of Goldstone bosons respect the unbroken symmetries of the theory. These symmetries are realized linearly; the broken part of the original symmetry is realized nonlinearly.

Exercise

1.1.1 Mayors of four towns located as shown in Fig. 1.6 decided to build a railroad connecting all four towns $\mathcal{A}$, $\mathcal{B}$, $\mathcal{L}$, and $\mathcal{M}$ with each other (possibly with connections). They also decided that its length must be minimal. The towns $\mathcal{A}$, $\mathcal{B}$, $\mathcal{L}$, $\mathcal{M}$ form a square on the map exhibiting a Z_4 symmetry – the symmetry with respect to rotations by $\pi/2$. What is the symmetry of the map with the minimal-length railroad?

1.2 Spontaneous Breaking of Gauge Symmetries

1.2.1 Abelian Theories

The simplest example of the spontaneous breaking of gauge symmetries is provided by the quantum electrodynamics (QED)[3] of a charged scalar field whose

[3] Strictly speaking, QED *per se* is under-defined at short distances, where the effective coupling grows and hits the Landau pole. Thus to make it consistent an ultraviolet completion is needed at short distances. For instance, one can embed QED into an asymptotically free theory. The Georgi–Glashow model, Section 4.1.1, gives an example of such an embedding. It is important to understand that different ultraviolet completions do not necessarily lead to the same physics in the infrared. For instance, Polyakov's confinement in three-dimensional QED illustrates this statement in a clear-cut manner; see Section 9.7.

self-interaction is described by the potential depicted in Fig. 1.5. This theory is obtained by gauging the model (1.13) with global U(1) symmetry that was studied in Section 1.1.6. In other words we add the photon field, whose interaction with the matter fields is introduced through a covariant derivative, giving

$$S = \int d^D x \left[-\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \left(\mathcal{D}_\mu \phi\right)^* \left(\mathcal{D}^\mu \phi\right) - U(\phi) \right], \tag{1.23}$$

where e is the electromagnetic coupling and the covariant derivative $\mathcal{D}$ is defined as

$$\mathcal{D}_\mu = \partial_\mu - i A_\mu . \tag{1.24}$$

The kinetic term of the photon field is standard. Now the Lagrangian is invariant under the *local* U(1) transformation

$$\phi(x) \to e^{i\alpha(x)} \phi(x), \quad A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x). \tag{1.25}$$

If the potential has the form (1.16), the field ϕ develops an expectation value and the gauge U(1) symmetry is spontaneously broken.

I hasten to add that the terminology "spontaneously broken gauge symmetry," although widely accepted, is, in fact, rather sloppy and confusing.[4] What exactly does one mean by saying that the gauge symmetry is spontaneously broken? The gauge symmetry, in a sense, is not a symmetry at all. Rather, it is a description of x physical degrees of freedom in terms of $x + y$ variables, where y variables are redundant and the corresponding degrees of freedom are physically unobservable. Only those points in the field space that are given by gauge-nonequivalent configurations are to be treated as distinct.

If we decouple the photon by setting $e = 0$, the action (1.23) is invariant under global phase rotations. The condensation of the scalar field breaks this invariance, but the invariance of the "family of models" is not lost. Under this phase transformation one vacuum goes into another that is physically equivalent. Say, if we start from the vacuum characterized by a real value of the order parameter ϕ, then in the "rotated" vacuum the order parameter is complex. The spontaneous breaking of any global symmetry leads to a set of degenerate (and physically equivalent) vacua.

Switching on the electromagnetic interaction (i.e., setting $e \neq 0$), we lose the vacuum degeneracy – the degeneracy associated with the spontaneous breaking of the global symmetry. Indeed, all states related by phase rotation are gauge equivalent. They are represented by a single state in the Hilbert space of the theory. In other words, one can always *choose* the vacuum value of ϕ to be real. This is nothing other than the (unitary) gauge condition. Thus, the spontaneous breaking of the gauge symmetry does not imply, generally speaking, the existence of a degenerate set of vacua as is the case for the global symmetries. Then what does it mean, after all?

Unitary gauge, first appearance of the Higgs field

By inspecting the action (1.23) it is not difficult to see that if ϕ has a nonvanishing (and constant) value in the vacuum, the spectrum of the theory does not contain any massless vector particles. The photon acquires three polarizations and a mass $m_V = \sqrt{2} e v$, where v is a real parameter, $v = \langle \phi \rangle$. The remaining degree of freedom is a real (rather than complex) scalar field, the Higgs field, with mass $m_H = \sqrt{2} g v$.

[4] At present theorists tend to say that the theory is "Higgsed" when there is a spontaneously broken gauge symmetry.

This is seen from the decomposition (1.18), where χ must be set to zero because the field ϕ is real in the unitary gauge. The theoretical discovery of the Higgs phenomenon goes back to [3–5]. This regime is referred to as the *Higgs phase*. One massless scalar field is eaten up by the photon field in the process of the transition to the Higgs phase. In the Higgs phase the electric charge is screened by the vacuum condensates. Probe (static) electric charges will see the Coulomb potential $\sim 1/R$ at distances less than m_V^{-1} and the Yukawa potential $\sim \exp(-m_V R)/R$ at distances larger than m_V^{-1}. Moreover, the gauge coupling runs, according to the standard Landau formula, only at distances shorter than m_V^{-1} and becomes frozen at m_V^{-1}.

1.2.2 Phases of the Abelian Theory

Quantum electrodynamics was historically the first gauge theory studied in detail. This model is simple, with no mysteries. Nevertheless, it is nontrivial exhibiting three different types of behavior at large distances.

We have just identified the Higgs regime, in which all excitations are massive. At large distances there is no long-range interaction between charges.

Now we replace the scalar charged matter fields by spinor fields (electrons) with mass m. The same probe charges will experience a totally different interaction at large distances, the Coulomb interaction, with potential proportional to

$$V(R) \sim \frac{e^2(R)}{R},$$

where R is the distance between the probe charges. Classically e^2 is a constant. Quantum corrections due to virtual electron loops make e^2 run.

Its behavior is determined by the well-known Landau formula, which tells us that at large distances e^2 decreases logarithmically:

$$e^2(R) \sim \frac{1}{\ln R}. \tag{1.26}$$

If m is finite, the logarithmic fall-off is frozen at $R \sim m^{-1}$. The corresponding limiting value of e^2 is

$$e_*^2 = e^2(R = m^{-1}).$$

The potential between two distant static charges is

$$V(R) \sim \frac{e_*^2}{R}, \qquad R \to \infty. \tag{1.27}$$

The dynamical regime having this type of long-distance behavior is referred to as the *Coulomb phase*. In the case at hand we are dealing with the Abelian Coulomb phase.[5]

Now let us ask ourselves what happens if the electron mass vanishes. Unlike the massive case, where the running coupling constant is frozen at $R = m^{-1}$, in the theory with $m = 0$ the logarithmic fall-off (1.26) continues indefinitely: at asymptotically large R the effective coupling becomes arbitrarily small.

[5] Behavior like (1.27) can occur in non-Abelian gauge theories as well, as we will see later. Such non-Abelian gauge theories, with long-range potential (1.27), are said to be in the non-Abelian Coulomb phase.

Thus, in the asymptotic limit of massless spinor QED we have a free photon and a massless electron whose charge is completely screened. The theory has no localized asymptotic states and no mass shell, nor S matrix in the usual sense of this word. Still, it is well defined in, say, a finite volume.

This phase of the theory is referred to as an *infrared-free phase*. Sometimes it is also called *the Landau zero-charge phase*.

Summarizing, even in the simplest Abelian example we encounter three different phases, or dynamical regimes: the Coulomb phase, the Higgs phase, and the free (Landau) phase, depending on the details of the matter sector. All these regimes are attainable in non-Abelian models too.

The non-Abelian gauge theories are richer since they admit more dynamical regimes, to be discussed in Section 1.3.

1.2.3 Higgs Mechanism in Non-Abelian Theories

The Higgs mechanism in QED, considered in Section 1.2.1, extends straightforwardly to non-Abelian theories. The only difference is that U(1) is replaced by a non-Abelian group, which is then gauged. The essence of the construction remains the same.

Instead of the single complex field ϕ of QED (see Eq. (1.23)), we start with a multiplet of scalar fields ϕ_i belonging to a representation R of a non-Abelian group G. The representation R may be reducible; for simplicity, however, we will assume R to be irreducible for the time being. The generators of the group G in the representation R will be denoted T^a, where

$$\left[T^a, T^b\right] = i f^{abc} T^c, \qquad \mathrm{Tr}\left(T^a T^b\right) = T(R)\delta^{ab}, \tag{1.28}$$

In the mathematical literature $T(R)$ is known as the Dynkin index.

and f^{abc} are the structure constants of the group G. In this book we will deal mostly with the unitary groups SU(N). Occasionally, the orthogonal groups O(N) will be involved.

Assume the self-interaction of the fields ϕ to be such that the lowest-energy state – the vacuum – breaks

$$G \to H, \tag{1.29}$$

where H is a subgroup of G. A particular case is $H = 1$, corresponding to the complete breaking of G. In accordance with the general Goldstone theorem, the spontaneous breaking (1.29) entails the occurrence of $\dim G - \dim H$ Goldstone bosons (here $\dim G$ is the dimension of the group, i.e., the number of its generators).

See Section 6.5.1.

Now, to gauge the theory, instead of the conventional derivative ∂_μ we introduce a covariant derivative

$$\mathcal{D}_\mu = \partial_\mu - i A_\mu, \tag{1.30}$$

where

$$A_\mu \equiv A_\mu^a T^a \tag{1.31}$$

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and A_μ^a are the gauge fields. If $\phi(x)$ transforms as $\phi \to U(x)\phi$ for any $U(x) \in G$ then $\mathcal{D}_\mu\phi$ must transform in the same way:

$$\mathcal{D}_\mu\phi(x) \to U(x)\left(\mathcal{D}_\mu\phi(x)\right). \tag{1.32}$$

This requirement defines the transformation law of the gauge fields:

$$A_\mu \to U\,A_\mu U^{-1} + i\,U\partial_\mu U^{-1}. \tag{1.33}$$

The gauge field strength tensor (to be denoted by $G_{\mu\nu}$ rather than $F_{\mu\nu}$, to distinguish the non-Abelian and Abelian cases) is defined as[6]

$$\begin{aligned} G_{\mu\nu} &\equiv i[\mathcal{D}_\mu,\mathcal{D}_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \\ &= \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc}A_\mu^b A_\nu^c\right)T^a \equiv G_{\mu\nu}^a T^a. \end{aligned} \tag{1.34}$$

The kinetic term of the gauge field is

$$\mathcal{L}_{\rm YM} = -\frac{1}{4g^2}G_{\mu\nu}^a\,G^{\mu\nu,a}, \tag{1.35}$$

while the scalar fields are described by the Lagrangian

$$\mathcal{L}_{\rm matter} = \mathcal{D}_\mu\phi^*\left(\mathcal{D}^\mu\phi\right) - U(\phi) \tag{1.36}$$

where summation over the multiplet-R index is implied. In what follows we will use the notations $\mathcal{D}_\mu\phi^*$ and $\mathcal{D}_\mu\bar\phi$ indiscriminately.

Reminder: The canonical form of the Yangs–Mills Lagrangian (corresponding to canonically normalized kinetic term) is

$$\mathcal{L}_{\rm YM} = -\frac{1}{4}G_{\mu\nu}^a G^{\mu\nu,a},\quad G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc}T^2 A_\mu^b A_\nu^c,\quad \mathcal{D}_\mu = \partial_\mu - igA_\mu.$$

In this form the coupling constant g appears in the interaction vertices. This convention is preferred in perturbation theory. In nonperturbative studies a more convenient convention is a (noncanonical) form obtained from the canonical one by the substitution $A_\mu \longrightarrow \frac{1}{g}A_\mu$. Then the coupling constant g disappears from all vertices, e.g.,

$$G_{\mu\nu}^a \longrightarrow \frac{1}{g}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc}A_\mu^b A_\nu^c\right).$$

The factor g^2 appears in the numerator of the gluon propagators. Equations (1.30), (1.34), and (1.35) refer to the noncanonical normalization. In this book I use both.

Now the $\dim G - \dim H$ Goldstone bosons that existed before gauging are paired up with the gauge bosons to produce $\dim G - \dim H$ three-component massive vector particles. In the unitary gauge one imposes $\dim G - \dim H$ gauge conditions. If instead of $\langle{\rm vac}|\phi|{\rm vac}\rangle$ we use the shorthand $\phi_{\rm vac}$ then $T^a\phi_{\rm vac} = 0$, provided that $T^a \in H$. The corresponding $\dim H$ gauge bosons stay massless. The masses of the remaining $\dim G - \dim H$ gauge bosons are obtained from the matrix

Mass formula for gauge bosons

$$m_{ab}^2 = 2g^2\left(\phi_{\rm vac}^* T^a T^b\phi_{\rm vac}\right),\qquad T^{a,b} \in G/H. \tag{1.37}$$

[6] It is obvious that the transformation law of $G_{\mu\nu}$ under the gauge transformation is

$$G_{\mu\nu} \to UG_{\mu\nu}U^{-1}.$$

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Referring to [6] for a more detailed discussion of the generalities, in the remainder of this section we will focus on two examples of particular interest.

1.2.3.1 From $SU(2)_{local}$ to $SU(2)_{global}$

The model discussed in this subsection is the Glashow–Weinberg–Salam (GWS) model of electroweak interactions – *part* of the Standard Model (SM) of particle physics.[7]

The gauge group is SU(2). The structure constants are $f^{abc} = \varepsilon^{abc}$, where ε^{abc} is the Levi–Civita tensor $(a, b, c = 1, 2, 3)$. The matter sector consists of an SU(2) doublet of complex scalar fields ϕ^i, where $i = 1, 2$. In other words, the ϕ^i are the scalar quarks in the fundamental representation. The covariant derivative acts on ϕ^i as follows:

$$\mathcal{D}_\mu \phi(x) \equiv \left(\partial_\mu - i\, A_\mu^a T^a\right)\phi, \qquad T^a = \tfrac{1}{2}\tau^a, \tag{1.38}$$

where the τ^a are the Pauli matrices. We will choose the ϕ self-interaction potential to be in the form

$$U = \lambda\left(\bar{\phi}\phi - v^2\right)^2. \tag{1.39}$$

Quite often it is said that this theory has just SU(2) gauge symmetry and nothing else. This is wrong. In fact, its symmetry is

$$SU(2)_{\rm gauge} \times SU(2)_{\rm global}. \tag{1.40}$$

One can prove this in a number of ways. Probably, the quickest proof is as follows. Let us introduce the 2×2 matrix

$$X = \begin{pmatrix} \phi^1 & -(\phi^2)^* \\ \phi^2 & (\phi^1)^* \end{pmatrix}. \tag{1.41}$$

The Lagrangian of the model rewritten in terms of X takes the form [7]

$$\mathcal{L} = -\frac{1}{4g^2} G_{\mu\nu}^a G^{\mu\nu,a} + \frac{1}{2}\mathrm{Tr}\left(\mathcal{D}_\mu X\right)^\dagger \left(\mathcal{D}_\mu X\right) - \lambda\left(\frac{1}{2}\mathrm{Tr}\, X^\dagger X - v^2\right)^2. \tag{1.42}$$

Note that the generators T^a in the covariant derivative $\mathcal{D}$ act on the matrix X through matrix multiplication from the left. This Lagrangian is obviously invariant under the transformation

$$X(x) \to U(x) X(x) M^{-1}, \tag{1.43}$$

supplemented by (1.33), where M is an arbitrary x-independent matrix from $SU(2)_{\rm global}$. The symmetry (1.40) is apparent. In the vacuum, $\frac{1}{2}\mathrm{Tr}\, X^\dagger X = v^2$. Using gauge freedom (three gauge parameters), one can always choose the unitary gauge in which the vacuum value of X is

$$X_{\rm vac} = v\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.44}$$

[7] The latter also includes QCD.

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This vacuum expectation value breaks the $\mathrm{SU}(2)_{\mathrm{gauge}}$ and $\mathrm{SU}(2)_{\mathrm{global}}$ symmetries, but the diagonal global SU(2) symmetry corresponding to $U = M$ remains unbroken. Thus, the symmetry-breaking pattern is

$$\mathrm{SU}(2)_{\mathrm{gauge}} \times \mathrm{SU}(2)_{\mathrm{global}} \to \mathrm{SU}(2)_{\mathrm{diag}}. \tag{1.45}$$

Three would-be Goldstone bosons are eaten up by the gauge bosons, transforming them into massive W bosons belonging to the triplet (adjoint) representation of the unbroken $\mathrm{SU}(2)_{\mathrm{diag}}$ symmetry. There are no massless particles in this model. The physically observable excitations are three W bosons with mass $m_W = gv/\sqrt{2}$ and one Higgs particle (a singlet with respect to $\mathrm{SU}(2)_{\mathrm{global}}$) with mass $2\sqrt{\lambda}v$.

This model will be discussed in more detail in Section 5.4.12 in the context of instanton calculus.

1.2.3.2 From $\mathrm{SU}(2)_{\mathrm{local}}$ to $\mathrm{U}(1)_{\mathrm{local}}$

Below, I will outline the Georgi–Glashow model [8]. If necessary, it can be easily generalized to $\mathrm{SU}(N)$, with the gauge-symmetry-breaking pattern

$$\mathrm{SU}(N) \to \mathrm{U}(1)^{N-1}.$$

The Lagrangian of the model is

$$\mathcal{L} = -\frac{1}{4g^2} G^a_{\mu\nu} G^{\mu\nu,a} + \frac{1}{2}(\mathcal{D}_\mu \phi^a)(\mathcal{D}^\mu \phi^a) - \lambda(\phi^a\phi^a - v^2)^2, \tag{1.46}$$

where ϕ^a is the triplet of real scalar fields in the adjoint representation; the covariant derivative in the adjoint acts as

$$\mathcal{D}_\mu \phi^a = \partial_\mu \phi^a + \varepsilon^{abc} A^b_\mu \phi^c. \tag{1.47}$$

One can always choose a gauge (the unitary gauge) in which

With matter fields in the adjoint representation, one can say that $\mathrm{O}(3) \to \mathrm{O}(2)$.

$$\phi^1 = \phi^2 \equiv 0, \qquad \phi^3 \neq 0. \tag{1.48}$$

The vacuum value of the field ϕ is

$$\phi^3_{\mathrm{vac}} = v, \tag{1.49}$$

which implies that the $\mathrm{SU}(2)_{\mathrm{gauge}}$ symmetry breaks down to $\mathrm{U}(1)_{\mathrm{gauge}}$. Since T^3 acts on ϕ_{vac} trivially, A^3_μ remains massless (a "photon"), while the two other gauge bosons become W bosons, acquiring mass $m_W = gv$, where

$$W^\pm = \frac{A^1_\mu \pm i A^2_\mu}{\sqrt{2}g}.$$

Besides the two W bosons and the photon there is another physical particle, the Higgs boson, with mass $m_H = 2\sqrt{2\lambda}v$. At distances much larger than m_W^{-1} the W bosons decouple and the theory reduces to QED.

This model will be discussed in Chapter 4.

Exercise

1.2.1 Assume we have Yang–Mills theory with the gauge group SU(3) and the Higgs sector consisting of one real scalar field in the *adjoint* representation of SU(3). The latter develops a generic vacuum expectation value (large compared to Λ, the dynamical scale of the theory). Determine the most general pattern of Higgsing and the masses of all gauge bosons.

1.3 Phases of Yang–Mills Theories

The phase structure of non-Abelian gauge theories is richer than that of QED. In addition to the three regimes described in Section 1.2.2, which were known already in the 1960s, Yang–Mills theories can exhibit confining and conformal phases, phases with or without chiral symmetry breaking, and so on.

1.3.1 Confinement

We will start by discussing the confining phase. Consider pure Yang–Mills theory (1.35), where the gauge group is assumed to be SU(N) for arbitrary N. At short distances the running coupling constant falls off logarithmically [9],

$$\frac{\alpha(p)}{2\pi} = \frac{1}{\beta_0 \ln(p/\Lambda)}, \qquad \beta_0 = \frac{11N}{3}, \tag{1.50}$$

Asymptotic freedom

the interaction switches off, and one can detect – albeit indirectly – the gluon degrees of freedom as described by (1.35). The parameter Λ is the so-called *dynamical scale*.

At large distances we enter a strong coupling regime. The physically observed spectrum is drastically different from what we see in the Lagrangian. In the case at hand an experimentalist, if he or she could exist in the world of pure Yang–Mills theories, would observe a spectrum of glueballs that are, generally speaking, nondegenerate in mass. One can visualize the glueballs as a closed string (or, better, a tube), in a highly quantum state, i.e., a string-like field configuration which wildly oscillates, pulsates, and vibrates; see Fig. 1.7. If we add nondynamical (i.e., very heavy) quarks into the theory and set the quark and antiquark at a large distance from each other, such a string will stretch between them (as shown in the figure on the

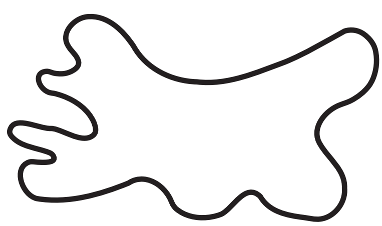

Fig. 1.7 A quantum closed string as a glueball.

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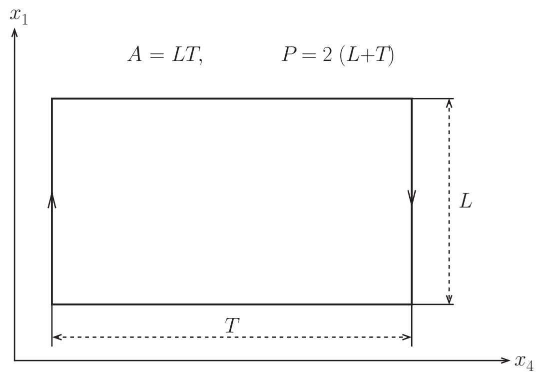

Fig. 1.8 A Wilson contour *C*, with area *A* and perimeter *P*. The probe quark is dragged along this contour.

opening page of this chapter), connecting the pair of probe quarks[8] in an inseparable configuration. What is depicted in that figure is a highly quantum (presumably, nonperturbative) open string configuration with quarks attached at the ends. If we try to pull the quarks apart we just make the string longer, while the energy of the configuration grows linearly with separation.

This phase of the theory, whose existence was conjectured in 1973 [9], is referred to as color confinement. Although there is no analytic proof of color confinement that could be considered exhaustive, there is ample evidence that this regime does, indeed, occur. First, a version of color confinement was observed in certain supersymmetric Yang–Mills theories [10]. Second, the formation of tube-like configurations connecting heavy probe quarks was demonstrated numerically, in lattice simulations. I will not dwell on the dynamics leading to color confinement (this topic will be postponed until we have learned more of the underlying physics; see Chapters 3 and 9). It is worth noting, however, that there are distinct versions of confinement regimes, such as oblique confinement [11], Abelian and non-Abelian confinement, both of which are found in Yang–Mills theories, etc. Some examples will be considered in Chapter 9. The impatient and curious reader is directed to the original literature or the review paper [12].

Kenneth Wilson was the first to suggest [13] a very convenient criterion indicating whether a given gauge theory is in the confinement phase. Consider a gauge theory in Euclidean space–time. Introduce a closed contour, as shown in Fig. 1.8. Assume that $T \gg L \gg \Lambda^{-1}$, i.e., the contour is large.[9] Consider the *Wilson operator*

$$W(C) = \frac{1}{\dim_R} \mathrm{Tr}\, P \exp\left[i \oint_C A^a_\mu(x) T^a_R \, dx^\mu \right], \tag{1.51}$$

where the subscript R indicates the representation of the gauge group to which the probe quark belongs (usually the fundamental representation).

[8] Probe quarks Q are those for which pair production in the vacuum can be ignored. This can be achieved by endowing them with a mass $m_Q \to \infty$. In contrast, dynamical quarks q are either massless or light, $m_q \ll \Lambda$, where Λ is the same scale parameter as in (1.50).

[9] Generally speaking the contour does not have to be rectangular, but for the rectangular contour the result is simpler to interpret.

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The asymptotic form of the vacuum expectation value of $W(C)$ is

$$\langle W(C)\rangle_{\text{vac}} \propto \exp[-(\mu P + \sigma A)], \tag{1.52}$$

where $A = LT$ is the area of the contour and $P = 2(L + T)$ is the perimeter; μ and σ are numerical coefficients of dimension mass and mass squared, respectively. If we have

$$\sigma \neq 0 \tag{1.53}$$

then the theory is in the confinement phase, while at $\sigma = 0$ the theory does not confine.[10] We refer to these cases as the area law and the perimeter law, respectively.

Why does the area law implies confinement? The reason is that, on general grounds,

$$\langle W(C)\rangle_{\text{vac}} \propto \exp[-V(L)T] \tag{1.54}$$

if the contour is chosen as in Fig. 1.8. Hence, the area law means that the potential $V(L)$ between distant probe quarks Q and $\bar{Q}$ is $V(L) = \sigma L$ at $L \gg \Lambda^{-1}$. The coefficient σ is the string tension (in many publications it is denoted by T rather than σ).

1.3.2 Adding Massless Quarks

From pure Yang–Mills theory we pass to theories with matter. Considering N_f massless quarks in the fundamental representation is the first step. Each quark is described by a Dirac spinor and the overall number of Dirac spinors is N_f. At $N = 3$ and $N_f = 3$ we obtain quantum chromodynamics (QCD), the accepted theory of strong interactions in nature.

The most obvious impact of adding massless quarks is the change in β_0, the first coefficient in the Gell-Mann–Low function. Instead of the expression of β_0 in (1.50) we now have

$$\beta_0 = \tfrac{11}{3}N - \tfrac{2}{3}N_f. \tag{1.55}$$

If $N_f > \frac{11}{2}N$ then the coefficient changes sign, we lose asymptotic freedom, and the Landau regime sets in. The theory becomes infrared-free, much like QED with massless electrons. From a dynamics standpoint this is a rather uninteresting regime.

Let us assume that $N_f \leq \frac{11}{2}N$. Now we will address the question: what happens if N_f is only slightly less than the critical value $\frac{11}{2}N$? To answer this we need to know the two-loop coefficient in the β function.

1.3.3 Conformal Phase

See Sections 1.4 and 8.4

The response of Yang–Mills theories to scale and conformal transformations is determined by the trace of the energy–momentum tensor

$$T^{\mu}_{\mu} \propto \beta(\alpha) G^{a}_{\mu\nu} G^{\mu\nu,a}, \tag{1.56}$$

[10] If $\sigma \neq 0$ the perimeter term is subleading. The parameter μ renormalizes the probe quark mass.

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where $\beta(\alpha)$ is the Gell-Mann–Low function (also known as the β function). In SU(N) Yang–Mills theory with N_f quarks it has the form

$$\beta(\alpha) = \frac{\partial \alpha(\mu)}{\partial \ln \mu} = -\beta_0 \frac{\alpha^2}{2\pi} - \beta_1 \frac{\alpha^3}{4\pi^2} - \cdots, \qquad \alpha = \frac{g^2}{4\pi}, \tag{1.57}$$

where β_0 is given in (1.55) while

$$\beta_1 = \frac{17}{3} N^2 - \frac{N_f}{6N}\left(13N^2 - 3\right). \tag{1.58}$$

At small α the term $\sim \beta_0$ in (1.57) dominates and so the β function is negative, implying asymptotic freedom at short distances. What is the large-distance behavior of the running coupling constant $\alpha(\mu)$?

Assume that

$$N_f = \tfrac{11}{2}N - \nu, \qquad 0 < \nu \ll \tfrac{11}{2}N. \tag{1.59}$$

Then the first coefficient, β_0, is anomalously small,

$$\beta_0 = \frac{2}{3}\nu. \tag{1.60}$$

The ratio β_1/β_0 is negative.

At the same time the second coefficient is not suppressed; it is of a normal order of magnitude,

$$\beta_1 = -\tfrac{25}{4}N^2 + \tfrac{11}{4} + \tfrac{1}{6}\nu N^{-1}\left(13N^2 - 3\right), \tag{1.61}$$

and *negative*!

Position of IR fixed point

As the scale μ decreases (at larger distances), the running gauge coupling constant grows and the second term in (1.57) eventually becomes important. Generally speaking, the second term takes over the first one at $N\alpha/\pi \sim 1$ (the strong coupling regime), when *all* terms in the α expansion of the β function are equally important and one cannot limit oneself to the first two terms. However, if N_f is only slightly less than $\frac{11}{2}N$ then the β function develops a zero at a value of α which is parametrically small,[11] namely, we have

$$\frac{N\alpha_*}{2\pi} = \frac{N\beta_0}{-\beta_1} = \frac{8}{75}\frac{\nu}{N f(N,\nu)}, \tag{1.62}$$

where

$$f(N,\nu) = 1 - \frac{11}{25N^2} - \frac{2\nu}{75N^3}\left(13N^2 - 3\right) \sim 1. \tag{1.63}$$

In other words, the second term catches up with the first one prematurely when $N\alpha/\pi \ll 1$. Hence we are at weak coupling and higher-order terms are inessential. The facts of the existence of this zero and its position are reliably established.

As an example, let me indicate that if $N = 3$ and $N_f = 15$ then

$$\frac{\alpha_*}{2\pi} = \frac{1}{44}. \tag{1.64}$$

The β function is shown in Fig. 1.9.

[11] By "parametrically" I mean that if, for instance, N is large while ν does not scale with N then $f(N,\nu) \to 1$, and $N\alpha_*/2\pi \to (8/75)(\nu/N)$.

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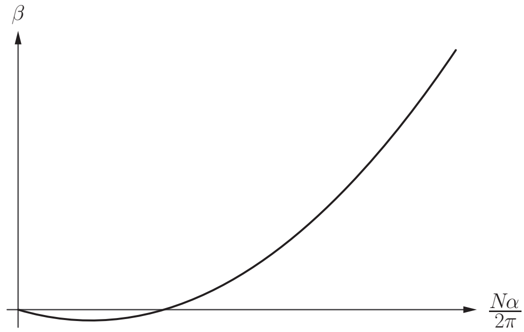

Fig. 1.9 The β function at N_f slightly less than $\frac{11}{2}N$. The horizontal axis presents $N\alpha/2\pi$. The zero of the beta function is at $\frac{8}{75}\nu/N \ll 1$.

The zero of the β function depicted in Fig. 1.9 is nothing other than the infrared fixed point of the theory. If we start from the value of α lying between 0 and α_* and let α run then it will hit α_* in the infrared (remember, in the ultraviolet $\alpha(\mu)$ tends to 0).

Hence at large distances $\beta(\alpha) = \beta(\alpha^*) = 0$, implying that the trace of the energy–momentum tensor of the theory vanishes and so the theory is in the *conformal* phase. There are no localized particle-like states in the spectrum; rather, we are dealing with massless unconfined *interacting* quarks and gluons. All correlation functions at large distances exhibit a power-like behavior.[12] As long as α^* is small, the interactions of the massless quarks and gluons in the theory are weak at all distances, short and large, and thus amenable to the standard perturbative treatment. In particular, the potential between two probe, static, quarks at a large separation R will behave approximately as α^*/R, reminding us of conventional QED with massive electrons.

Since we are absolutely certain that, slightly below $N_f = \frac{11}{2}N$, we are in the conformal phase, on increasing ν (i.e., decreasing N_f) we cannot leave this phase straight away. There should exist a critical value N_f^* of the number of flavors above which the theory is conformal in the infrared. The interval

Conformal window

$$N_f^* \le N_f \le \tfrac{11}{2}N \tag{1.65}$$

is referred to as a *conformal window*.[13] The exact value of N_f^* is unknown. From experiment we know that $N_f^* > 3$ at $N = 3$. On general grounds one can argue that $N_f^* \sim cN$, where c is a numerical constant of the order of unity. Of course, near the left-hand (lower) edge of the conformal window one should expect $N\alpha_*/2\pi \sim 1$ so that the theory, albeit conformal in the infrared, is strongly coupled. In particular, in this case there is no reason for the anomalous dimensions to be small.

Summarizing, if N_f lies in the interval (1.65) then the theory is in the conformal phase. For N_f close to the right-hand (upper) edge of the conformal window the theory is weakly coupled and all anomalous dimensions are calculable. Belavin and

[12] We will see in Chapter 8, Section 8.4, that the trace of the energy–momentum tensor in Yang–Mills theories with massless quarks is proportional to $\beta(\alpha)G^a_{\mu\nu}G^{\mu\nu,a}$. Basic data on conformal symmetry are collected in Appendix section 1.4. A more detailed discussion of the implications of conformal invariance in four and two dimensions can be found, e.g., in [14].

[13] This terminology was suggested in [12], and it took root.

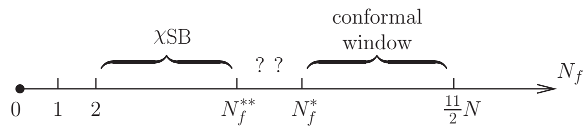

Fig. 1.10 Dynamical regimes change with the number of massless quarks N_f.

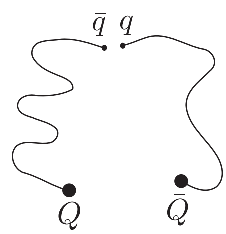

Fig. 1.11 The string between two probe quarks Q and $\bar{Q}$ can break through $\bar{q}q$ pair creation in Yang–Mills theories with dynamical quarks.

Migdal considered this model in the early 1970s [15]. Somewhat later, it was studied thoroughly by Banks and Zaks [16].

1.3.4 Chiral Symmetry Breaking

Next, in our journey along the N_f axis (Fig. 1.10) let us descend to $N_f = 1, 2, 3, \ldots$ Strictly speaking, dynamical quarks (in the fundamental representation) negate confinement understood in the sense of Wilson's criterion – the area law for the Wilson loop disappears. Indeed the string forming between the probe quarks can break, through $\bar{q}q$ pair creation, when the energy stored in the string becomes sufficient to produce such a pair (Fig. 1.11). As a result, sufficiently large Wilson loops obey the perimeter law rather than the area law. However, intuitively it is clear that, in essence this is the same confinement mechanism, although in the case at hand it is natural to call it *quark confinement*. The dynamical quarks are identifiable at short distances in a clear-cut manner and yet they never appear as asymptotic states. Experimentalists detect only color-singlet mesons of the type $\bar{q}q$ or baryons of the type qqq.

Theoretically, if necessary, one can suppress $\bar{q}q$ pair creation by sending N to ∞; see Chapter 9.

At $N_f \geq 2$ a new and interesting phenomenon shows up. The global symmetry of Yang–Mills theories with more than one massless quark flavor is

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_V. \tag{1.66}$$

The vectorial U(1) symmetry is simply the baryon number, while the axial U(1) is anomalous (see Chapter 8) and hence is not shown in (1.66). The origin of the chiral $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ symmetry is as follows. The quark part of the Lagrangian has the form

Massless quark sector

$$\mathcal{L}_{\text{quark}} = \sum_f \bar{\Psi}_f i \not{\mathcal{D}} \Psi^f, \tag{1.67}$$

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where Ψ^f is the Dirac spinor of a given flavor f and $\mathcal{D}\!\!\!/ = \gamma^\mu \mathcal{D}_\mu$. Each Dirac spinor is built from one left- and one right-handed Weyl spinor,

$$\Psi_i^f = \begin{pmatrix} \xi^f_{\alpha,i} \\ \bar{\eta}_i^{\dot{\alpha},f} \end{pmatrix}, \tag{1.68}$$

Dirac spinor from two Weyl spinors

where i is the color index (i.e., the index of the fundamental representation of $\mathrm{SU}(N)_{\mathrm{color}}$) while f is the flavor index, $f = 1, 2, \ldots, N_f$. The left- and right-handed Weyl spinors in the kinetic term (1.67) totally decouple from each other. Hence, $\mathcal{L}_{\mathrm{quark}}$ is invariant under the independent global rotations

$$\xi \to U\xi \quad \text{and} \quad \bar{\eta} \to U'\bar{\eta}, \qquad U \in \mathrm{SU}(N_f)_L, \quad U' \in \mathrm{SU}(N_f)_R. \tag{1.69}$$

Experimentally it is known that the chiral $\mathrm{SU}(N_f)_{\mathrm{L}} \times \mathrm{SU}(N_f)_R$ symmetry is spontaneously broken at $N = 3$ and $N_f = 2, 3$, leaving the diagonal $\mathrm{SU}(N_f)_V$ subgroup unbroken. $N_f^2 - 1$ massless Goldstone bosons – the pions – emerge as a result of this spontaneous breaking. This phenomenon bears the name *chiral symmetry breaking* (χSB). In Chapter 8 we will outline theoretical arguments demonstrating χSB in the limit $N \to \infty$ with N_f fixed.

There are qualitative arguments showing that in four-dimensional Yang–Mills theory χSB may be a consequence of quark confinement plus some general features of the quark–gluon interaction. In particular, a well-known picture is that of Casher [17] "explaining"[14] why in Yang–Mills theories with massless quarks (no scalar fields!) color confinement entails a Goldstone-mode realization of the global axial symmetry of the Lagrangian. A brief outline is as follows. If we deal with massless quarks, the left-handed quarks are decoupled from the right-handed quarks in the QCD Lagrangian. If spontaneous breaking of the chiral symmetry does not take place, this decoupling becomes an exact property of the theory: the quark chirality (helicity) is exactly conserved. Assume that we produce an energetic quark–antiquark pair in, say, e^+e^- annihilation. Let us place the origin at the annihilation point. If the quarks' energy is high then they can be treated quasiclassically. Let us say that in the given event the quark produced is right-handed and moves off in the positive z direction; the antiquark will then move off in the negative z direction. If the quark energy is high ($E \gg \Lambda$, where Λ is the QCD scale parameter) the distance L that the quark travels before confining effects become critical is large, $L \sim E/\Lambda^2$. Color confinement means that the quark cannot move indefinitely in the positive z direction; at a certain time $T \sim E/\Lambda^2$ it should turn back and start moving in the negative z direction. Let us consider this turning point in more detail. Before the turn, the quark's spin projection on the z axis is $+1/2$. Since by assumption the quark's helicity is conserved, after the turn, when p_z becomes negative, the quark's spin projection on the z axis must be $-1/2$ (Fig. 1.12). In other words, $\Delta S_z = -1$. The total angular momentum is conserved, consequently, $\Delta S_z = -1$ must be compensated. At the time of the turn, the quark is far from the antiquark and so they do not "know" what their respective partners are doing; conservation of angular momentum must be achieved locally. The only object that could be responsible for

Casher's argument

[14] I have used quotation marks since Casher's discussion could be said to be a little nebulous and imprecise.

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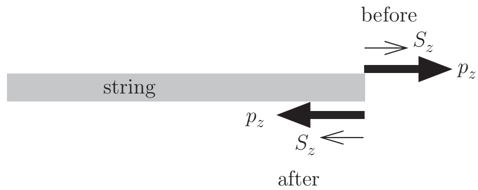

Fig. 1.12 Right-handed quark before and after the turning point.

compensating the quark ΔS_z is a QCD string that stretches in the z direction between the quark and the antiquark. The QCD string provides color confinement but it does not have L_z (more exactly, it is *presumed* to have no L_z) and, thus, cannot support the conservation of angular momentum in this picture. Thus, either the quark never turns (no confinement) or, if it does, chiral symmetry *must* be spontaneously broken.

The relation between quark confinement and χSB is a deep and intriguing dynamical question. Since I have nothing to add, let me summarize. There is a phase of QCD in which quark confinement and χSB coexist. On the N_f axis this phase starts at $N_f = 2$ and extends to some upper boundary $N_f = N_f^{**}$. We do not know whether N_f^{**} coincides with the left-hand edge of the conformal window N_f^*. It may happen that $N_f^{**} < N_f^*$, and the interval $N_f^{**} < N_f < N_f^*$ is populated by some other phase or phases (e.g., confinement without χSB) . . .

1.3.5 A Few Words on Other Regimes

Using various ingredients and mixing them in various proportions to construct a matter sector with the desired properties, one can reach other phases of Yang–Mills theories. For instance, by Higgsing the theory, as in Section 1.2.3.2, and breaking SU(N) down to U(1)$^{N-1}$ we can implement the Coulomb phase. Let us ask ourselves what happens if this Higgsing is implemented through the scalar fields in the fundamental representation, as in Section 1.2.3.1. If the vacuum expectation value $v \gg \Lambda$ then the theory is at weak coupling; it resembles the standard model. However, if $v \ll \Lambda$ then the theory is at strong coupling. Our intuition tells us that in this case it should resemble QCD, with a rich spectrum of composite color-singlet mesons having all possible quantum numbers.

There are convincing arguments [18] that there is no phase transition between these two regimes. Indeed, if the scalar fields are in the fundamental representation then the color-singlet interpolating operators that can be built from these fields and their covariant derivatives, and the gluon field strength tensor, span the space of physical (color-singlet or gauge-invariant) states in its *entirety*. All possible quantum numbers are covered. As the vacuum expectation value v changes from small to large, the strong coupling regime gives place smoothly to the weak coupling regime, possibly with a crossover in the middle. Each state existing at strong coupling is mapped onto its counterpart at weak coupling.

For instance, consider the operator

$$\mathrm{Tr}\left(\bar{X} i \overset{\leftrightarrow}{\mathcal{D}}_\mu X \tau^a\right). \qquad (1.70)$$

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At $v \ll \Lambda$ this operator produces a ρ meson and its excitations. The low-lying excitations could be seen as resonances. As v increases and becomes much larger than Λ the very same operator obviously reduces to $v^2 W_\mu$ plus small corrections. It produces a W boson from the vacuum. It produces excitations, too, but they are no longer resonances; rather, they are states that contain a number of W bosons and Higgs particles with the overall quantum numbers of a single W boson. Note that the global SU(2) symmetry of the model of Section 1.2.3.1 is respected in both regimes. All states appear in complete representations of SU(2), e.g., triplets, octets, and so on.

In the general case the following conjecture can be formulated (Fradkin and Shenker [18]):

Suppose that, in addition to gauge fields, a given non-Abelian theory contains a set of Higgs fields in the fundamental representation, which, by developing vacuum expectation values (VEVs) can "Higgs" the gauge group completely while the set of gauge-invariant operators built from the fields of the theory spans the space of all possible global quantum numbers (such as spin, isospin, and all other global symmetries of the Lagrangian). Then on decreasing all the above VEVs in proportion to each other from large to small values we do not pass through a Higgs-confinement phase transition. Rather, a crossover from weak to strong coupling takes place. If in addition there are massless fermions coupled to the gauge fields then there could be a phase transition separating the chirally symmetric and chirally asymmetric phases. This would be an example of χSB without confinement.[15] *The opposite – confinement without χSB – is impossible in the absence of couplings between the fermion and scalar fields.*

Contrived matter sectors can lead to more "exotic" phases. I have already mentioned oblique confinement. In supersymmetric Yang–Mills theories with matter in the adjoint representation a number of unconventional phases were found in [19]. We will not consider them here, as this aspect goes far beyond our scope in the present text.

Exercise

1.3.1 In QED with one massless Dirac fermion, identify the only one-loop diagram that determines charge renormalization. Calculate this diagram and show that the following relation holds for the running coupling constant:

$$\frac{1}{e^2(p)} = \frac{1}{e^2(\mu)} - \frac{1}{6\pi^2} \ln \frac{p}{\mu}.$$

Landau formula

Regardless of the value of $e^2(\mu)$, at $p \ll \mu$ (i.e., at large distances) we have $e^2(p) \to 0$. This phenomenon is known as the Landau zero-charge or infrared freedom. However, at large p namely, $p = \mu \exp[6\pi^2/e^2(\mu)]$, we hit the Landau pole in $e^2(p)$. When one approaches this pole from below, perturbation theory fails.

[15] Such examples are known in supersymmetric Yang–Mills theories.

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1.4 Appendix: Basics of Conformal Invariance

Its generalization, superconformal symmetry, is briefly discussed in Section 10.19.3.

In this appendix we will review briefly some general features of conformal invariance. For a comprehensive consideration of conformal symmetry and its applications the reader is directed to [14, 20, 21].

In D-dimensional Minkowski space we have

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu,$$

where for $D = 4$, for example,

$$g_{\mu\nu} = \text{diag}\{1, -1, -1, -1\} \equiv \eta_{\mu\nu}. \tag{1.71}$$

Under the general coordinate transformation

$$x \to x'$$

the original metric $g_{\mu\nu}$ is substituted by

$$g_{\mu\nu} \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x), \tag{1.72}$$

so that the interval ds^2 remains intact. Clearly, the general coordinate transformations form a very rich class that includes, as a subclass, transformations that change only the scale of the metric:

$$g'_{\mu\nu}(x') = \omega(x) g_{\mu\nu}(x). \tag{1.73}$$

All transformations belonging to this subclass form, by definition, the *conformal group*. It is obvious that, for instance, the global *scale* transformations

$$x \to x' = \lambda x, \quad \lambda \text{ is a number}, \tag{1.74}$$

is a conformal transformation. Moreover, the Poincaré group (of translations plus Lorentz rotations of flat space) is always a subgroup of the conformal group. The Minkowski metric (1.71) is invariant with respect to translations and Lorentz rotations.

In general, conformal algebra in four dimensions includes the following 15 generators:

- P_μ (four translations);
- K_μ (four special conformal transformations);
- D (dilatation);
- $M_{\mu\nu}$ (six Lorentz rotations).

Below, a few simple facts concerning the action of the conformal group in four dimensions are summarized. The set of 15 transformations given above forms a 15-parameter Lie group, the conformal group. This is a generalization of the 10-parameter Poincaré group, that is formed from 10 transformations generated by P_α and $M_{\alpha\beta}$. By considering the combined action of various infinitesimal transformations taken in a different order, the Lie algebra of the conformal group can be shown to be as follows:

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$$\begin{aligned}
i[P^\alpha, P^\beta] &= 0,\\
i\left[M^{\alpha\beta}, P^\gamma\right] &= g^{\alpha\gamma}P^\beta - g^{\beta\gamma}P^\alpha,\\
i\left[M^{\alpha\beta}, M^{\mu\nu}\right] &= g^{\alpha\mu}M^{\beta\nu} - g^{\beta\mu}M^{\alpha\nu} + g^{\alpha\nu}M^{\mu\beta} - g^{\beta\nu}M^{\mu\alpha},\\
i\,[D, P^\alpha] &= P^\alpha,\\
i\,[D, K^\alpha] &= -K^\alpha,\\
i\left[M^{\alpha\beta}, K^\gamma\right] &= g^{\alpha\gamma}K^\beta - g^{\beta\gamma}K^\alpha,\\
i\left[P^\alpha, K^\beta\right] &= -2g^{\alpha\beta}D + 2M^{\alpha\beta},\\
i[D, D] = i[D, M^{\alpha\beta}] &= i[K^\alpha, K^\beta] = 0.
\end{aligned} \tag{1.75}$$

Conformal algebra

The first three commutators define the Lie algebra of the Poincaré group. The remaining commutators are specific to the conformal symmetry. If they were exact in our world this would mean, in particular, that

$$e^{i\alpha D}P^2 e^{-i\alpha D} = e^{2\alpha}P^2. \tag{1.76}$$

The latter relation would imply, in turn, either that the mass spectrum is continuous or that all masses vanish. In neither case can one speak of the S matrix in the usual sense of this word. Instead of the on-shell scattering amplitudes, the appropriate objects for study in conformal theories are n-point correlation functions of the type

$$\langle O_1(x_1), \ldots, O_n(x_n)\rangle$$

whose dependence on $x_i - x_j$ is power-like. The powers, also known as critical exponents, depend on a particular choice of the operators O_i (and, certainly, on the theory under consideration).

Before establishing the conditions under which a given Lagrangian $\mathcal{L}$, which depends on the fields ϕ, is scale invariant or conformally invariant, we must decide how these fields ϕ transform under dilatation and conformal transformations. For translations and Lorentz transformations the rules are well known:

$$\begin{aligned}
\delta_T^\alpha \phi(x) &= -i\left[P^\alpha, \phi(x)\right] = \partial^\alpha \phi(x),\\
\delta_L^{\alpha\beta} \phi(x) &= -i\left[M^{\alpha\beta}, \phi(x)\right] = \left(x^\alpha\partial^\beta - x^\beta\partial^\alpha + \Sigma^{\alpha\beta}\right)\phi(x),
\end{aligned} \tag{1.77}$$

where $\Sigma^{\alpha\beta}$ is the spin operator. For the remaining five operations forming the conformal group, the following choice is consistent with (1.75):

$$\delta_D\phi(x) = (d + x\partial)\phi(x), \tag{1.78}$$

$$\delta_C^\alpha\phi(x) = \left(2x^\alpha x^\nu - g^{\alpha\nu}x^2\right)\partial_\nu\phi(x) + 2x_\nu\left(g^{\nu\alpha}d - \Sigma^{\nu\alpha}\right)\phi(x), \tag{1.79}$$

where d is a constant called the scale dimension of the field ϕ.

We can describe the generators of the conformal group in a slightly different language. Consider the infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x); \tag{1.80}$$

then

$$\partial x^\beta/\partial x'^\rho = \delta^\beta_\rho - \partial\epsilon^\beta/\partial x^\rho$$

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and to ensure that (1.73) holds one must take $\partial^\rho \varepsilon^\beta + \partial^\beta \varepsilon^\rho$ as being proportional to $\eta^{\beta\rho}$, namely, that

$$\partial^\rho \epsilon^\beta + \partial^\beta \epsilon^\rho = \frac{2}{D}(\partial\epsilon)\,\eta^{\beta\rho} \tag{1.81}$$

where $\eta^{\beta\rho}$ is the flat Minkowski metric. For $D > 2$ the maximal information one can extract from this relation is as follows:

(i) $\epsilon^\beta(x)$ is at most a quadratic function of x;
(ii) $\epsilon^\beta(x)$ can include a constant part

$$\epsilon^\beta = a^\beta$$

corresponding to ordinary x-independent translations;
(iii) the linear part can be of two types, either $\epsilon^\mu(x) = \lambda x^\mu$, where λ is a small number (dilatation), or $\epsilon^\mu(x) = \omega^\mu_\nu x^\nu$, where $\omega^{\mu\nu} = -\omega^{\nu\mu}$ (Lorentz rotations);
(iv) finally, the quadratic term satisfying Eq. (1.81) has the form

$$\epsilon^\mu(x) = b^\mu x^2 - 2x^\mu(bx), \tag{1.82}$$

where b^μ is a constant vector. Equation (1.82) corresponds to special conformal transformations. It is rather easy to see that the latter actually presents a combination of an inversion and a constant translation,

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} + b^\mu. \tag{1.83}$$

Loosely speaking, in three or more dimensions conformal symmetry does not contain more information than Poincaré invariance plus scale invariance. If one is dealing with a local Lorentz- and scale-invariant Lagrangian, its conformal invariance will ensue.

A digression about the possible existence of "abnormal" theories

Caveat: The above assertion lacks the rigor of a mathematical theorem and, in fact, need not be true in subtle instances (such instances will *not* be considered in this book). In "normal" theories the scale and conformal currents are of the form [22]

$$S^\mu = x_\nu T^{\mu\nu}, \quad C^\mu = \left[b_\nu x^2 - 2x_\nu(bx)\right] T^{\mu\nu}, \tag{1.84}$$

The vector b_ν is the same as in (1.82)

respectively. Here $T^{\mu\nu}$ is the conserved and symmetric energy–momentum tensor[16] that exists in any Poincaré-invariant theory and defines the energy–momentum operator of the theory:

$$P^\mu = \int d^{D-1}x T^{0\mu}, \qquad \dot{P}^\mu = 0. \tag{1.85}$$

Then the scale invariance implies that

$$\partial_\mu S^\mu = 0, \tag{1.86}$$

[16] Note that in some theories $T^{\mu\nu}$ is not unique. This allows for the so-called *improvements*, extra terms which are conserved by themselves and do not contribute to the spatial integral in (1.85). For instance, in the complex scalar field theory one can add

$$\Delta T^{\mu\nu} = \text{const} \times \left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)\phi^\dagger\phi;$$

this improvement does not change P^μ but it does have an impact on the trace T^μ_μ.

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which, in turn, entails[17]

$$T^{\mu}_{\mu} = 0. \tag{1.87}$$

Equation (1.87) then ensures that the conformal current is also conserved,

$$\partial_{\mu} C^{\mu} = 0. \tag{1.88}$$

Logically speaking, the representation (1.84) need not be valid in "abnormal" theories.[18]

For instance, Polchinski discusses [21] a more general extended representation in which[19]

$$S^{\mu} = x_{\nu} T^{\mu\nu} + \mathcal{S}^{\mu}, \tag{1.89}$$

where $\mathcal{S}^{\mu}$ is an appropriate local operator without an explicit dependence on x_{ν}. Then, (1.86) implies that

$$T^{\mu}_{\mu} = -\partial_{\mu} \mathcal{S}^{\mu}, \tag{1.90}$$

and the energy–momentum tensor is not traceless provided that $\partial_{\mu}\mathcal{S}^{\mu} \neq 0$. Generally speaking, the absence of a traceless energy–momentum tensor (possibly improved) is equivalent to the absence of *conformal* symmetry. Thus, "abnormal" scale-invariant theories need not be conformal.

After this digression, let us return to "normal" theories – those treated in this book. In such theories Eq. (1.84) is satisfied and scale invariance entails conformal invariance.

Applying the requirement of conformal invariance is practically equivalent to making all dimensional couplings in the Lagrangian vanish. In particular, all mass terms must be set to zero.

Warning: this last assertion is valid at the classical level and is, in fact, a necessary but not sufficient condition. Moreover classical conformal invariance may be (and typically is) broken at the quantum level owing to the scale anomaly; see Chapter 8. There are notable exceptions: for example $\mathcal{N} = 4$ super-Yang–Mills theory (Section 10.18.3) is conformally invariant at the classical level. It remains conformally invariant at the quantum level too.

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[17] In theories in which improvements are possible one should analyze the set of all conserved and symmetric energy–momentum tensors to verify that there exists a traceless tensor in this set.

[18] The word "abnormal" is in quotation marks because so far we are unaware of explicit examples of local and Lorentz-invariant field theories of this type with not more than two derivatives. For an exotic example with four or more derivatives see [20]. Moreover, if the requirement of scale and Lorentz invariance in four dimensions is supplemented by *unitarity* the class of "abnormal" theories, in which scale invariance does not necessarily entail conformal invariance, is essentially empty. It is likely to be exhausted by theories reducible to free field theories of a special type [23].

[19] Here C^{μ} must also be extended compared to the expression in (1.84).

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