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Smash nilpotent cycles on varieties dominated by products of curves

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Abstract

Voevodsky conjectured that numerical equivalence and smash equivalence coincide on a smooth projective variety. We prove the conjecture for 1-cycles on varieties dominated by products of curves.

1. Introduction

Throughout this article we work over an algebraically closed field and with Chow groups tensored with $\mathbb{Q}$.

Voevodsky introduced in [Voe95] the relation of smash nilpotence. Let X be a smooth projective variety. An algebraic cycle α on X is *smash nilpotent* if there exists $n > 0$ such that α^n is rationally equivalent to 0 on X^n. Voevodsky proved in [Voe95, Corollary 3.3], and Voisin in [Voi96, Lemma 2.3], that any cycle algebraically equivalent to 0 is smash nilpotent. Because of the multiplicative property of the cycle class map, any smash nilpotent cycle is homologically equivalent to 0 and so numerically equivalent to 0; Voevodsky conjectured that the converse is true as well [Voe95, Conjecture 4.2].

The first general result giving examples of smash nilpotent cycles is the following result of Kimura: if M and N are finite-dimensional motives of different parity and $f : M \to N$ is a morphism of motives, then f is smash nilpotent [Kim05, Proposition 6.1]. This was used in [KS09] to show that on abelian varieties of dimension less than or equal to 3, homological equivalence and smash equivalence coincide. The author is not aware of any nontrivial examples or general results on smash nilpotence of morphisms between motives of the same parity. In this article we provide the first fairly general results in this direction.

On abelian varieties of dimension 4 or greater, the Griffiths group of symmetric cycles can have infinite rank, as shown by Fakhruddin in [Fak96, Theorem 4.4] (note that β is said to be *symmetric* if $[-1]^*\beta = \beta$). Symmetric cycles can be viewed as morphisms between motives of the same parity. The methods in [Kim05, KS09] do not directly apply to symmetric cycles, and it is of interest to know whether such cycles are smash nilpotent.

The main results in this article are the following theorem and its corollary.

Theorem 9. *Numerical equivalence and smash equivalence coincide for* 1*-cycles on a product of curves.*

Corollary 10. *Let Y be a smooth projective variety which is dominated by a product of curves X. Then numerical equivalence and smash equivalence coincide for* 1*-cycles on Y.*

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In particular, the cycles constructed by Fakhruddin in [Fak96, Theorem 4.4 and Corollary 4.6] are smash nilpotent.

It was brought to the author's attention by the referee that some of the above results can be obtained by using results of Herbaut and Marini, in particular [Her07, Lemma 4] and [Mar08, Corollary 24].

The proof of Theorem 9 proceeds by induction on the number of factors in the product. For any 1-cycle α, we show that $\alpha \sim_{\mathrm{sm}} \sum \alpha_i$, where each α_i is obtained in a canonical way from α and comes from a smaller product of curves. Let $d_0 \in C$ be a base point. To prove the above assertion, we are led to consider 1-cycles on C^m of the form

$$\Delta_\nu = \sum_{k=1}^{m} r_k \bigg(\sum_{T \subset S, \# T = k} \Delta_T \bigg).$$

Here S denotes the set $\{1, 2, \dots, m\}$, and Δ_T denotes the curve C embedded diagonally into the factor given by T and the base point d_0 in the remaining factors. The above 'modified diagonal' cycle is inspired by [GS95]. If m is small, then it is not clear whether this cycle is smash nilpotent. However, for $m \gg 0$, since this is a symmetric cycle and $S^m C$ is a projective bundle over $J(C)$, it is easy to deduce sufficient conditions on the integer coefficients $r_1, r_2, \dots, r_m$ so that:

- Δ_ν is smash nilpotent;
- projecting Δ_ν to a smaller product C^N yields a nontrivial relation of the type $\alpha \sim_{\mathrm{sm}} \sum \alpha_i$.

Solving for the coefficients boils down to showing that certain linear homogeneous polynomials are linearly independent, which is done in § 4.

2. The cycle Δ_ν is smash nilpotent

Let $N \geqslant 3$ and t be positive integers, and consider the following set of linear homogeneous polynomials in the m variables $r_1, r_2, \dots, r_m$:

(i) $l_1 = \sum_{k=1}^{m} \binom{m}{k} k r_k$;

(ii) $l_{2,2i} = \sum_{k=1}^{m} \binom{m}{k} k^{2i} r_k$ for every i from the set $\{1, \dots, t\}$;

(iii) $l_3 = \sum_{k=N}^{m} \binom{m-N}{k-N} r_k$.

PROPOSITION 1. *Fix integers $N \geqslant 3$ and t. If $m > \max\{N, 2t\}$, then l_3 is not in the span of $\{l_1, l_{2,2i}\}_{i \in \{1, \dots, t\}}$.*

This proposition is proved in § 4 (see Lemma 11).

Let D be a smooth projective curve of genus g and let $J(D)$ denote its Jacobian. Fix $N \geqslant 3$ and take t to be $\lfloor (g+1)/2 \rfloor$. Fix an integer $m > \max\{N, 2g+2\}$ (then clearly $m > \max\{N, 2t\}$ and so we may apply Proposition 1), and fix a collection $\{r_1, r_2, \dots, r_m\}$ such that the following conditions are satisfied.

(S1) $\sum_{k=1}^{m} \binom{m}{k} k r_k = 0$.

(S2) For every *even* integer i from the set $\{0, 1, 2, \dots, g-1\}$,

$$\bigg(\sum_{k=1}^{m} \binom{m}{k} k^{2+i} r_k \bigg) = 0.$$

(S3) $\sum_{k=N}^{m} \binom{m-N}{k-N} r_k \neq 0$.

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We fix the above data $\{m, N, t = \lfloor (g+1)/2 \rfloor$ and r_k for $k = 1, \dots, m\}$ for the remainder of this section. Let S denote the set $\{1, 2, \dots, m\}$. Let $p_i : D^m \to D$ denote the projection onto the ith factor. Let $d_0 \in D$ be a base point; for every nonempty $T \subset S$ consider the morphism $\phi_T : D \to D^m$ given by

$$p_i \circ \phi_T(d) = \begin{cases} d & \text{for } i \in T, \\ d_0 & \text{for } i \notin T, \end{cases} \tag{2.1}$$

and define Δ_T to be $\phi_{T*}(D)$.

Let $f : D^m \to S^m D$ denote the quotient by the group S_m. Since $m > 2g-2$, $S^m D$ is the projective bundle associated to a locally free sheaf on $J(D)$. Fix a point $d_1 \in D$ which is different from d_0. We may take $\mathscr{O}(1)$ to be the line bundle associated to the (reduced) divisor $f(p_1^{-1}(d_1))$. On $S^m D$ consider the (reduced) subvariety $\Delta_k^s := f(\Delta_T)$ for some T with $\#T = k$ (clearly, this depends only on $\#T$).

PROPOSITION 2. *In $CH_1(D^m)$, we have $f^*\Delta_k^s = k!(m-k)! \sum_{\#T=k} \Delta_T$.*

Proof. Since $f : D^m \to S^m D$ is a finite flat Galois morphism, it is clear that $f^*\Delta_k^s$ is a multiple of $\sum_{\#T=k} \Delta_T$. Applying f_* to both cycles and using the fact that $f_* f^* = m!$, we get the desired result. □

PROPOSITION 3. *We have $\deg(f_*(\Delta_T) \cap c_1(\mathscr{O}(1))) = (m-1)!(\#T)$.*

Proof. It suffices to compute the degree of $\Delta_T \cap f^* c_1(\mathscr{O}(1))$. Arguing as in the proof of Proposition 2, we get that $f^*(c_1(\mathscr{O}(1))) = (m-1)! \sum_{i=1}^m p_i^{-1}(d_1)$. Taking the set-theoretic intersection of Δ_T with $\sum_{i=1}^m p_i^* \mathscr{O}_D(d_1)$ gives the desired result. □

Define a 1-cycle on $S^m D$ by

$$\Gamma_\nu := \sum_{k=1}^m \binom{m}{k} r_k \Delta_k^s.$$

LEMMA 4. *The cycle Γ_ν is smash nilpotent.*

Proof. Using the base point d_0, obtain a map $\pi : S^m D \to J(D)$. The cycle Γ_ν, which is a 1-cycle, can be written as

$$\Gamma_\nu = c_1(\mathscr{O}(1))^{m-g-1} \cap \pi^* \beta_0 \oplus c_1(\mathscr{O}(1))^{m-g} \cap \pi^* \beta_1;$$

see [Ful97, Theorem 3.3(b) and Proposition 3.1(a)(i)]. In the above equation, the $\beta_i \in CH_i(J(D))$ are given by

$$\beta_0 = \pi_*(\Gamma_\nu \cap c_1(\mathscr{O}(1))) - \pi_*(c_1(\mathscr{O}(1))^{m-g+1} \cap p^* \beta_1) \quad \text{and} \quad \beta_1 = \pi_*(\Gamma_\nu). \tag{2.2}$$

First, we show that β_1 is smash nilpotent. Using the base point d_0, embed the curve D into $J(D)$ and denote its Beauville components (see [Bea86]) by α_i, where the α_i are such that:

- $[n]_* \alpha_i = n^{2+i} \alpha_i$;
- $\alpha_i = 0$ for $g-1 < i < 0$.

We have

$$\beta_1 = \pi_*(\Gamma_\nu) = \sum_{k=1}^m \binom{m}{k} r_k \pi_*(\Delta_k^s) = \sum_{k=1}^m \binom{m}{k} r_k [k]_*([\tilde{D}])$$
$$= \sum_{i=0}^{g-1} \left(\sum_{k=1}^m \binom{m}{k} r_k k^{2+i} \right) \alpha_i.$$

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Using [KS09, Proposition 1], α_i is smash nilpotent for i odd. Since the r_k satisfy (S2), it follows that β_1 is smash nilpotent. In particular, β_1 is numerically trivial.

Next, we compute the degree of β_0. Since β_1 is numerically trivial,

$$\deg(\beta_0) = \deg(\pi_*(\Gamma_\nu \cap c_1(\mathscr{O}(1)))).$$

A cycle of dimension 0 on an abelian variety is smash nilpotent if and only if its degree is 0. Using Proposition 3, it is easily checked that for β_0 to be smash nilpotent, we need that $\sum_{k=1}^{m} \binom{m}{k} r_k k = 0$, which is true as (S1) is satisfied, and this proves the lemma. □

The modified diagonal cycle was introduced in [GS95]. We define a more general modified diagonal cycle Δ_ν in the Chow group of D^m, and then use Proposition 2 to get

$$\Delta_\nu := \frac{1}{m!} f^* \Gamma_\nu = \sum_{k=1}^{m} r_k \bigg(\sum_{T \subset S, \# T = k} \Delta_T \bigg). \tag{2.3}$$

COROLLARY 5. *The cycle Δ_ν is smash nilpotent.*

Let $X := C_1 \times C_2 \times \cdots \times C_N$ be a product of N smooth projective curves. Let $j : E \hookrightarrow X$ be a reduced and irreducible curve and let $h : D \to E$ denote its normalization. Denote the composite $j \circ h$ by $\tilde{j} : D \to X$. Let $q_i : X \to C_i$ denote the projection onto the ith factor, and define a morphism $\psi : D^m \to X$ as

$$\psi := (q_1 \circ \tilde{j}) \times (q_2 \circ \tilde{j}) \times \cdots \times (q_N \circ \tilde{j}).$$

Recall that S denotes the set $\{1, 2, \ldots, m\}$. The morphisms $\tilde{j}$ factor as

$$\begin{tikzcd} & D^m \arrow[d, "\psi"] \\ D \arrow[ur, "\phi_S"] \arrow[r, "\tilde{j}"'] & X \end{tikzcd} \tag{2.4}$$

where ϕ_S is as defined in (2.1) (and in this case is simply the diagonal embedding). The 1-cycle $\psi_*(\Delta_\nu)$ on X is smash nilpotent. Let $S_0 := \{1, 2, \ldots, N\}$. We will let

$$\underline{x} = (x_1, x_2, \ldots, x_N)$$

denote the closed points of X. Let $\underline{c} \in X$ be a closed point. For $T \subset S_0$, define

$$\zeta_T^{\underline{c}} : X \to X$$

by

$$q_i \circ \zeta_T^{\underline{c}}(\underline{x}) = \begin{cases} x_i & \text{for } i \in T, \\ c_i & \text{for } i \notin T. \end{cases}$$

Remark 6. The map $\zeta_T^{\underline{c}}$ is the identity if and only if $T = S_0$. If $T \subsetneq S_0$, then $\zeta_T^{\underline{c}}$ is the composite of a projection onto the coordinates in T followed by an inclusion into X.

Remark 7. It is clear that if $\underline{v}, \underline{w} \in X$ are two closed points, then for any cycle α, the cycles $\zeta_{T*}^{\underline{v}}(\alpha)$ and $\zeta_{T*}^{\underline{w}}(\alpha)$ are algebraically equivalent.

Define

$$\kappa := \sum_{k=N}^{m} \binom{m-N}{k-N} r_k. \tag{2.5}$$

As the r_k satisfy (S3), we get that $\kappa \neq 0$.

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Lemma 8. *The* 1*-cycle* $[E]$ *on* X *is smash equivalent to a sum of cycles coming from a smaller product of curves.*

Proof. Observe that $\psi_*(\Delta_T)=0$ if $T\cap S_0=\emptyset$. Define $\underline{v}:=\tilde{j}(d_0)$. It follows that

$$\psi_*(\Delta_\nu)=\psi_*\bigg(\sum_{k=1}^m r_k\bigg(\sum_{T\subset S,\#T=k}\Delta_T\bigg)\bigg)=\sum_{k=1}^m r_k\bigg(\sum_{T\subset S,\#T=k}\psi_*\Delta_T\bigg)$$
$$=\sum_{k=1}^m r_k\bigg(\sum_{T\subset S,\#T=k}\zeta^{\underline{v}}_{(T\cap S_0)*}(\tilde{j}_*(D))\bigg).$$

Let $\mathscr{S}$ denote the collection of subsets of S with the property that $T\cap S_0\neq\emptyset$, and denote by $\mathscr{U}$ the collection of subsets of S which contain S_0. Then we have

$$\psi_*(\Delta_\nu)=\sum_{k=N}^m r_k\bigg(\sum_{T\in\mathscr{U},\#T=k}\zeta^{\underline{v}}_{(T\cap S_0)*}(\tilde{j}_*(D))\bigg)+\sum_{k=1}^m r_k\sum_{T\in\mathscr{S}\setminus\mathscr{U},\#T=k}\zeta^{\underline{v}}_{(T\cap S_0)*}(\tilde{j}_*(D))$$
$$=[E]\bigg(\sum_{k=N}^m\binom{m-N}{k-N}r_k\bigg)+\sum_{k=1}^m r_k\sum_{T\in\mathscr{S}\setminus\mathscr{U},\#T=k}\zeta^{\underline{v}}_{(T\cap S_0)*}(\tilde{j}_*(D)). \tag{2.6}$$

In the above equation we have used that there are exactly $\binom{m-N}{k-N}$ many subsets T in $\mathscr{U}$ with $\#T=k$. Corollary 5 and the above calculation show that the following cycle on X is smash nilpotent:

$$[E]+\frac{1}{\kappa}\sum_{k=1}^m r_k\sum_{T\in\mathscr{S}\setminus\mathscr{U},\#T=k}\zeta^{\underline{v}}_{(T\cap S_0)*}(\tilde{j}_*(D)). \tag{2.7}$$

The second term consists of cycles coming from a smaller product of curves, which proves the lemma. □

3. Smash nilpotent 1-cycles

We now prove the main theorem of this article. We use similar notation to that in the previous section. In particular, we fix an integer $N\geqslant 3$, consider $X=C_1\times C_2\times\cdots\times C_N$, and let $q_i:X\to C_i$ denote the projection onto the ith factor.

Theorem 9. *Numerical equivalence and smash equivalence coincide for* 1*-cycles on a product of curves.*

Proof. The proof proceeds by induction on the number of factors in the product. Let α be a one-dimensional cycle on X such that

$$\alpha=\sum_{i=1}^s n_iE_i,$$

where the E_i are distinct reduced and irreducible components. Let D_i denote the normalization of E_i and define $\tilde{j}_i$ as the composite

$$D_i\xrightarrow{\text{normalization}}E_i\hookrightarrow X.$$

Choose a base point $d_i\in D_i$ and define closed points $\underline{v}^i\in X$ by $\underline{v}^i:=\tilde{j}_i(d_i)$. Let $t:=\max\{\lfloor(g(D_i)+1)/2\rfloor\}_{i\in\{1,2,\ldots,s\}}$. Now fix an integer $m>\max\{N,2g(D_i)+2\}_{i\in\{1,2,\ldots,s\}}$. Then it is clear that $m>\max\{N,2t\}$, so that we can apply Proposition 1 to find integers $r_1,r_2,\ldots,r_m$

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such that (S1), (S2) and (S3) are satisfied. Define κ as in (2.5). Using Lemma 8, more specifically (2.7), we get that for each i, the following cycle is smash nilpotent:

$$[E_i] + \frac{1}{\kappa}\sum_{k=1}^{m} r_k \sum_{T\in\mathscr{S}\setminus\mathscr{U},\#T=k} \zeta^{\underline{v}^i}_{(T\cap S_0)*}(\tilde{j}_{i*}(D_i)). \tag{3.1}$$

Upon multiplying by n_i and summing over i, we get that the following cycle is smash nilpotent:

$$\alpha + \frac{1}{\kappa}\sum_{k=1}^{m} r_k \sum_{T\in\mathscr{S}\setminus\mathscr{U},\#T=k} \sum_{i=1}^{s} n_i \zeta^{\underline{v}^i}_{(T\cap S_0)*}(\tilde{j}_{i*}(D_i)).$$

Modulo algebraic equivalence, using Remark 7 we obtain

$$\begin{aligned} &\alpha + \frac{1}{\kappa}\sum_{k=1}^{m} r_k \sum_{T\in\mathscr{S}\setminus\mathscr{U},\#T=k} \sum_{i=1}^{s} n_i \zeta^{\underline{v}^i}_{(T\cap S_0)*}(\tilde{j}_{i*}(\tilde{D}_i)) \\ &\quad = \alpha + \frac{1}{\kappa}\sum_{k=1}^{m} r_k \sum_{T\in\mathscr{S}\setminus\mathscr{U},\#T=k} \sum_{i=1}^{s} n_i \zeta^{\underline{v}^1}_{(T\cap S_0)*}(\tilde{j}_{i*}(\tilde{D}_i)) \\ &\quad = \alpha + \frac{1}{\kappa}\sum_{k=1}^{m} r_k \sum_{T\in\mathscr{S}\setminus\mathscr{U},\#T=k} \zeta^{\underline{v}^1}_{(T\cap S_0)*}(\alpha). \end{aligned}$$

Since α is numerically trivial, it follows that $\zeta^{\underline{v}^1}_{(T\cap S_0)*}(\alpha)$ is numerically trivial. By induction on N, since $\zeta^{\underline{v}^1}_{(T\cap S_0)*}(\alpha)$ is the pushforward of a cycle from a smaller product of curves, we may assume that it is smash nilpotent. Thus α, being the sum of smash nilpotent cycles, is smash nilpotent. The base case for the induction is the $N=3$ case. In this case, we would get that α is smash equivalent to a sum of cycles coming from a product of two curves, which is a surface. On a surface, numerical equivalence and algebraic equivalence coincide, and so numerical equivalence and smash equivalence coincide; see [Voe95, Corollary 3.3]. This proves that α is smash nilpotent. □

COROLLARY 10. *Let Y be a smooth projective variety and let $h : X = C_1 \times C_2 \times \cdots \times C_N \to Y$ be a dominant morphism. Then numerical equivalence and smash equivalence coincide for 1-cycles on Y.*

Proof. Let $l \in CH^1(Y)$ be a relatively ample line bundle. The relative dimension of h is $r := N - \dim(Y)$; define d by $h_*(l^r) = d[Y]$. Then, by the projection formula, we have that for all $\alpha \in CH^*(Y)$,

$$h_*(l^r \cdot h^*\alpha) = d\alpha.$$

If α is a numerically trivial 1-cycle on Y, then $l^r \cdot h^*\alpha$ is a numerically trivial 1-cycle on X and hence is smash nilpotent. The above equation shows that α is smash nilpotent. □

4. Solving the equations

In this section we give a proof of Proposition 1.

Let $N \geqslant 3$ and t be positive integers, and consider the following set of linear homogeneous polynomials in $r_1, r_2, \ldots, r_m$:

(i) $l_1 = \sum_{k=1}^{m} \binom{m}{k} k r_k$;

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(ii) $l_{2,2i} = \sum_{k=1}^{m} \binom{m}{k} k^{2i} r_k$ for every i from the set $\{1, \dots, t\}$;
(iii) $l_3 = \sum_{k=N}^{m} \binom{m-N}{k-N} r_k$.

Consider also the following collection of polynomials in one variable:

(i) $r_i(x) = \sum_{k=1}^{m} \binom{m}{k} k^i x^k$;
(ii) $x^N(1+x)^{m-N} = \sum_{k=N}^{m} \binom{m-N}{k-N} x^k$.

If T denotes the operator $x(d/dx)$, then

$$r_i(x) = T^i(1+x)^m.$$

To prove Proposition 1, it suffices to prove the following lemma.

Lemma 11. *The following polynomials are linearly independent if $N \geqslant 3$ and $m > \max\{N, 2t\}$:*

$$\{x^N(1+x)^{m-N},\ r_1(x),\ r_{2i}(x)_{i=1,\dots,t}\}.$$

Proof. Suppose the assertion is not true; then there exists a nontrivial linear dependence

$$\begin{aligned}\beta_1 r_1(x) &+ \sum_{i=1}^{t} \beta_{2i} r_{2i}(x) + a x^N(1+x)^{m-N} \\ &= \beta_1 T(1+x)^m + \sum_{i=1}^{t} \beta_{2i} T^{2i}(1+x)^m + a x^N(1+x)^{m-N} = 0. \qquad (4.1)\end{aligned}$$

Since $2t < m$, it is clear that for $1 \leqslant i \leqslant t$ the highest power of $(1+x)$ which divides $T^{2i}(1+x)^m$ is $m - 2i$. Looking at the smallest power of $(1+x)$ which divides all the terms in (4.1), we find that $\beta_i = 0$ for $m - 2i < m - N$, that is, for $i > N/2$. Thus, we may rewrite the above relation as

$$\beta_1 r_1(x) + \sum_{i=1}^{\lfloor N/2 \rfloor} \beta_{2i} r_{2i}(x) = -a x^N(1+x)^{m-N}. \qquad (4.2)$$

This shows that

$$x^N \Bigg| \left(\beta_1 r_1(x) + \sum_{i=1}^{\lfloor N/2 \rfloor} \beta_{2i} r_{2i}(x)\right). \qquad (4.3)$$

By equating to 0 the coefficient of x^k in

$$\beta_1 r_1(x) + \sum_{i=1}^{\lfloor N/2 \rfloor} \beta_{2i} r_{2i}(x)$$

for every $k \in \{1, 2, \dots, N-1\}$, we get

$$\beta_1 k + \sum_{i=1}^{\lfloor N/2 \rfloor} \beta_{2i} k^{2i} = 0 \quad \text{for } k \in \{1, 2, \dots, N-1\}. \qquad (4.4)$$

It is clear that all $\beta_i = 0$ if $N - 1 \geqslant \lfloor N/2 \rfloor + 1$, that is, for $N \geqslant 3$. □

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