THE HULLS OF REPRESENTABLE l-GROUPS AND f-RINGS

Dedicated to the memory of Hanna Neumann

PAUL CONRAD

(Received 11 September 1971)
Communicated by G. B. Preston

1. Introduction and statements of the main results

A lattice-ordered group ("l-group") \( G \) will be called
a \( P \)-group if \( G = g^+ \oplus g^- \) for each \( g \in G \) (projectable)
an \( SP \)-group if \( G = C \oplus C' \) for each polar \( C \) of \( G \) (strongly projectable)
an \( L \)-group if each disjoint subset has a l. u. b. (laterally complete)
an \( O \) group if it is both an \( L \)-group and a \( P \)-group (orthocomplete).

\( G \) is representable if it is an \( l \)-subgroup of a cardinal product of totally ordered
groups. It follows that a \( P \)-group must be representable and hence \( SP \)-groups
and \( O \)-groups are also representable.

\( G \) is a large \( l \)-subgroup of an \( l \)-group \( H \) or \( H \) is an essential extension of \( G \)
if \( G \) is an \( l \)-subgroup of \( H \) and for each non-zero convex \( l \)-subgroup \( S \) of \( H \) we
have \( S \cap G \neq 0 \).

We show that if \( G \) is a large \( l \)-subgroup of an \( X \)-group \( H \), where \( X = P, SP, \)
\( L \) or \( O \), then the intersection \( K \) of all \( l \)-subgroups of \( H \) that contain \( G \) and are
\( X \)-groups is an \( X \)-group. Thus \( K \) is a minimal essential extension of \( G \) that is an
\( X \)-group and we shall call such an extension of \( G \) an \( X \)-hull of \( G \).

**Theorem 2.6.** There exists a unique \( X \)-hull \( G^X \) of a representable \( l \)-group
\( G \). Moreover, \( G \) is dense in \( G^X \), \( G^X \) is representable and if \( G \) is archimedean or
abelian, then so is \( G^X \).

We then show that if \( G \) is a representable \( l \)-group then each \( 0 < g \in G^O \) is
the join of a disjoint subset of \( G^P \). Thus

\[
G \subseteq G^P \subseteq G^{SP} \subseteq (G^{SP})^L = (G^P)^L = G^O \quad \text{and}
\]

\[
G \subseteq G^L \subseteq (G^L)^P = (G^L)^{SP} \subseteq G^O.
\]

but \( (G^L)^{SP} \) need not equal \( G^O \).
A rather natural direct limit construction provides the existence and uniqueness of $G^X$.

If $G$ is a $D_f$-module, $f$-ring or $f$-algebra then there is a unique way of extending the multiplication so that $G^X$ is a $D_f$-module, $f$-ring or $f$-algebra that contains $G$ as a submodule, subring or subalgebra. Thus the multiplicative structure of $G^X$ is completely determined by its additive structure. This phenomenon is due to the fact that each polar preserving endomorphism ("p-endomorphism") of $G$ has a unique extension to a $p$ endomorphism of $G^X$.

If $G$ is a vector lattice then $G^p$ is the $p$-extension of $G$ defined by Amemiya [1], but Amemiya's definition of a $p$ extension is fairly complicated and so are his proofs of the existence and uniqueness of $G^p$. However, he does mention that $G^p$ is the minimal $P$-group in which $G$ is dense.

Now suppose that $G$ is a representable $l$-group. Then $G^p$ is the Stone extension $\Sigma(G)$ of $G$ that is defined by Speed [21]. His definition of $\Sigma(G)$ is categorical, but the maps involved are rather special $l$-homomorphisms. Speed also defines $G^p$ categorically and makes a rather thorough investigation of $P$-groups. $G^L$ is the lateral completion of $G$ defined in [9]. There the definition required that $G$ be dense in $G^L$. Finally $G^o$ is the orthocompletion of $G$ defined by Bernau [3]. Here again the definition of $G^o$ is somewhat complicated being modelled after the definition used by Amemiya for countably laterally complete vector lattice $p$ extensions.

If $F$ is a (real) $f$-algebra then Amemiya remarks that his $p$-extension is also an $f$-algebra. Bernau proves that if $G$ is an $f$-ring or a vector lattice then so is its orthocompletion.


An archimedean $l$ group $A$ is a

- $d$-group if it is divisible
- $v$-group if it is a vector lattice
- $c$-group if it is a (conditionally) complete lattice
- $e$-group if it is essentially closed in the class of archimedean $l$ groups.

If $A$ is a large $l$-subgroup of an archimedean $y$ group $H$, where $y = d$, $v$, $c$ or $e$, then the intersection $K$ of all $l$ subgroups of $H$ that contain $A$ and are $y$-subgroups is a $y$ group. Thus $K$ is a minimal essential extension of $A$ that is a $y$ group. We shall call such an extension of $A$ a $y$ hull.

**Theorem 5.2.** Each archimedean $l$-group $A$ admits a unique $y$-hull $A^y$ for $y = d$, $v$, $c$ or $e$. $A^e$ is the Dedekind MacNeille completion of $A$ and $A$ is dense in $A^e$. $A^e$ is the $l$ subspace of $(A^d)^c$ that is generated by $A$. $A^e = ((A^e)^c)^L$ is the essential closure of $A$.

Once again if $A$ is an $f$-ring then there is a unique extension of the multipli-
cation of \( A \) to a multiplication of \( A^\prime \) so that \( A^\prime \) is an \( f \)-ring and \( A \) is a subring of \( A^\prime \). Thus the multiplicative structure of \( A^\prime \) is completely determined by its additive structure.

In Section 6 we completely characterize the structure of an archimedean essentially closed \( f \)-ring and this gives quite a bit of information about the structure of an arbitrary \( f \)-ring.

In Section 7 we get a nice representation of the orthocompletion of an \( f \)-ring with a basis and this leads to information about the structure of an arbitrary \( f \)-ring with a basis.

**NOTATION.** Throughout \( G \) will denote an \( l \)-group and for each \( 0 < g \in G \), \( G(g) \) will denote the convex \( l \)-subgroup of \( G \) generated by \( g \). \( G \) is a dense \( l \)-subgroup of an \( l \)-group \( H \) if for each \( 0 < h \in H \) we have \( 0 < g \leq h \) for some \( g \in G \). \( \Pi A_\lambda \) will denote the cardinal product of \( l \)-groups \( A_\lambda \) and \( \Sigma A_\lambda \) will denote the cardinal sum. The cardinal sum of a finite number of \( l \)-groups will be denoted by \( A_1 \oplus \cdots \oplus A_n \).

For each subset \( S \) of \( G \)

\[
S' = \{ g \in G \mid \| g \| \wedge | s | = 0 \text{ for all } s \in S \}
\]

is the polar of \( S \). Sik [20] has shown that the set \( P(G) \) of all polars in \( G \) is a complete Boolean algebra and that an \( l \)-group is representable if and only if each polar is normal.

**2. The existence and uniqueness of \( X \)-hulls**

**Lemma 2.1.** If \( G \) is a \( P \)-group and \( L \)-group then \( G \) is an \( SP \)-group.

**Proof.** If \( C \in P(G) \) and \( \{ a_\lambda \mid \lambda \in \Lambda \} \) is a maximal disjoint subset of \( C \) then \( a = \lor a_\lambda \) is a weak order unit in \( C \) and so \( a'' = C \). Thus

\[
G = a'' \oplus a' = C \oplus C'.
\]

\( G \) is an \( \mathcal{L} \)-subgroup of an \( l \)-group \( H \) if \( G \) is an \( l \)-subgroup of \( H \) and for each disjoint subset \( \{ a_\lambda \mid \lambda \in \Lambda \} \) of \( G \) for which \( \lor a_\lambda \) exists we have \( \lor a_\lambda = \lor a_\lambda \). Note that the intersection of laterally complete \( \mathcal{L} \)-subgroups of \( H \) is a laterally complete \( \mathcal{L} \)-subgroup.

**Lemma 2.2.** If \( G \) is a large \( l \)-subgroup of an \( l \)-group \( H \) then \( G \) is an \( \mathcal{L} \)-subgroup of \( H \).

**Proof.** Suppose that \( \{ a_\lambda \mid \lambda \in \Lambda \} \) is a disjoint subset of \( G \) and \( a = \lor a_\lambda \) exists. If \( h \) is an upper bound for the \( a_\lambda \) in \( H \) then \( a \geq a \wedge h = k \geq a_\lambda \) and so it suffices to show that \( a = k \). For each \( \lambda \in \Lambda \), \( a^\lambda = \lor a_\lambda ( \neq \lambda ) \) exists, \( a_\lambda \wedge a^\lambda = 0 \) and \( a = a_\lambda + a^\lambda \). Thus

\[
H(a) = H(a_\lambda) \oplus H(a^\lambda).
\]
Now \( k = k_x + k_y \), where \( k_x \in H(a_x) \) and \( k_y \in H(a_y) \) and since \( a \leq k \geq a \) we have \( a_x \leq k_x \leq a_y \). Therefore \( a - k = a - k \in H(a^2) = K \). But \( K \cap G = \bigcap \Lambda G(a^0) \subseteq G(a) \) and so if \( 0 \leq x \in K \cap G \) then \( x \wedge a_x = 0 \) for all \( \Lambda \in \Lambda \). Thus \( x \wedge a = x \wedge \bigvee_G a_x = \bigvee_G x \wedge a_x = 0 \) and since \( a \) is a unit in \( G(a) \), \( x = 0 \). Therefore \( K \cap G = 0 \) and since \( G \) is large in \( H \), \( K = 0 \).

Let \( G \) be an \( l \)-subgroup of \( H \) and denote the polar operation in \( G(H) \) by \( ' \) (\( * \)). For \( B \in P(G) \) and \( C \in P(H) \) define

\[
B\mu = (B')* \quad \text{and} \quad C\nu = C \cap G.
\]

1) \( B\mu \nu = (B')* \cap G = B** \cap G = B** \nu = B \).

**Proof.** Since \( B' \subseteq B^* \) we have \( (B')* \supseteq B** \supseteq B \) and so \( (B')* \cap G \supseteq B** \cap G \supseteq B \). If \( 0 < x \in (B')* \cap G \) then \( x \in G \) and \( x \wedge B' = 0 \) and so \( x \in B'' = B \).

2) If \( \nu \) is one-to-one then \( B\mu = B** \).

3) ([9] p. 455). If \( G \) is large in \( H \) then \( \mu \) is an isomorphism of \( P(G) \) onto \( P(H) \) and \( \nu \) is the inverse.

4) ([10] p. 156). If \( H \) is archimedean then the following are equivalent.

i) \( G \) is large in \( H \).

ii) \( \nu \) is an isomorphism of \( P(H) \) into \( P(G) \) and \( \mu \) is the inverse.

iii) If \( 0 \neq C \in P(H) \) then \( C \cap G \neq 0 \).

iv) If \( 0 < h \in H \) then \( h'' \cap G \neq 0 \).

5) If \( G \) is large in \( H \) and \( X \) is an \( l \)-subgroup of \( G \) or just a non-void subset of \( G \) then

i) \( (X'')** = X** \) and \( X** \cap G = X'' \)

ii) \( (X')** = X* \) and \( X* \cap G = X' \).

**Proof.** Since \( X \subseteq X'' \) we have \( X** \subseteq (X'')** \). Also \( X** \nu \) is a polar of \( G \) that contains \( X \) and so \( X** \nu = X** \cap G \supseteq X'' \). Thus \( X'' \subseteq X** \) and hence \( (X'')** \subseteq X** \).

\[
X** \cap G = (X')** \cap G = X'' \mu \nu = X''.
\]

From (i) and (2) we have \( X* = (X'')* = (X')** \). Finally \( X* \cap G = \{ g \in G | g \wedge X = 0 \} = X' \) holds for any \( l \)-subgroup \( G \) of \( H \).

6) If \( \alpha \) is an \( l \)-automorphism of \( H \) that induces the identity on \( P(G) \) then \( \alpha \) induces the identity on \( P(H) \) provided that \( G \) is large in \( H \).

**Proof.** If \( C \in P(H) \) then \( C\nu = C\nu \alpha = (G \cap C)\alpha = G\alpha \cap C\alpha = G \cap C\alpha = C\nu \alpha \), so that \( C = C\alpha \) by (3).

**Proposition 2.3.** Let \( G \) be a convex \( l \)-subgroup of an \( l \)-group \( H \).

i) If \( H \) is an \( SP \)-group so is \( G \).

ii) If \( H \) is a \( P \)-group so is \( G \).
PROOF. (i) If \( A \in P(G) \) then \( H = A^{**} \oplus A^* \) and hence \( G = (A^{**} \cap G) \oplus (A^* \cap G) = A \oplus A' \).

(ii) Pick \( g \in G \). Then \( H = g^{**} \oplus g^* \) and so \( G = (G \cap g^{**}) \oplus (G \cap g^*) = g'' \oplus g' \). For \( g' \subseteq g^* \) implies \( (g'')^* = g'^* \supseteq g^{**} \) and so \( g'' = (G \cap (g'')^*) \supseteq G \cap g^{**} \supseteq g'' \).

Note that a polar in an \( L \)-group is an \( L \)-group, but an \( l \)-ideal \( C \) of an \( L \)-group \( G \) need not be an \( L \)-group.

EXAMPLE. \( C = \sum_{i=1}^\infty R_i \subseteq \prod_{i=1}^\alpha R_i = G \).

This also shows that an \( l \)-ideal of an \( O \) group need not be an \( O \)-group.

THEOREM 2.4. If \( H \) is an \( X \)-group and an essential extension of \( G \) and \( \{H_\lambda \mid \lambda \in \Lambda \} \) is the set of all \( l \)-subgroups of \( H \) that contain \( G \) and are \( X \)-groups then \( K = \bigcap \Lambda H_\lambda \) is an \( X \)-hull of \( G \), where \( X = P, SP, L \) or \( O \).

PROOF. If \( H \) is an \( L \)-group then by Lemma 2.2 each \( H_\lambda \) is a laterally complete \( L \)-subgroup of \( H \) and so \( K \) is an \( L \)-group.

Suppose that \( H \) is a \( P \)-group, \( 0 < k \in K \) and denote the polar operation in \( H, K, \) and \( H_\lambda \) by \( * \), \( \neq \) and \( ^\lambda \) respectively. If \( 0 < x \in K \subseteq H_\lambda \) then \( x = x_1 + x_2 \in k^\lambda \oplus k^{\lambda \lambda} \) and by (5) \( k^\lambda = k^* \cap H_\lambda \) and \( k^{\lambda \lambda} = k^{**} \cap H^\lambda \). Thus \( x_1 + x_2 \) is the unique decomposition of \( x \) in \( H = k^* \oplus k^{**} \). This holds for all \( \lambda \) so \( x_1, x_2 \in \cap H_\lambda = K \). Thus \( x_1 \in K \cap k^* = k^* \) and \( x_2 \in K \cap k^{**} = k^{**} \). Therefore \( x \in k^* \oplus k^{**} \) and hence \( K = k^* \oplus k^{**} \).

If \( H \) is an \( SP \)-group then an entirely similar argument shows that \( K \) is also an \( SP \)-group.

LEMMA 2.5. An \( L \)-hull \( K \) of a representable \( l \)-group \( G \) is representable.

PROOF. Theorem 2.8 in [9] asserts that if \( G \) is dense in \( K \) then \( K \) is also representable. The only place in the proof where the hypothesis of denseness is used is to infer that if \( ( - a_a + (a_a \wedge b) + a_a) \wedge (a_a \wedge b) = 0 \) and \( a_a \wedge b > 0 \) then \( a_a \wedge b \geq g > 0 \) for some \( g \in G \) and so \( ( - a_a + g + a_a) \wedge g = 0 \). But since \( G \) is large in \( K \) we can conclude that \( n(a_a \wedge b) \geq g > 0 \) for some \( n > 0 \) and \( g \in G \). Thus \( 0 = n((- a_a + (a_a \wedge b) + a_a) \wedge (a_a \wedge b)) = ( - a_a + n(a_a \wedge b) + a_a) \wedge (a_a \wedge b) \) \( \wedge (a_a \wedge b) \geq ( - a_a + g + a_a) \wedge g \geq 0 \) and so \( ( - a_a + g + a_a) \wedge g = O \).

COROLLARY. An \( X \)-hull of a representable \( l \)-group is representable, where \( X = P, SP, L \) or \( O \).

THEOREM 2.6. There exists a unique \( X \)-hull \( G^X \) of a representable \( l \)-group \( G \) for \( X = P, SP, L \) or \( O \). Moreover \( G \) is dense in \( G^X \) and \( G^X \) is representable and if \( G \) is abelian or archimedean then so is \( G^X \).
PROOF. The existence follows from Theorem 2.4 provided that we can embed $G$ as a large $l$-subgroup in an $X$-group. In order to do this we make use of the direct limit construction developed in [9].

Let $D(G)$ be the set of all maximal disjoint subsets of the Boolean algebra $P(G)$ of polars of $G$. If $\mathcal{A}, \mathcal{C} \in D(G)$ then we define $\mathcal{A} \subseteq \mathcal{C}$ if each $A \in \mathcal{A}$ is contained in some $C \in \mathcal{C}$. Then $D(G)$ is a lower directed partially ordered set. For each $\mathcal{C} \in D(G)$ let $G_{\mathcal{C}}$ be the $l$-group

$$G_{\mathcal{C}} = \prod_{C \in \mathcal{C}} G/C'.$$

If $\mathcal{A} \subseteq \mathcal{C} \in D(G)$ and $C \in \mathcal{C}$ then $C = (\cap A_i)'$ the polar join of the $A_i \in \mathcal{A}$ that are contained in $C$. Thus $C' = \cap A_i'$ and so the natural map

$$G/C \rightarrow \prod_{C \in \mathcal{C}} G/A_i'.$$

is an $l$-isomorphism. Thus there is a natural $l$ isomorphism $\pi_{\mathcal{C}}$ of $G_{\mathcal{C}}$ into $G_{\mathcal{C}}$ obtained by combining the above maps for each $G/C'$, where $C \in \mathcal{C}$. Let $\mathcal{O}(G)$ be the direct limit of the $l$-groups $G$ with connecting $l$-isomorphisms $\pi_{\mathcal{C}}$. Define $k \in \mathcal{O}(G)$ to be positive if $k = 0$ or $k_\mathcal{C} > 0$ for some $\mathcal{C} \in D(G)$. For each $g \in G$ let $\mathcal{g}$ be the element in $\mathcal{O}(G)$ with $\mathcal{g} = (\cdots, C' + g, \cdots)$ for each $\mathcal{C} \in D(G)$.

In [9] it is shown that $\mathcal{O}(G)$ is a representable laterally complete $l$-group and if $G$ is abelian or archimedean then so is $\mathcal{O}(G)$. Also the map $g \rightarrow \mathcal{g}$ is an $l$-isomorphism of $G$ into $\mathcal{O}(G)$ and $G$ is dense in $\mathcal{O}(G)$. Thus to complete the proof of existence it suffices to show that $\mathcal{O}(G)$ is a $P$-group. Thus we must show that if $\theta < l \in \mathcal{O}$ then $\mathcal{O} = l^* \oplus l^*$. Consider $\theta < k \in \mathcal{O}(G)$ and pick $\mathcal{C} \in D(G)$ such that $l_\mathcal{C} \neq 0 \neq k_\mathcal{C}$. Then $l_\mathcal{C} = (\cdots, C' + l(C), \cdots)$, where $0 \leq l(C) \in G$. Let $l(C)$ be the convex $l$-subgroup of $G$ that is generated by $l(C)$ and pick $\mathcal{C} \subseteq \mathcal{A} \in D(G)$ so that each $(C \cap l(C))'' \neq 0$ belongs to $\mathcal{A}$.

$$G_{\mathcal{A}} = \prod G/(C \cap l(C))' \oplus \prod G/A_i'$$

Let $x (y)$ be the element in $\mathcal{O}(G)$ with $\mathcal{A}$-th component $x_\mathcal{A}$ if $x_\mathcal{A} \neq 0$ ($y_\mathcal{A}$ if $y_\mathcal{A} \neq 0$) and $\theta$ otherwise. Then $k = x + y$. It is shown in [9] that the only non-zero components of $l_\mathcal{A}$ are of the form $(C \cap l(C))' + l(C)$. Thus $l_\mathcal{A} \wedge y_\mathcal{A} = 0$ and so $y \in l^*$. Thus we need only prove that $x \in l^*$. Consider $\theta < t \in \mathcal{O}(G)$ such that $l \wedge t = \theta$. To complete the proof of existence we need to show that $x \wedge t = \theta$.

Pick $\mathcal{B} \in D(G)$ so that $0 \neq t_\mathcal{B} = (\cdots, D' + t(D), \cdots)$. Now ([9] p. 456) $(C \cap l(C))'' \cap (D \cap l(D))'' = 0$ and so we may choose a $\mathcal{B} \in D(G)$ that contains the $(C \cap l(C))'' \neq 0$ and the $(D \cap l(D))'' \neq 0$. Let

$$\mathcal{A} \cap \mathcal{B} = \{ A \cap B \neq 0 \mid A \in \mathcal{A} \text{ and } B \in \mathcal{B} \}$$

Then $\mathcal{A} \cap \mathcal{B} \in D(G)$ and so we have
Now \( x_{df} \) has nonzero components of the form \((C \cap \mathfrak{l}(C))' + z\) and \( t_{df} \) has non-zero components of the form \((D \cap \mathfrak{l}(D))' + \mathfrak{i}(D)\). These do not change under the maps into \( G_{df} \cap \mathfrak{g} \) and so \( x \wedge t = \theta \). Thus there exists an \( X \)-hull of \( G \).

Let \( H \) be an \( X \)-hull of \( G \) and let \( \alpha(\beta) \) the the natural \( l \)-isomorphisms of \( G \) \((H)\) into \( \mathfrak{G}(G) \)(\( \mathfrak{G}(H) \)). We complete the proof by showing that \( \alpha \) can be extended to an \( l \)-isomorphism \( \rho \) of \( H \) onto the \( X \)-hull \( K \) of \( G \alpha = \mathfrak{G}_r \) in \( \mathfrak{G}(G) \).

\[
\begin{array}{ccc}
G & \xrightarrow{\alpha} & \mathfrak{G}_r \\
\downarrow \quad \rho \quad \downarrow & & \mathfrak{G}_r \subseteq K \subseteq \mathfrak{G}(G)
\end{array}
\]

Thus if \( H_1 \) and \( H_2 \) are \( X \)-hulls of \( G \) then \( \rho_1 \rho_2^{-1} \) is an \( l \)-isomorphism of \( H_1 \) onto \( H_2 \) that induces the identity on \( G \). It follows from Theorem 2.7 that \( \rho_1 \rho_2^{-1} \) is unique.

Since \( G \) is large in \( H \) for each \( C \in \mathcal{P}(G) \) we have \( C = G \cap C^{**} \) and \( C' = G \cap C^* \). Thus \( C' + g \longrightarrow C^* + g \) is an \( l \)-isomorphism of \( G/C' \) into \( H/C^* \). For each \( \mathfrak{c} \in \mathcal{D}(G) \) let \( \mathfrak{c} = \{C^{**} \mid C \in \mathfrak{c} \} \). Then \( \mathfrak{c} \in \mathcal{D}(H) \) and thus there is a natural \( l \)-isomorphism \( \tau_{\mathfrak{c}} \) of \( G_{\mathfrak{c}} \) onto \( H_{\mathfrak{c}} \). Moreover if \( \mathfrak{a} \leq \mathfrak{c} \) in \( \mathcal{D}(G) \)

\[
\begin{array}{ccc}
G_{\mathfrak{a}} & \xrightarrow{\tau_{\mathfrak{a}}} & H_{\mathfrak{a}} \\
\pi_{\mathfrak{a},df} \downarrow & & \pi_{\mathfrak{a},df} \downarrow \\
G_{df} & \xrightarrow{\tau_{df}} & H_{df}
\end{array}
\]

commutes, where \( \pi_{\mathfrak{a},df} \) is the \( l \)-isomorphism used in the construction of \( \mathfrak{G}(H) \). Thus (see [9]) the \( \tau_{\mathfrak{c}} \) determine an \( l \)-isomorphism \( \tau \) of \( \mathfrak{G}(G) \) into \( \mathfrak{G}(H) \)

\[
\begin{array}{ccc}
H & \xrightarrow{\beta} & \mathfrak{G}(H) \\
\uparrow \tau \quad \downarrow \beta & & \uparrow \tau \\
G & \xrightarrow{\alpha} & G \alpha = \mathfrak{G}_r \subseteq K \subseteq \mathfrak{G}(G)
\end{array}
\]

If \( g \in G \) and \( \mathfrak{c} \in \mathcal{D}(H) \) then \((g \mathfrak{a} \mathfrak{c})_{\mathfrak{c}} = (g \mathfrak{a})_{\mathfrak{c}} \mathfrak{c} = (\cdots, C' + g, \cdots)_{\mathfrak{c}} = (\cdots, C^* + g, \cdots) = (g \beta)_{\mathfrak{c}} \). Thus \( g \mathfrak{a} \mathfrak{c} = g \beta \) and hence \( G \beta = G \mathfrak{a} \mathfrak{c} \subseteq \mathfrak{G}(G) \tau \) which is an \( X \)-group and \( G \beta \) is large in \( \mathfrak{G}(H) \). Thus \( H \beta \cap \mathfrak{G}(G) \tau \) is an \( X \)-group and contains \( G \beta \) and so since \( H_\beta \) is an \( X \)-hull of \( G \beta \) we have
Thus $H \beta \tau^{-1}$ is an $X$-group that contains $G_x$ and so

$$G \alpha = G \beta \tau^{-1} \subseteq K \subseteq H \beta \tau^{-1} \subseteq \mathcal{O}(G)$$

and since $H \beta \tau^{-1}$ is an $X$-hull of $G \beta \tau^{-1}$ we have $K = H \beta \tau^{-1}$. This completes the proof of Theorem 2.6.

REMARK. We can, of course, define countably laterally complete $l$-groups in the obvious way and then it follows from the above proof that each representable $l$-group admits a unique $CL$-hull. Also $G$ admits a unique minimal essential extension $H$ that is both a $P$-group and a $CL$-group. For the vector lattice case $H$ is the “completion” of Amemiya [1]. See also Vulich [25].

THEOREM 2.7. If $\alpha$ is an $l$-isomorphism of $G_1$ onto $G_2$, where the $G_i$ are representable $l$-groups, then there exists a unique extension of $\alpha$ to an $l$-isomorphism of $G_1^X$ onto $G_2^X$ for $X = P, SP, L$ or $O$.

PROOF. $\alpha$ induces an isomorphism of $P(G_1)$ onto $P(G_2)$ and hence an isomorphism of $D(G_1)$ onto $D(G_2)$. Also for $C \in P(G_1)$ we have the natural map $C' \to (C\alpha)' + g\alpha$ of $G_1/C'$ onto $G_2/(C\alpha)'$. Thus there is a natural map $\alpha_C$ of $G_1^C$ onto $G_2^{C\alpha}$ such that

$$\begin{array}{ccc} G_1^C & \xrightarrow{\alpha_C} & G_2^{C\alpha} \\ \downarrow \pi_{C,\alpha} & & \downarrow \pi_{C,\alpha}\alpha_C \\ G_1^C \alpha & \xrightarrow{\alpha_C} & G_2^{C\alpha}\alpha \end{array}$$

commutes. These maps $\alpha_C$ generate an isomorphism $\bar{\alpha}$ of $\mathcal{O}(G_1)$ onto $\mathcal{O}(G_2)$ and the following diagram commutes

$$\begin{array}{ccc} & & G_1^X \\ & \downarrow & \downarrow \bar{\alpha} \\ G_1 \xrightarrow{\alpha} & \bar{G}_1 \subseteq \bar{G}_1^X \subseteq \mathcal{O}(G_1) & \bar{G}_2 \subseteq \bar{G}_2^X \subseteq \mathcal{O}(G_2) \\ & \downarrow \alpha & \uparrow \bar{\alpha} \\ & G_2 \xrightarrow{\alpha} & G_2^X \\
\end{array}$$

Also it is easy to see that $\bar{G}_1^X \bar{\alpha} = \bar{G}_2^X$. Thus $\alpha$ can be extended to an $l$-isomorphism of $G_1^X$ onto $G_2^X$. 

Downloaded from https://www.cambridge.org/core. IP address: 54.70.40.11, on 15 Mar 2019 at 14:15:36, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446788700015391
For the uniqueness it suffices to show that if \( \alpha \) is an \( l \)-automorphism of \( G^X \) that induces the identity on \( G \) then \( \alpha \) is the identity. Since \( \alpha \) induces the identity on \( P(G) \) it must also induce the identity on \( P(G^X) \). Thus we may assume that \( \alpha \) is an \( l \)-automorphism of \( \mathcal{O}(G) \) that induces the identity on \( G \) and \( P(\mathcal{O}(G)) \). Consider \( l \in \mathcal{O}(G) \) with \( l_\varphi = (\ldots, C' + g, \ldots) \) and suppose (by way of contradiction that \( (l_\varphi) = (\ldots, C' + x, \ldots) \), where \( C' + x \neq C' + g \). Then

\[
|g - l| \varphi \wedge (0, \ldots, 0, C' + |g - x|, 0, \ldots, 0) = 0 \text{ but } \\
(|g - l| \varphi \wedge (0, \ldots, 0, C' + |g - x|, 0, \ldots, 0) \neq 0.
\]

Thus \( \alpha \) does not induce the identity on \( P(\mathcal{O}(G)) \), a contradiction.

**Proposition 2.8.** Suppose that \( G \) is a representable \( l \)-group, \( \alpha \) is an \( l \)-automorphism of \( G^0 \) and \( X = P, SP, L \) or \( 0 \).

i) \( G^X \alpha = (G\alpha)^X \) and so if \( G\alpha = G \), then \( G^X \alpha = G^X \).

ii) If \( G\alpha \subseteq G \) then \( G^X \alpha \subseteq G^X \).

**Proof.** \( G\alpha \) is large in \( G^0 \) and hence in \( G^X\alpha \). Also \( G^X\alpha \) is an \( X \)-group. If \( G\alpha \subseteq K \subseteq G^X\alpha \), where \( K \) is an \( l \)-subgroup of \( G^X\alpha \) and an \( X \)-group then \( G \subseteq K\alpha^{-1} \subseteq G^X \) which contradicts the minimality of \( G^X \). Thus \( G^X\alpha \) is the \( X \)-hull of \( G\alpha \) and so \( G^X\alpha = (G\alpha)^X \). If \( G\alpha \subseteq G \) then \( G^X\alpha = (G\alpha)^X \subseteq G^X \). The following example shows that we may or may not have equality.

**Example.** Let \( G \) be the \( l \)-ideal in \( \prod_{i=1}^\omega R_i \) generated by \((1, 2, 3, \ldots)\). Then \( G^0 = \prod R_i \). Let \( \alpha \) be the multiplication of \( G^0 \) by \((1, 1/2, 1/3, \ldots)\). Then \( G\alpha \) is the \( l \)-ideal of \( G^0 \) generated by \((1, 1, 1, \ldots)\). Thus \( G\alpha \subseteq G \) and both \( G \) and \( G\alpha \) are \( SP \)-groups.

\[
G^p\alpha = G\alpha \subseteq G = G^p \text{ and } \\
G^L\alpha = (G\alpha)^L = G^0 = G^L.
\]

**Corollary.** If \( \alpha \) is an \( l \)-endomorphism of \( G^X \) that induces an automorphism on \( G \) then \( \alpha \) is an automorphism of \( G^X \).

**Proof.** Since \( G \) is large in \( G^X \) it follows that \( \alpha \) is one-to-one on \( G^X \) and by the minimality of \( G^X\alpha \) must be an \( l \)-automorphism of \( G^X \).

**Theorem 2.9.** If \( G \) is a \( P \)-group then each \( \theta < l \in \mathcal{O}(G) \) is the join of a disjoint subset of \( \hat{G} \). In particular, \( \hat{G}^L = \mathcal{O}(G) \) and hence \( G^L \) is an \( SP \)-group.

**Proof.** Consider \( \theta < l \in \mathcal{O} \) and \( l_\varphi \neq 0 \). In each \( C \in \mathcal{C} \) pick a maximal disjoint set \( \{a_\lambda \mid \lambda \in \Lambda \} \) of elements of \( G \). Then \( C = (\cap a_\lambda)' = (\cup a_\lambda)^{''} \) and so there is a partition \( \mathcal{A} \leq \mathcal{C} \) that consists of principal polars of \( G \).

\[
\mathcal{A} = \{a_\lambda \mid \lambda \in \Lambda \}
\]
Thus $0 \neq l_{st} = (\cdots, a_{\lambda} + l(\lambda), \cdots)$. Now $G = a_{\lambda}' \oplus a_{\lambda}''$ and so we may assume that $0 \leq l(\lambda) \in a_{\lambda}''$ for each $\lambda \in \Lambda$. In particular, the $l(\lambda)$ are disjoint in $G$.

Thus $\bigvee l(\lambda)_{st} = l_{st}$ and so $\bigvee l(\lambda) = l$.

**Corollary I.** If $G$ is an $O$-group then $\bar{G} = O(G)$.

**Corollary II.** If $G$ is a representable $l$-group then

$$G \subseteq \bar{G} \subseteq \bar{G}^{sp} \subseteq (\bar{G}^{sp})^L = (\bar{G}^P)^L = \bar{G}^O = O(G)$$

where the indicated $X$-hulls are all in $O(G)$. In particular, $G^O = O(G)$ and so $G^O$ is the orthocompletion defined by Bernau.

**Proof.** Clearly $G \subseteq \bar{G}^P \subseteq \bar{G}^{sp} \subseteq (\bar{G}^P)^L \subseteq (\bar{G}^{sp})^L \subseteq \bar{G}^O \subseteq O(G)$ and so it suffices to show that $(\bar{G}^P)^L = O(G)$. Let $H$ be the $P$-hull of $G$ and let $\alpha, \beta, \tau$ be as in the proof of Theorem 2.6.

$$H \xrightarrow{\beta} \bar{H} \subseteq \bar{H}^L = O(H)$$

$$\downarrow \tau$$

$$G \xrightarrow{\alpha} \bar{G} \subseteq \bar{G}^P(\bar{G}^P)^L \subseteq O(G)$$

Then $\bar{H} = \bar{G}^P \tau \subseteq (\bar{G}^P)^L \tau \subseteq O(H)$ and $(\bar{G}^P)^L \tau$ is an $L$-group. Thus $(\bar{G}^P)^L \tau = O(H)$ and so $(\bar{G}^P)^L = O(G)$.

Also it follows that

$$\bar{G} \subseteq \bar{G} \subseteq (\bar{G}^L)^P \subseteq (\bar{G}^L)^{sp} \subseteq \bar{G}^O = O(G)$$

but as the next example shows $(\bar{G}^L)^{sp}$ need not equal $\bar{G}^O$. Thus the operators $SP$ and $L$ need not commute.

**Example.** Let $\Lambda$ be the po-set

\[ \bullet \bullet \bullet \bullet \bullet \bullet \bullet \]

Denote the set of maximal (minimal) elements in $\Lambda$ by $A$ ($B$). Let $V$ be the set of all functions from $\Lambda$ into the reals. Then $V$ is a real vector lattice if we define addition pointwise and define $v \in V$ to be positive if each non-zero maximal component is positive. Next let

$$G = \{ v \in V \mid v \text{ is constant on } A \}.$$ 

Note that $G$ is laterally complete but not a $P$-group. Let
Then $H$ is not laterally complete and $H^L = V$. We show that

$$H = G^{SP} = G^p.$$  

Clearly $G$ is large in $H$ and $H$ is an $SP$-group. Suppose that $G \subseteq K \subseteq H$, where $K$ is a $P$-group. Let '(*)' denote the polars in $K (H)$. Let $S$ be a subset of $B$ and let $s \in G$ be the characteristic function on $S$. Let $a \in G$ be the characteristic function on $A$.

$$K = s'' \oplus s', H = s^{**} \oplus s^*$$

and $s^{**} \cap K = s''$ and $s^* \cap K = s'$.

Thus $a = a_1 + a_2 \in s'' \oplus s' = K$ and this is also the decomposition in $H = s^{**} \oplus s^*$. Thus $a_1$ is the characteristic function of the elements in $A$ above $S$, but such elements generate the group of functions on $A$ with finite range. Therefore $K = H$ and hence $H = G^p$.

**PROPOSITION 2.10.** If $G$ is a representable $l$-group then $(G^L)^p = (G^L)^{SP}$.

**PROOF.** Take $C \in P((G^L)^p)$; then $C \cap G^L = C \in P(G^L)$: so as in Lemma 2.1, $C \vee a''$, and thus $C = C \alpha \mu = a'' \alpha \mu = (a'')^{**} = a^{**}$, by (3) and (5). Thus $(G^L)^p$ is an $SP$-group and so $(G^L)^p = (G^L)^{SP}$.

**COROLLARY.** Let $G$ be a representable $l$-group.

i) $(G^X)^0 = (G^X)^0$ for $X = P$, $SP$ or $L$ and $(G^p)^{SP} = (G^p)^p = G^{SP}$.

ii) $(G^L)^p = (G^L)^{SP} \subseteq (G^p)^L = (G^{SP})^L$ and equality need not hold.

### 3. The $X$-hulls of $D_f$-modules and $f$-rings

A $p$-endomorphism of an $l$-group $G$ is an endomorphism $\alpha$ of the group such that

$$x \wedge y = 0 \text{ implies } xx \wedge y = 0 \text{ for all } x, y \in G.$$  

It is easy to show that this is equivalent to $G^+ \alpha \subseteq G^+$ and $C \alpha \subseteq C$ for each $C \in P(G)$ (see [13]). Thus the $p$-endomorphisms of $G$ are the $l$-endomorphisms that preserve polars. In Section 4 we shall show that each $p$-endomorphism of a representable $l$-group $G$ has a unique extension to the $X$-hull $G^X$ of $G$.

Let $D$ be a directed po-ring. $G$ is a $D_f$-module (see [22]) if $G$ is an abelian $l$-group and a $D$-module such that for each $d \in D^+$ the map

$$g \rightarrow gd$$

is a $p$-endomorphism of $G$. Steinberg [22] shows that such a $G$ is isomorphic to a subdirect sum of totally ordered modules. Note that each polar of $G$ is a submodule. Note also that each abelian $l$-group $A$ is a $D_f$-module with respect
to the ring \( \mathbb{Z} \) of integers and also with respect to the directed ring \( D \) of all polar preserving endomorphisms of \( A \).

**Proposition 3.1.** If \( G \) is a vector lattice over a totally ordered division ring \( D \) then \( G \) is a \( D_f \)-module.

**Proof.** We are given that \( G \) is an abelian \( l \)-group and \( G^+ D^+ \subseteq G^+ \). If \( d \in D^+ \) and \( g \in G \) then \( (g \lor 0)d = gd \lor 0 \). For \( (g \lor 0)d \geq gd \) and 0 and if \( z \geq gd \) and 0 then \( zd^{-1} \geq g \) and 0 and so \( zd^{-1} \geq g \lor 0 \). Therefore \( z \geq (g \lor 0)d \).

Now suppose that \( x \land y = 0 \), where \( x, y \in G \) and \( d \in D^+ \). If \( 1 \geq d \) then \( x \geq xd \) and hence \( 0 = x \land y \geq xd \land y = 0 \). If \( d > 1 \) then \( 1 > d^{-1} \) and so \( x \land y = 0 \). Thus \( 0 = (x \land y) d = xd \land y \).

Suppose that \( G \) is a \( D_f \)-module. Then each \( C \in P(G) \) is a submodule and hence \( G/C' \) is a \( D_f \)-module. Thus each of the \( l \)-groups \( G/C \) used in the construction of \( \mathcal{O}(G) \) is an \( D_f \)-module and each of the connecting \( l \)-isomorphisms \( \pi_q \) also preserves scalar multiplication by elements of \( D \). Consider \( \mathcal{L} \in \mathcal{O}(G) \) and \( C \in D(G) \) such that

\[
0 \neq \mathcal{L} = (\ldots, C' + \mathcal{L}(C), \ldots) \quad \text{where} \quad \mathcal{L}(C) \in G.
\]

Define \( \mathcal{L}d \) to be the element in \( \mathcal{O}(G) \) with \( (\mathcal{L}d)_q = (\ldots, C' + \mathcal{L}(C)d, \ldots) \). It follows that \( \mathcal{O}(G) \) is a \( D_f \)-module and the natural map \( g \rightarrow \tilde{g} \) of \( G \) into \( \mathcal{O}(G) \) also preserves scalar multiplication by elements of \( D \).

**Theorem 3.2.** There exists a unique minimal essential extension \( G^{X_D} \) of the \( D_f \)-module \( G \) that is an \( X \)-group and also a \( D_f \)-module. \( G^{X_D} \) is isomorphic to the intersection of all \( X \)-subgroups of \( \mathcal{O}(G) \) that contain \( G \) and are \( D_f \)-modules.

The proof is analogous to the proof of Theorem 2.6. We shall show that \( G^X = G^{X_D} \) as \( l \)-groups and there exists a unique extension of the scalar multiplication of \( G \) to a scalar multiplication of \( G^X \) by \( D \).

Recall that an \( f \)-ring \( G \) is a lattice ordered ring such that

\[
x \land y = 0 \quad \text{implies} \quad xd \land y = dx \land y = 0 \quad \text{for all} \quad x, y, d \in G^+.
\]

Thus each polar of \( G \) is a ring ideal and so it follows that \( \mathcal{O}(G) \) is also an \( f \)-ring and the natural \( l \)-isomorphism of \( G \) into \( \mathcal{O}(G) \) is a ring isomorphism.

**Theorem 3.3.** There exists a unique minimal essential extension \( G^{X_f} \) of the \( f \)-ring \( G \) that is an \( X \)-group and also an \( f \)-ring. Moreover, \( G^{X_f} \) is isomorphic to the intersection of all \( X \)-subgroups of \( \mathcal{O}(G) \) that contain \( G \) and are sub-\( f \)-rings of \( \mathcal{O}(G) \).

Again the proof is analogous to the proof of Theorem 2.6. We shall show that \( G^X = G^{X_f} \) as \( l \)-groups and there exists a unique \( f \)-ring structure for \( G^X \) so that \( G \) is a subring.
4. Lifting \( p \)-endomorphisms from \( G \) to \( G^x \)

Let \( G \) be a representable \( l \)-group and let \( G^x \) be the \( X \)-hull of \( G \) in \( \mathcal{C}(G) \).

**Theorem A.** (Chambless [7]) \( G^{sp} = \{l \in \mathcal{C}(G) \mid l = 0 \text{ or } l_s \neq 0 \text{ for some finite partition of } P(G)\} \). Thus \( G^{sp} \) is the direct limit of the groups \( G_s \) for finite \( s \in D(G) \) and hence is the join of the directed set of \( l \)-groups \( G_s \pi_s \), where \( \pi_s \) is the natural map of \( G_s \) into \( \mathcal{C}(G) \).

**Theorem B.** (Chambless [7]). Let \( S \) be the subalgebra of \( P(G) \) generated by elements of the form \( g' \) and \( g'' \). Then
\[
G^p = \{l \in \mathcal{C}(G) \mid l = 0 \text{ or } l_s \neq 0 \text{ for some finite partition of } P(G) \text{ such that } s \subseteq S\}
\]
Thus \( G^p \) is a direct limit.

Now, as we have seen, if \( G \) is an \( f \)-ring then so are the \( G_s \) and so it follows that \( G^p \) and \( G^{sp} \) are subrings of \( \mathcal{C}(G) \). We shall also show that \( G^c \) is a subring of \( \mathcal{C}(G) \).

Amemiya [1] mentions that if \( G \) is a vector lattice or an \( f \)-ring then under his construction \( G^c \) is also a vector lattice or an \( f \)-ring.

If \( G \) is an \( f \)-ring then each minimal prime subgroup of \( (G, +) \) is a ring ideal and so \( T = \prod G/M_s \) for all minimal prime subgroups \( M \), is an \( f \)-ring. is a subring constructs \( G^p \) in \( T \). Here it is hard to determine whether or not \( G^p \) is large in \( T \).

**Lemma 4.1.** If \( \sigma \) is a polar preserving endomorphism of an \( l \)-group \( G \), \( \{a_x \mid x \in A\} \) is a disjoint subset of \( G \) and \( \vee a_x \) exists, then \( \{a_x \sigma \mid x \in A\} \) is disjoint and \( (\vee a_x) \sigma = \vee a_x \sigma \).

**Proof.** Clearly \( (\vee a_x) \sigma \geq a_x \sigma + x \) for all \( \beta \in A \). Suppose that \( d \geq a_x \sigma + x \) for all \( \beta \). Then \( (\vee a_x) \sigma \geq (\vee a_x) \sigma \wedge d \geq a_x \sigma \) for each \( \beta \) and hence
\[
(\vee a_x) \sigma - x = (\vee a_x) \sigma \wedge d \geq a_x \sigma
\]
for all \( \beta \), where \( x \geq 0 \). Therefore \( (\vee a_x) \sigma \geq a_x \sigma + x \) for all \( \beta \). To complete the proof it suffices to show that \( x = 0 \). Now \( (\vee a_x) \sigma \geq a_x \sigma + x \wedge a_{\gamma} \) for all \( \beta \); so \( (\vee a_x \sigma )\sigma \geq x \wedge a_{\beta} \) for each \( \beta \). But \( (x \wedge a_{\beta}) \wedge a_{\gamma} = 0 \) for all \( \gamma \neq \beta \), and so
\[
0 = (x \wedge a_{\beta}) \wedge (\vee a_x \sigma ) = (x \wedge a_{\beta}) \wedge ((\vee a_x \sigma )\sigma ) = x \wedge a_{\beta}
\]
for each \( \beta \); hence \( x \wedge (\vee a_x) = 0 \), and thus \( 0 = x \wedge (\vee a_x) = x \).

**Corollary I.** If \( \{a_x \mid x \in A\} \) is a disjoint subset of a \( D_f \)-module \( G \) over a directed po-ring \( D \), \( \vee a_x \) exists and \( 0 < c \in D \) then \( (\vee a_x) c = \vee a_x c \).

**Corollary II.** If \( \{a_x \mid x \in A\} \) is a disjoint subset of an \( f \)-ring \( G \) and \( \vee a_x \) exists then \( (\vee a_x) c = \vee a_x c \) and \( c(\vee a_x) = \vee a_x c \) for each \( c \in G^+ \).
LEMMA 4.2. (Henriksen and Isbell [15]). If $Y$ is a multiplicative sub-
semigroup of an $f$-ring $F$ then the $l$-subgroup $T$ of $(F, +)$ that is generated
by $Y$ is a subring.

PROOF. Let $[Y] = \{ e_1 y_1 + \cdots + e_n y_n \mid y_i \in Y, e_i = \pm 1 \text{ and } n \geq 0 \}$ be the
subgroup of $(F, +)$ generated by $Y$. Then
\[
T = \{ \bigvee_A \bigwedge_B s_{a\beta} \mid s_{a\beta} \in [Y] \text{ and } A \text{ and } B \text{ are finite} \}.
\]
But $[Y]$ is a subring of $F$ and if $a = \bigvee A a_{\beta}$ and $b = \bigvee B b_{\gamma}$ belong to $T$ then
\[
a^+ = \bigvee (a_{\beta} \vee 0) \text{ and } b^+ = \bigvee (b_{\gamma} \vee 0)
\]
and since positive elements distribute multiplicatively over $\vee$ and $\wedge$ it follows that $a^+ b^+ \in T$ and hence $T$ is a subring of $F$.

PROPOSITION 4.3. Suppose that $G$ is an $f$-ring and also a subring of the
$f$-ring $H$. If $H$ is laterally complete and an essential extension of $G$ then the
lateral completion $G^L$ of $(G, +)$ in $H$ is a subring.

PROOF. Consider \{ $a_\alpha \mid \alpha \in A$ \} and \{ $b_\beta \mid \beta \in B$ \} disjoint subsets of $G$. Then by
Corollary II of Lemma 4.1
\[
(\bigvee a_\alpha)(\bigvee b_\beta) = \bigvee a_\alpha b_\beta.
\]
Thus the set of all such $\bigvee a_\alpha$ is a subsemigroup of $H$. It follows from Lemma 4.2
that the $l$-subgroup $G(1)$ of $H$ generated by these elements $\bigvee a_\alpha$ is a subring.
Then by transfinite induction it follows that $G^L$ is a subring of $H$, (see [9]).

THEOREM 4.4. Let $G$ be a representable $l$-group and let $X = P, SP, L$ or $O$.

1) A $p$-endomorphism $\sigma$ of $G$ has a unique extension to a $p$ endomorphism
$\sigma^X$ of $G^X$.

2) If $\sigma$ is one to one then so is $\sigma^X$. If $\sigma$ is onto then so is $\sigma^X$ for $X = P, SP$
or $O$.

3) If $\alpha$ is a $p$ endomorphism of $G^0$ such that $G\alpha \subseteq G$ then $G^X\alpha \subseteq G^X$.

PROOF. If $C \in D(G)$ and $C \in C$ then $C' + g \rightarrow C' + g\sigma$ is an $l$-endomorphism
of $G/C'$ and hence
\[
(\cdots, C' + g(C), \cdots)\xrightarrow{\sigma^X} (\cdots, C' + g(C)\sigma, \cdots)
\]
is an $l$-endomorphism of $G^X$. If $C \supseteq A \in D(G)$ then
\[
\begin{array}{ccc}
G^X = \prod G/C' & \xrightarrow{\sigma^X} & G^X \\
\pi_{q,A} \downarrow & & \downarrow \pi_{q,A}
\end{array}
\]
\[
\begin{array}{ccc}
G^X = \prod G/A' & \xrightarrow{\sigma^X} & G^X
\end{array}
\]
commutes. For \((\ldots, A' + g(C), \ldots)\sigma_{\xi, \eta} = (\ldots, C' + g(C)\sigma, \ldots)\pi_{\xi, \eta} = (\ldots, A' + g(C)\sigma, \ldots) = (\ldots, A' + g(C), \ldots)\sigma_{\xi, \eta} = (\ldots, C' + g(C), \ldots)\pi_{\xi, \eta}\sigma_{\xi, \eta}\) where of course \(A \subseteq C\).

Thus \(\sigma\) determines an \(l\)-endomorphism \(\tilde{\sigma}\) of \(\mathcal{O}(G)\). Let \(\pi\) be the natural map of \(G\) onto \(\tilde{G} \subseteq \mathcal{O}(G)\). Then \((g\pi)_{\xi} = (\ldots, C' + g, \ldots)\) for all \(\xi \in D(G)\), and \(\pi\tilde{\sigma} = \sigma\pi\) on \(G\) and so \(\tilde{\sigma}\) is an extension to \(\mathcal{O}(G)\) of the \(p\) endomorphism \(\pi^{-1}\sigma\pi\) of \(\tilde{G}\).

We next show that \(\tilde{\sigma}\) is a \(p\)-endomorphism of \(\mathcal{O}(G)\). If \(\theta \neq l, k \in \mathcal{O}(G)\) and \(\wedge k = \theta\) then there exist \(\xi \in D(G)\) such that \(l_{\xi} \neq 0 \neq k_{\xi}\) and such that their supports are disjoint. If \(l_{\xi}\sigma_{\xi} = 0\) then \(l\tilde{\sigma} = 0\) and hence \(l\tilde{\sigma} \wedge k = \theta\). In any case the support of \(l_{\xi}\sigma_{\xi} \subseteq\) support of \(l_{\xi}\) and hence \(l_{\xi}\sigma_{\xi} \wedge k_{\xi} = 0\) and so \(l\tilde{\sigma} \wedge k = \theta\). Therefore \(\tilde{\sigma}\) is a \(p\)-endomorphism of \(\mathcal{O}(G)\).

We next show that if \(\alpha\) is a \(p\)-endomorphism of \(\mathcal{O}(G)\) that induces \(\pi^{-1}\sigma\pi\) on \(\tilde{G}\) then \(\alpha = \tilde{\sigma}\). Consider \(l_{\xi} = (\ldots, C' + g, \ldots)\) and suppose that \((l\alpha)_{\xi} = (\ldots, C' + x, \ldots)\) where \(C' + x \neq C' + g\sigma\). Then

\[
\left|\tilde{\alpha} - l_{\xi}\right| \wedge (0, \ldots, 0, C' + \left|g\sigma - x\right|, 0, \ldots, 0) = 0 \text{ but }
\left|\tilde{\alpha} - l_{\xi}\right| (0, \ldots, 0, C' + \left|g\sigma - x\right|, 0, \ldots, 0) \neq 0
\]

and thus \(\alpha\) is not a \(p\) endomorphism, a contradiction.

Therefore \(\sigma\) has a unique extension to a \(p\)-endomorphism of \(G^G\). Now if \(\rho\) is an extension of \(\sigma\) to say \(G^P\) then it can be extended to \(G^G\) and so \(\rho\) is unique. Thus to complete the proof of (1) it suffices to verify (3). So suppose that \(\alpha\) is a \(p\) endomorphism of \(G^G\) such that \(G\alpha \subseteq G\).

a) \(G^G\alpha \subseteq G^L\). For if \(\{a_\theta \mid \theta \in \Lambda\}\) is a disjoint subset of \(G\) then by Lemma 4.1 \((\wedge a_\theta)\alpha = \wedge a_\theta\alpha\) and so \(G(1)\alpha \subseteq G(1)\), where \(G(1)\) is the \(l\)-subgroup of \(G^L\) that is generated by all the elements \(\wedge a_\theta\). Thus it follows by transfinite induction that \(G^L\alpha \subseteq G^L\).

b) \(G^P\alpha \subseteq G^P\). Here we assume that \(G = \tilde{G}\) and \(G^G = \mathcal{O}(G)\). Then we know exactly how \(\alpha\) operates on \(\mathcal{O}(G)\). Consider \(\theta \neq l \in G^SP\). Then \(l_{\xi} \neq 0\) for some finite partition \(\xi\) of \(P(G)\). If \((l\alpha)_{\xi} = 0\) then \(l\alpha = \theta\) and if \((l\alpha)_{\xi} \neq 0\) then clearly \(l\alpha \in G^SP\) by Chambless' Theorem A.

c) \(G^P\alpha \subseteq G^P\). This is a simple application of Chambless' Theorem B. This completes the proof of (1) and (3).

(2) If \(\sigma\) is one to one then \(\sigma^X\) is one to one since \(G\) is large in \(G^X\). Now suppose that \(\sigma\) is onto. Then the map \(C' + g \rightarrow C' + g\sigma\) is an \(l\)-homomorphism of \(G/C'\) onto itself. Thus \(\sigma^G\) is clearly onto and using our representations of \(G^P\) and \(G^SP\) it follows that \(\sigma^P\) and \(\sigma^{SP}\) are also onto.

QUESTION. Is \(\sigma^X\) onto provided that \(\sigma\) is onto?

**Theorem 4.5.** If \(G\) is a \(D_f\)-module over the directed po-ring \(D\) then there
exists a unique extension of the scalar multiplication by elements of $D$ so that $G^X$ is also a $D$-module. Moreover $G^X$ with this scalar multiplication equals $G^{X_D}$ for $X = P, SP, L$ or $O$.

**Proof.** The first part follows from the fact that each $p$-endomorphism of $G$ has a unique extension to a $p$ endomorphism of $G^X$. Now (without loss of generality) $G \subseteq G^X \subseteq G^{X_D} \subseteq \mathcal{O}(G)$ and $G^X$ is a submodule of $G^{X_D}$. Therefore $G^X = G^{X_D}$.

**Theorem 4.6.** If $G$ is an $f$-ring then there is a unique multiplication on $G^X$ so that $G^X$ is an $f$-ring and $G$ is a subring. Moreover, $G^X$ with this ring structure equals $G^{X_f}$ for $X = P, SP, L$ or $O$.

**Proof.** We first verify the result for $X = O$. Now as we have seen $\mathcal{O}(G)$ is a ring and the natural map $g \to \tilde{g}$ is a ring $l$-isomorphism. So all we need show is that the multiplication of $\mathcal{O}(G)$ is uniquely determined by that of $\tilde{G}$. Suppose that $\cdot$ is a multiplication on $\mathcal{O}(G)$ so that $\mathcal{O}(G)$ is an $f$-ring and $\cdot$ induces the given multiplication on $\tilde{G}$.

If $0 < \tilde{g} \in \tilde{G}$ then the right multiplication of $\tilde{G}$ by $\tilde{g}$ is a $p$-endomorphism of $\tilde{G}$ and so has a unique extension to a $p$ endomorphism of $\mathcal{O}(G)$. Therefore

$$x \cdot \tilde{g} = x\tilde{g} \text{ for all } x \in \mathcal{O}(G).$$

Suppose that $x_\Phi = (0, \ldots, 0, C' + t, 0, \ldots, 0)$. Now

$$\tilde{g}_\Phi = (0, \ldots, 0, C' + g, 0, \ldots, 0) + (\text{the other non-zero components}) = a + b.$$

Now $x_\Phi \cdot b = 0$ since they are disjoint and so $(0, \ldots, 0, C' + tg, 0, \ldots, 0) = x_\Phi \tilde{g}_\Phi = x_\Phi \cdot (a + b) = x_\Phi \cdot a = (0, \ldots, 0, C' + t, 0, \ldots, 0) \cdot (0, \ldots, 0, C' + g, 0, \ldots, 0)$.

Now consider $x, y \in \mathcal{O}(G)$ with $x_\Phi \neq 0 \neq y_\Phi$.

$$x_\Phi = (\ldots, C' + x(C), \ldots) = \vee x_C, \text{ where } x_C = (0, \ldots, 0, C' + x(C), 0, \ldots, 0)$$

$$y_\Phi = (\ldots, C' + y(C), \ldots) = \vee y_C, \text{ where } y_C = (0, \ldots, 0, C' + y(C), 0, \ldots, 0).$$

Thus by Lemma 4.1 and the above

$$x_\Phi \cdot y_\Phi = \vee x_C \cdot y_C = \vee x_C \cdot y_C = \vee x_C y_C = x_\Phi y_\Phi.$$

Therefore $\cdot$ is the natural multiplication on $\mathcal{O}(G)$ and so there is a unique $f$-ring structure on $G^O$ so that $G$ is a subring of the $f$-ring $G^O$.

Finally we have shown that $\tilde{G}^P$, $\tilde{G}^{SP}$ and $\tilde{G}^L$ are all subrings of $\mathcal{O}(G)$. Also any ring structure on $G^X$ that induces the given one on $G$ can be extended to a ring structure on $G^O$. Therefore the ring structures of $G^P$, $G^{SP}$ and $G^L$ are also determined by their additive structures.
5. The $y$-hulls of archimedean $l$-groups and $f$-rings

An archimedean $l$-group $A$ is called a

$d$-group if it is divisible,
$v$-group if it is a vector lattice,
$c$-group if it is a conditionally complete lattice,
$e$-group if it is essentially closed in the class of archimedean $l$-groups.

It is well known that an abelian $l$-group $A$ is contained in a unique minimal divisible abelian $l$-group $A^d$. For there is exactly one way of extending the order of $A$ to a lattice-order of its injective hull $A^d$ so that $(A^d)^+ \cap A = A^+$. Also if $A$ is archimedean then so is $A^d$.

**Theorem 5.1.** If $A$ is a large $l$-subgroup of an archimedean $y$-group $H$, where $y = d, v, c$ or $e$, then the intersection $K$ of all the $l$-subgroups of $H$ that contain $A$ and are $y$-groups is a $y$-group. Thus $K$ is a minimal essential extension of $A$ that is a $y$-group and we shall call such an extension a $y$-hull of $A$.

**Theorem 5.2.** Each archimedean $l$ group $A$ admits a unique $y$-hull $A^y$ for $y = d, v, c$ or $e$. $A^c$ is the Dedekind MacNeille completion $A^\wedge$ of $A$ and $A$ is dense in $A^c$. $A^e$ is the $l$-subspace of $(A^d)^{\tilde{c}}$ that is generated by $A$. $A^e = ((A^d)^{\tilde{c}})^e$ is the essential closure of $A$.

**Remarks.** A minimal essential extension of an archimedean $l$-group that is a vector lattice is necessarily archimedean [11]. Bleier [6] has shown that a minimal archimedean vector lattice that contains $A$ is necessarily an essential extension of $A$ and hence is $A^e$. Also, of course, any complete $l$-group is archimedean.

**Proof of Theorem 5.1.** If $y = d$ or $v$ then clearly the theorem holds. For the intersection of divisible subgroups (subspaces) is again divisible (a subspace). If $A$ is a large $l$-subgroup of an archimedean $e$-group $H$ then clearly $H$ is an $e$-hull of $A$. To prove the theorem for $y = c$ we make use of the following two lemmas.

**Lemma 5.3.** (Bernau [3]). If $G$ is a dense $l$-subgroup of an $l$-group $H$ then all joins and intersections in $G$ agree with those in $H$.

**Lemma 5.4.** If $A$ is a large $l$-subgroup of an abelian $l$ group $B$ then all joins and intersections in $A$ agree with those in $B$.

**Proof.** $A$ is large in $B^d$ and so $A^d$ is dense in $B^d$. Suppose that $\{a_\lambda | \lambda \in \Lambda \} \subseteq A$ and $\vee_A a_\lambda$ exists. If $\{a_\lambda | \lambda \in \Lambda \} \subseteq y \in A^d$ then $ny \in A$ for some $n > 0$ and so $ny \geq \vee_A na_\lambda = n \vee_A a_\lambda$. Thus $y \geq \vee_A a_\lambda$ and hence $\vee_{A^y} a_\lambda = \vee_A a_\lambda$.

Next $\vee_{A^d} a_\lambda = \bigvee_B a_\lambda$ since $A^d$ is dense in $B^d$. Finally $\bigvee_{B^d} a_\lambda = \bigvee_B a_\lambda$ since $\{a_\lambda | \lambda \in \Lambda \} \subseteq B$ and $\bigvee_{B^d} a_\lambda = \bigvee_A a_\lambda \in A \subseteq B$. Thus $\bigvee_A a_\lambda = \bigvee_B a_\lambda$. 

Downloaded from https://www.cambridge.org/core. IP address: 54.70.40.11, on 15 Mar 2019 at 14:15:36, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446788700015391
COROLLARY. If $A$ is a large $l$-subgroup of a complete $l$-group $H$, then the intersection of all $c$ subgroups of $H$ that contain $A$ is a $c$ subgroup.

QUESTION. Is Lemma 5.4 true for non abelian $l$ groups?

PROOF OF THEOREM 5.2. Clearly the theorem holds for $y = d$. In [11] it is shown that $A$ admits a unique $v$ hull $A^v$ and that $A^v$ is the $l$ subspace of $(A^v)^\wedge$ that is generated by $A$.

In [10] it is shown that $A$ admits a unique essential closure $A^e$ and that $A^e = ((A^e)^\wedge)^L$.

The existence of $A^e$ for a complete vector lattice $A$ was proven by Pinsker [19] and Jakubík [16] showed that $A^e$ can be constructed solely from the underlying lattice structure of $A$.

We now show that there exists a unique $c$ hull $A^c$ and that $A^c = A^\wedge$. Note that $A^\wedge$ is the unique minimal complete $l$ group in which $A$ is dense [12]. Also if $A$ is an $l$-subgroup of a complete $l$-group $H$ then $H$ need not contain a copy of $A^\wedge$ [12].

LEMMA 5.5. If $A$ is a large $l$-subgroup of a complete $l$ group $H$ then $A^\wedge \subseteq H$.

PROOF. We shall show that there exists an $l$-isomorphism of $A^\wedge$ into $H$ that is the identity on $A$. If $x \in A^\wedge$ then

$$x = \vee \{x \in A \mid x \leq x\} = \wedge \{\tilde{x} \in A \mid \tilde{x} \geq x\}.$$ 

Since $\tilde{x} \geq \{x \in A \mid x \leq x\}$ we have that $\vee_H \tilde{x}$ exists. In particular for $0 < x \in A^\wedge$, $x = \vee \{x \in A^+ \mid x \leq x\}$ and $\vee_H \{\tilde{x} \in A^+ \mid \tilde{x} \leq x\}$ exists. Define $x\sigma = \vee_H \{\tilde{x} \in A^+ \mid \tilde{x} \leq x\}$.

1) If $a \land b = 0$ in $A^\wedge$ then $a\sigma \land b\sigma = 0$.

For $a = \vee a$ and $b = \vee b$, where $a \land b = 0$ and hence

$$0 \leq a\sigma \land b\sigma = \vee_H a \land \vee_H b = \vee_H (a \land b) = 0.$$ 

2) If $a, b \in (A^\wedge)^+$ then $a\sigma + b\sigma = (a + b)\sigma$.

For $a\sigma + b\sigma = \vee_H a + \vee_H b = \vee_H (a + b) = \vee_H X$, where

$$X = \{a + b \mid a, b \in A^+, a \leq a \text{ and } b \leq b\},$$

and

$$(a + b)\sigma = \vee_H a + b = \vee_H Y,$$

where

$$Y = \{y \in A^+ \mid y \leq a + b\}.$$

Now if $x \in X$ then $x = a + b \leq a + b$ and so $x \in Y$. Thus $X \subseteq Y$ and hence $\vee_H X \subseteq \vee_H Y$. 


If \( y \in Y \) then \( 0 \leq y \leq a + b \) and hence \( y = u + v \) where \( u, v \in A^* \), \( 0 \leq u \leq a \) and \( 0 \leq v \leq b \). Thus \( u = \vee u \) and \( v = \vee v \) and hence \( y = \vee (u + v) = \vee A^* S \) where \( S \subseteq X \subseteq A \) and \( y \in A \). Therefore \( y = \vee A^* S = \vee A S = \vee H S \) since by Lemma 5.4 joins in \( A \) agree with those in \( H \). Thus \( y \leq \vee H X \) and so \( \vee H Y \leq \vee H X \).

Therefore \( \sigma \) is a map of \((A^*)^+\) into \( H^+ \) that preserves addition and disjointness and induces the identity on \( A^+ \). For \( g = a - b \in A^* \), where \( a, b \in (A^*)^+ \) define \( g \tau = a \sigma - b \sigma \). Then \( \tau \) is a group homomorphism of \( A^* \) into \( H \) that preserves disjointness and so it is an \( l \)-homomorphism. Since \( \tau \) induces the identity on the large \( l \) subgroup \( A \) of \( A^* \) it follows that \( \tau \) is an \( l \)-isomorphism.

**Corollary I.** \( A^* \subseteq (A^d)^* \).

**Corollary II.** If \( A \) is a large \( l \)-subgroup of a complete \( l \)-group \( H \) and no proper \( l \)-subgroup of \( H \) contains \( A \) and is complete, then \( H = A^* \). In particular \( A \) is dense in \( H \).

**Corollary III.** \( A = A^c \) is unique.

This completes the proof of Theorem 5.2.

If follows at once from Lemma 5.4 that if \( A \) is a large \( l \)-subgroup of a \( \sigma \)-complete \( l \)-group \( H \) then the intersection \( K \) of all the \( \sigma \)-complete \( l \)-subgroups of \( H \) that contain \( A \) is \( \sigma \) complete. Thus \( K \) is a \( \sigma \) complete hull of \( A \). Since \( A \) is large in \( K^* \) it follows from Lemma 5.5 that \( A \subseteq A^* \subseteq K^* \). Now \( A^* \cap K \) is \( \sigma \)-complete and contains \( A \) and so since \( K \) is minimal we have \( A \subseteq K \subseteq A^* \). Thus \( K \) is the intersection of all \( \sigma \)-complete \( l \)-subgroups of \( A^* \) that contain \( A \) and hence \( K \) is unique. Therefore each archimedean \( l \)-group \( A \) admits a unique \( \sigma \)-complete hull \( A^\sigma \).

It is well known that \( A^\sigma \) is a \( P \) group but need not be an \( SP \)-group (see for example [25] p. 85). If each bounded disjoint subset of an archimedean vector lattice \( A \) is countable then since \( A \) is dense in \( A^\sigma \) it follows that each bounded disjoint subset of \( A^\sigma \) is also countable. Thus ([25] p. 156) \( A^\sigma \) is complete and hence \( A^\sigma = A^* \). These spaces \( A^\sigma \) of "countable type" were introduced by Pinsker and have many nice properties (see [25] pp. 156–160).

**Theorem 5.6.** If \( \alpha \) is a \( p \)-endomorphism of an archimedean \( l \)-group \( A \) then there exists a unique extension of \( \alpha \) to a \( p \) endomorphism \( \bar{\alpha} \) of the \( y \)-hull \( A^y \) of \( A \), where \( y = d, v, c \) or \( e \).

**Proof.** The proof for \( y = c \) is contained in [13]. Suppose that \( y = d \) and consider \( a \in A^p \). Then \( n a \in A \) for some \( n > 0 \). Define \( a\bar{\alpha} = ((na)\alpha)/n \). A straightforward computation shows that \( \bar{\alpha} \) is a \( p \) endomorphism of \( A^y \) and an extension
of $\alpha$. If $\beta$ is an extension of $\alpha$ to a $p$-endomorphism of $A^p$ then
\[ n(\alpha \beta) = (na)\beta = (na)\alpha = (na)\bar{a} = n(a\bar{a}) \]
and hence $a\beta = a\bar{a}$.

Combining the above we get a unique extension of $\alpha$ to a $p$-endomorphism $\gamma$ of $(A^d)^c$. Also $\gamma$ is linear [13] and maps $A$ into $A$. Thus $\gamma$ maps the $l$-subspace $A^c$ of $(A^d)^c$ that is generated by $A$ into $A^c$.

Finally since $A^e = ((A^d)^c)^L$ it follows from Theorem 4.4 that $\alpha$ has a unique extension to a $p$-endomorphism of $A^e$.

**Corollary.** If $A$ is an archimedean $D_f$-module over the directed po-ring $D$ then there exists a unique extension of the scalar multiplication by elements of $D$ so that $A^p$ is also a $D_f$-module, where $y = d, v, c$ or $e$.

**Remarks.** Since $A$ is large in $A^p$ it follows that $\alpha$ is one-to-one if and only if $\bar{a}$ is one-to-one. It can be shown that if $y = d, v$ or $c$ then $\bar{a}$ is onto provided that $a$ is onto. The proof for $y = c$ is given in [13]. Bleier [6] shows that an $l$-automorphism of $A$ has a unique extension to an $l$-automorphism of $A^e$.

**Theorem 5.7.** If $A$ is an archimedean $l$-group and $\alpha$ is an $l$-automorphism of $A$ then there exists a unique extension to an $l$-automorphism $\bar{a}$ of $A^p$, where $y = d, v, c$ or $e$.

**Proof.** For $y = d$ the map $\bar{a}$ defined in the proof of the last theorem is an $l$-automorphism of $A^d$. We have shown that the theorem holds for $y = L$. Thus to complete the proof it suffices to show that $\alpha$ can be extended uniquely to an $l$-automorphism of $A^c$. For $h \in (A^d)^c$, $h = \vee \{h \in A^+ \mid h \leq h\}$. Define
\[ h\bar{a} = \vee h\alpha. \]
A straightforward computation shows that $\bar{a}$ determines an $l$-automorphism of $A^c$ that is the unique extension of $\alpha$ (see the proof of Lemma 5.5).

**Lemma 5.8.** (Bernau [2]). If $F$ is an archimedean $f$-ring, $x \in F^+$, $\{\lambda \lambda \in \Lambda\} \subseteq F$ and $\vee a_\lambda$ exists then $\vee (xa_\lambda)$ exists and $\vee (xa_\lambda) = x(\vee a_\lambda)$, and dually.

**Theorem 5.9.** Suppose that $A$ is an archimedean $f$-ring, and $A^p$ is the $y$-hull of $(A, +)$ for $y = d, v, c$ or $e$. Then there is a unique multiplication on $A^p$ so that $A^p$ is an $f$-ring and $A$ is a subring. Thus the additive structure of $A^p$ completely determines the ring structure.

**Proof.** For $a, b \in A^d$ there exists an integer $n > 0$ such that $na$ and $nb$ belong to $A$. Define
\[ ab = ((na)(nb))/n^2. \]
A routine check shows that $A^d$ is an $f$-ring and this is the unique extension of the multiplication of $A$ to an $f$-ring multiplication of $A^d$.

For $a, b \in ((A^d)^o)^+$ define

$$ab = \wedge \{xy \mid x \geq a, y \geq b \text{ and } x, y \in A^d\}$$

and for $x = x_1 - x_2$ and $y = y_1 - y_2$ in $(A^d)^c$ where $x_i, y_i \in ((A^d)^c)^+$ define

$$xy = x_1y_1 + x_2y_2 - (x_1y_2 + x_2y_1).$$

A rather long messy computation shows that $(A^d)^c$ is an $f$-ring. This construction is "well known".

Now suppose that $\cdot$ and $\times$ are two multiplications of $(A^d)^c$ so that it is an $f$-ring and $A^d$ is a subring and consider $a, b \in ((A^d)^c)^+$.\[a = \wedge \{x \in A^d \mid x \geq a\} \text{ and } b = \wedge \{y \in A^d \mid y \geq b\}\]

and hence by Lemma 5.8

$$a \cdot b = (\wedge x) \cdot (\wedge y) = \wedge (x \cdot y) = \wedge (x \times y) = (\wedge x) \times (\wedge y) = a \times b.$$  

Thus there is only one such multiplication. Of course the same result holds for $A^e$.

Now we have shown that the ring structure of $(A^d)^c$ has a unique extension to $((A^d)^c)^L = A^e$ (see Theorem 4.6). To complete the proof it suffices to show that $A^p$ is a subring of $A^e$. Consider $x, y \in A$ and $r, s \in R$. Then $rx, sy \in A^e$ and $xy \in A$. Thus since $A^e$ is a real algebra (see Section 6)

$$(rx)(sy) = rs(xy) \in A^e.$$ It follows that the subspace $S$ of $A^e$ that is generated by $A$ is a subring of $A^e$. Now

$$A^p = \{\bigvee_U \bigwedge_V a_{\alpha \beta} \mid a_{\alpha \beta} \in S, \alpha \in U, \beta \in V \text{ and } U \text{ and } V \text{ are finite}\}.$$ Thus by Lemma 4.2 $A^p$ is a subring of $A^e$.

**REMARKS.** If $A$ is an archimedean $f$-ring and $H$ is a minimal essential extension of $A$ that is an archimedean $f$-ring and a $y$-group then $H = A^p$. For clearly $A \subseteq A^p \subseteq H$ as $l$-groups by Theorems 5.1 and 5.2. If $y = e$ then $A^e$ is essentially closed and large in $H$ and so $A^e = H$. If $y = d$ then an easy computation shows that $A^d$ is a subring of $H$ and so $A^d = H$.

If $y = c$ or $r$ then a rather messy proof shows that $A^r$ is a subring of $H$ and so once again $A^r = H$.

**6. The structure of an archimedean $f$-ring**

Let $A$ be an archimedean $f$-ring and let $X$ be the Stone space of the complete Boolean algebra $P(A)$ of polars of $A$. Then $X$ is compact, Hausdorff and extremally disconnected. Let $D(X)$ be the ring of continuous functions from $X$
into the extended reals \((R, \pm \infty)\) that are finite on a dense open subset of \(X\). Then as \(l\) groups \(A^*\) and \(D(X)\) are isomorphic \([10]\). So let us examine the ways in which \(D(X)\) can be made into an \(f\)-ring with pointwise addition and order.

Suppose that \(D = D(X)\) has a multiplication \(\cdot\) so that it is an \(f\)-ring. Then for \(a \in D^+\) the map \(d \rightarrow d \cdot a\), for all \(d \in D\), is a \(p\)-endomorphism of \((D, +)\) and so (see \([13]\)) there is an element \(\tilde{a} \in D^+\) such that

\[
d \cdot a = d\tilde{a} \quad \text{for all } d \in D.
\]

We investigate the map \(a \rightarrow \tilde{a}\). Consider \(a, b \in D^+\).

1) \(a \cdot \tilde{b} = \tilde{a} \cdot b\).

For \(\tilde{d}(a + b) = d \cdot (a + b) = d \cdot a + d \cdot b = d\tilde{a} + d\tilde{b} = d(\tilde{a} + \tilde{b})\) for all \(d \in D\) and so for \(d = 1\), \(a + b = \tilde{a} + \tilde{b}\).

2) \(\tilde{a} \cdot \tilde{b} = \tilde{a} \cdot \tilde{b}\).

3) \(a\tilde{a} = b\tilde{a}\).

4) \(a \cdot \tilde{b} = \tilde{a} \cdot b\).

5) \(a\tilde{a} = b\tilde{a}\).

Here we use the fact that an archimedean \(f\)-ring is commutative.

4) Put \(\bar{I} = p\); then for \(u, v \in D^+, u \cdot v = uvp\).

For, for \(a \in D^+\), we have \(\tilde{a} = 1\tilde{a} = a\bar{I} = ap\). Now, \(v = a - b\), where \(a, b \in D^+\), and so \(u \cdot v = u \cdot (a - b) = u \cdot a - u \cdot b = u\tilde{a} - u\tilde{b} = uap - ubp = u(a - b)p = uvp\).

5) If \(\cdot\) is a multiplication on \(D(X)\) such that \(D(X)\) is an \(f\)-ring with componentwise addition and order then there exists an element \(p \in D^+\) so that \(a \cdot b = abp\) for all \(a, b \in D\), and conversely.

Now \(D\) is complete and hence a \(P\) group. Thus

\[
D = p'' \oplus p'.
\]

Clearly \(p''\) is a subring with respect to the \(\cdot\) multiplication and \(p'\) is a zero subring.

Consider \(d = u + v \in p'' \oplus p'\) and define

\[
d\tau = pu + v.
\]

Then for \(d_1 = u_1 + v_1\) and \(d_2 = u_2 + v_2\) in \(D\) we have

\[
(d_1 \cdot d_2)\tau = (pu_1u_2)\tau = pu_1pu_2 = d_1\tau d_2\tau
\]

and so we have an \(l\)-isomorphism of the \(f\)-ring \((D, +, \cdot, \leq)\) onto the \(f\)-ring \(D = p'' \oplus p'\), where \(p''\) is a ring with respect to the pointwise multiplication of \(D\) and \(p'\) has the zero multiplication.

**Theorem 6.1.** Let \(X\) be a Stone space and suppose that \(D(X)\) is an \(f\)-ring with componentwise addition and order. Then there exist clopen subsets \(Y\) and
Z of X such that \( X = Y \cup Z \), \( Y \cap Z = \emptyset \) and \( D(X) = D(Y) \oplus D(Z) \), where \( D(Y) \) has the pointwise multiplication and \( D(Z) \) has the zero multiplication.

Thus we have the structure of an arbitrary essentially closed archimedean \( f \)-ring. Recall that the radical of an \( f \)-ring \( A \) consists of the nilpotent elements.

**Corollary I.** (Henricksen and Isbell [15]). An archimedean \( f \)-ring is a subdirect sum of a ring with zero multiplication and one with radical zero.

**Corollary II.** If \( A \) is an archimedean \( f \)-ring then \( \text{rad} \; A = \{ a \in A \mid aA = 0 \} \) the set of annihilators of \( A \). In particular, \( \text{rad} \; A \) is a polar.

**Proof.** \( A \subseteq D(Y) \oplus D(Z) \) and if \( a = u + v \in A \) is nilpotent, where \( u \in D(Y) \) and \( v \in D(Z) \) then \( u = 0 \) and so \( a = v \) is an annihilator. Thus \( \text{rad} \; A = A \cap D(Z) \). Now \( D(Z) \) is a polar in \( D(X) \) and \( A \) is large in \( D(X) \). Thus \( \text{rad} \; A \) is a polar in \( A \).

**Corollary III.** If \( A \) is an archimedean \( f \)-ring and also an SP-group, then \( \text{rad} \; A \) is a cardinal summand. In particular, \( \text{rad} \; A \) is a cardinal summand of a complete \( f \)-ring \( A \).

Note that Corollaries II and III follow directly from Corollary I.

**Corollary IV.** If \( A \) is an archimedean \( f \)-ring with a weak order unit \( u \) and also a P-group, then \( \text{rad} \; A \) is a cardinal summand.

**Proof.** Since \( A \) is large in \( A^e \), \( u \) is also a weak unit of \( A^e \) and without loss of generality we may assume that as \( l \)-groups \( A^e = D(X) \) and \( 1 = u \in A \). Then \( 1 \cdot 1 = p \in A \) and so \( A = p'' \oplus p' \), where the polars are taken in \( A \).

**Corollary V.** For an archimedean \( f \)-ring \( A \) the following are equivalent.

i) \( \text{rad} \; A = 0 \).

ii) \( A^e \) contains an identity.

iii) \( \text{rad} \; A^e = 0 \).

**Proof.** \((\text{rad} \; A^e) \cap A = \text{rad} \; A \) and hence since \( A \) is large in \( A^e \) it follows that i) and iii) are equivalent. From the Theorem iii) and ii) are equivalent.

Let \( A \) be an archimedean \( f \)-ring with identity \( u \). Then \( u \) is a weak unit in \( A \) \((u \wedge a = 0 \) implies \( a = ua = 0 \)) and hence in \( A^e \). Let \( X \) be the Stone space of \( P(A) = P(A^e) \). Then there is a \( l \)-group isomorphism of \( A^e \) onto \( D(X) \) so that \( u \) maps upon 1. Thus without loss of generality, \( 1 \in A \subseteq A^e = D(X) \) as \( l \)-groups. It follows from the next theorem that \( A \) and \( A^e \) are both subrings of \( D(X) \). Thus, once again, the additive structure of \( A \) determines the ring structure.

**Theorem 6.2.** Suppose that \( A \) is an \( l \)-subgroup of \((D(X), +)\) and \( 1 \in A \), where \( X \) is a Stone space. If \( A \) is an \( f \)-ring with identity 1 then \( A \) is a subring of \( D(X) \).
PROOF. Let $\cdot$ be the multiplication in $A$. Then by (6)

$$1 = 1 \cdot 1 = 1p = p.$$ 

Thus $\cdot$ agrees with the pointwise multiplication of $D(X)$.

COROLLARY I. (Birkhoff and Pierce [5]). An archimedean f-ring with identity has radical zero.

COROLLARY II. If $A$ is an archimedean f-ring with identity $u$ then $u$ is also an identity for the f-ring $A^y$, where $y = d, v, c$ or $e$.

COROLLARY III. If $A$ is an archimedean f-ring with identity then each p-endomorphism of $A$ is a multiplication by a positive element.

PROOF. We may assume that $A$ is a subring of $D(X)$, where $D(X)$ has the pointwise multiplication, and $1 \in A$. Thus any $p$-endomorphism of $A$ has a unique extension to a $p$-endomorphism of $D(X)$, but each $p$-endomorphism of $D(X)$ is a multiplication by an element $d \in D^+ [13]$. Thus since $1 \in A$ it follows that $d \in A$.

We give two examples of archimedean f-rings for which the radical is not a cardinal summand.

I. Let $A = C[0,1]$ and let

$$p(x) = \begin{cases} -x + \frac{1}{2} & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Define $g \cdot f = gfh$ for $g, f \in A$. Then $A$ is an f-ring with

$$\text{rad } A = \{f \in A \mid f(x) = 0 \text{ for } 0 \leq x \leq \frac{1}{2}\}$$

but $(A, +)$ is cardinally indecomposable and so rad $A$ is not a summand.

II. Let $H = \prod_{i=1}^{\infty} Q_i$, where $Q_i$ is the additive group of rationals. In the even components use zero multiplication and in the odd components use the natural multiplication. Let $a = (1/2, 1/4, 1/8, \ldots, 1/2^n, \ldots)$, and let $S$ be the subring generated by $a$. Thus $S$ is the ring of polynomials without constant terms in $a$ and with integral coefficients. Let $A$ be the subring of $H$ generated by $S$ and $\sum Q_i$.

$$A = \{h \in H \mid h \text{ is a polynomial in } a \text{ except at a finite number of places}\}.$$ 

Then $A$ is an f-ring with a basis and a strong order unit, $a$ but rad $A$ is not a cardinal summand. Note that $a^2 = (1/4, 0, 1/64, 0, \ldots)$ but $a$ does not split into a "zero part and a radical zero part".

The next two examples show the well known fact that the class of f-rings with zero radical is not equationally definable.

III. Let $S$ be the semigroup of negative integers. Let $A$ be the semigroup ring of $S$ over the integers and define an element in $A$ to be positive if its largest non-zero component is positive. Then $A$ is a totally ordered integral domain

Downloaded from https://www.cambridge.org/core. IP address: 54.70.40.11, on 15 Mar 2019 at 14:15:36, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446788700015391
and so \( \text{rad} \, A = 0 \). Let \( J \) be the set of elements in \( A \) with support included in \(-2, -3, \ldots\). Then \( J \) is a convex ring ideal and \( A/J \) is a zero ring. Thus \( \text{rad} \, A/J = A/J \).

IV. Let \( A \) be the set of all bounded rational sequences with cardinal order. Then \( \text{rad} \, A = 0 \). Let \( a = (1, 1/4, 1/9, \ldots, 1/n^2, \ldots) \) and

\[ \langle a \rangle = \{ x \in A \mid |x| < na \text{ for some } n > 0 \}. \]

Then \( J/\langle a \rangle \) is an \( f \)-ring and \( 0 \neq \langle a \rangle + (1, 1/2, 1/3, \ldots) \in \text{rad} \, J/\langle a \rangle \).

The following example is due to Roger Bleier and shows that if \( G \) is an \( l \)-subgroup of an essentially closed archimedean \( l \)-group \( H \) then \( H \) need not contain a copy of the essential closure \( G^e \) of \( G \).

V. Pick a Stone space \( Y \) so that \( D(Y) \) cannot be represented as a subdirect sum of reals. Let \( C(Y) \) be the \( l \)-group of all continuous real valued functions on \( Y \). Then \( C(Y) \subseteq \prod R \) and \( C(Y)^e = D(Y) = C(Y)^l \).

7. The structure of an \( f \)-ring with a basis

A strictly positive element \( s \) in an \( f \)-ring \( A \) is called basic if \( s^n \) is totally ordered or equivalently if \( A/s' \) is a totally ordered ring. A basis for \( A \) is a maximal disjoint subset \( \{s_\lambda \mid \lambda \in \Lambda \} \) where in addition each \( s_\lambda \) is basic. Let \( S = \{s_\lambda \mid \lambda \in \Lambda \} \) be a basis for \( A \). Then there exists a natural ring \( l \)-isomorphism \( \sigma \) of \( A \) into \( K = \prod A/s_\lambda' \)

\[ a \xrightarrow{\sigma} (s_\lambda' + a, \ldots). \]

**Theorem 7.1.** \( K = (A\sigma)^0 \) and if \( S \) is finite then \( K = (A\sigma)^p \). In either case \( A \) is dense in \( A^0 \).

**Proof.** Consider \( 0 < x = (\ldots, s_\lambda' + x_\lambda, \ldots) \in K \) with say \( s_\lambda' + x_\lambda > s_\lambda' \). Then we may assume \( 0 < x_\lambda \neq s_\lambda' \) and so \( 0 < a = x_\lambda \wedge s_\lambda \in (\bigcap_{\lambda \neq \lambda} s_\lambda') \setminus s_\lambda' \). Thus \( 0 < a\sigma \leq x \) and so \( A\sigma \) is dense in \( K \). Thus since \( K \) is a \( P \)-group

\[ A\sigma \subseteq (A\sigma)^p \subseteq K. \]

We next show that \( s_\lambda' + x_\lambda = (0, \ldots, 0, s_\lambda' + x_\lambda, 0, \ldots, 0) \in (A\sigma)^p \) and hence \( (A\sigma)^p \supseteq \Sigma A/s_\lambda' \). Let \( * (\neq) \) be the polar operation in \( (A\sigma)^p \) (K).

\[ (A\sigma)^p = s_\lambda' + s_\lambda** \oplus s_\lambda' + s_\lambda* = s_\lambda\sigma** \oplus s_\lambda\sigma* \]

\[ x_\lambda\sigma = c + d \]

but this is also the unique decomposition of \( x_\lambda\sigma \) in \( K = s_\lambda' + s_\lambda \neq \oplus s_\lambda' + s_\lambda \neq = A/s_\lambda' \oplus \prod_{\lambda \neq \lambda} A/s_\lambda \).

Thus \( c = s_\lambda' + x_\lambda \in (A\sigma)^p \).
Clearly $K$ is the lateral completion of $\Sigma A/s'_\lambda$ and hence of $(A\sigma)^p$. Thus $K$ is the orthocompletion of $A\sigma$. If $S$ is finite then $K = \Sigma A/s'_\lambda$ and so $(A\sigma)^p = K$.

**COROLLARY I.** Each $s'_\lambda$ is a prime ring ideal if and only if $\text{rad } A = 0$.

**PROOF.** ($\rightarrow$) Each stalk $A/s'_\lambda$ is an integral domain and so $\text{rad } A = 0$.

($\leftarrow$) Suppose that $x, y \in A$, and $xy \in s'_\lambda$, then $|x||y| = |xy| \in s'_\lambda$ and so without loss of generality $0 < x \leq y$ and $xy \in s'_\lambda$. Then by convexity $x^2 \in s'_\lambda$. Suppose (by way of contradiction) that $x \notin s'_\lambda$. Then $0 < a = x \wedge s_\lambda \in (\cap_{\lambda \neq \lambda'} s'_\lambda)$.

**REMARK.** Chambliss [7] has shown that if $A$ is an $f$-ring with $\text{rad } A = 0$ then each minimal prime subgroup of $(A, +)$ is a prime ring ideal.

Let $A$ be an $f$-ring and suppose that $A$ satisfies

(F) each bounded disjoint subset of $A$ is finite.

Then $A$ has a basis $S = \{s_\lambda | \lambda \in \Lambda\}$ and the mapping of $a$ onto $(\cdots, s_\lambda + a, \cdots)$ is a ring $l$-isomorphism of $A$ into $\Sigma A/s'_\lambda$.

**COROLLARY II.** $\Sigma A/s'_\lambda = (A\sigma)^p$.

**PROOF.** Since $A\sigma$ is dense in $H = \Sigma A/s'_\lambda$ we have $A\sigma \subseteq (A\sigma)^p \subseteq H$ and we have shown that $H \subseteq (A\sigma)^p$.

**COROLLARY III.** For an $f$-ring $A$ the following are equivalent.

1) $A = \Sigma A\lambda$, where each $A\lambda$ is a totally ordered ring.
2) $A$ satisfies (F) and is a $P$-group.

**PROOF.** Clearly 1) implies 2). If 2) holds then by Corollary II we have $A \cong \Sigma A/s'_\lambda$.

**COROLLARY IV.** For an $f$-ring $A$ the following are equivalent.

1) $A = \Sigma A\lambda$, where each $A\lambda$ is a totally ordered integral domain.
2) $A$ satisfies (F), $A$ is a $P$-group and $\text{rad } A = 0$.

**PROOF.** Once again it is clear that 1) implies 2). Suppose that 2) is true. By Corollary III, $A \cong \Sigma A/s'_\lambda$ and by Corollary I each stalk $A/s'_\lambda$ is a totally ordered ring.

A convex $l$-subgroup $C$ of an $f$-ring $A$ will be called an $L$-ideal if $C$ is also an ideal of the ring $A$ and a $P$-ideal if $C$ is a ring ideal and $A/C$ is totally ordered. If $0 < s \in A$ is basic, then $s'$ is a $P$-ideal.

**THEOREM 7.2.** For an $f$-ring the following are equivalent.

1) $A = \Sigma A\lambda$, where each $A\lambda$ is an $o$-simple totally ordered integral domain.
2) $A$ satisfies (F), $\text{rad } A = 0$ and the $P$-ideals of $A$ satisfy the DCC.

If this is the case then the $P$-ideals of $A$ are trivially ordered by inclusion.
PROOF. 1 → 2. For λ ∈ Λ let $M_λ = \{a \in A \mid a_λ = 0\}$. We shall show that these are the only $P$-ideals of $A$ and hence the $P$-ideals are trivially ordered. For let $M$ be a $P$-ideal of $A$. If for each $λ ∈ Λ$ there exists $0 < a_λ ∈ M$ with $a_λ > 0$ then it follows that $M = \sum A_λ$ a contradiction. Thus $M ⊆ M_λ$ for some $λ$. Pick $0 < a_λ ∈ A_λ$. Then $a = (0, \ldots, 0, a_λ, 0, \ldots, 0) \notin M$ and since $M$ is a prime subgroup of $(A, +)$ we have $M_λ = a' ⊆ M$. Thus $M = M_λ$.

2 → 1. Let $\{s_λ \mid λ ∈ Λ\}$ be a basis for $A$. Since $A$ satisfies (F) the mapping $σ$ of $a$ upon $(\ldots, s_λ' + a, \ldots)$ is an $l$-isomorphism of $A$ into $\sum A/s_λ'$. $s_λ'$ is a $P$ ideal and hence the $P$-ideals of $A/s_λ'$ satisfy the DCC. Let $σ = I/s_λ'$ be the minimal convex ring ideal of $A/s_λ'$. By Corollary I of Theorem 7.1 we have that $A/s_λ'$ is an integral domain and hence $σ^2 ≠ 0$. Thus by a theorem of Johnson (see [14] p. 132) $A/s_λ'$ is $o$-simple and so $s_λ'$ is a maximal $L$-ideal of $A$. Now $s_λ' ∈ \cap λ ≠ α s_λ' \setminus s_α'$ and hence since $s_α'$ is a maximal $L$-ideal we have

$A = \bigcap λ ≠ α s_λ' + s_α'$.

If $0 < a ∈ A$ then $a = x + t$, where $x ∈ \bigcap λ ≠ α s_λ'$ and $t ∈ s_α'$. Thus $s_α' + x = s_α' + a$ and $s_λ' + x = s_λ'$ for all $λ ≠ α$. Therefore

$xσ = (0, \ldots, 0, s_α' + a, 0, \ldots, 0)$

and so $Aσ = \sum A/s_λ'$.

COROLLARY. (Birkhoff and Pierce [5]). For an $f$-ring $A$ the following are equivalent.

1) $A = \sum_{i=1}^n A_i$, where each $A_i$ is an $o$-simple totally ordered integral domain.

2) The $L$-ideals of $A$ satisfy the DCC and $\text{rad} A = 0$.

3) There are only a finite number of $L$-ideals of $A$ and $\text{rad} A = 0$.

PROOF. 1 → 3. If $T$ is an $L$-ideal then $T = \sum (A_i ∩ T)$ and since each $A_i$ is $o$-simple $A_i ∩ T = A_i$ or 0. Thus there are only a finite number of $L$-ideals.

3 → 2. Trivial.

2 → 1. Let $P_1, P_2, \ldots$ be the minimal prime subgroups of $(A, +)$. Then $P_1 ⊃ P_1 ∩ P_2 ⊃ P_1 ∩ P_2 ∩ P_3 ⊃ \ldots$; for if $a_1 ∈ P_1 \setminus P_3$ and $a_2 ∈ P_2 \setminus P_3$ then $a_1 \land a_2 ∈ (P_1 ∩ P_2) \setminus P_3$. Thus there are only a finite number of $P_i$ and hence $A$ has a finite basis and so satisfies (F).

Commutative laws for the various operators

Throughout this section $y$ will denote $d, v, c$ or $e$, $X$ will denote $P, SP, L$ or $O$ and $W$ will denote $d, v, c, e, P, SP, L$ or $O$. We shall investigate when two of these operators commute.

1) For an archimedean $l$-group $G$, $(G'^y)^e = (G^e)^y = G^e$. 
2) For an archimedean \( l \)-group \( G, (G^W)^d \leq (G^d)^W \). For \( W = v, e, P \) or \( SP \) we have equality, but for \( W = c, L \) or \( O \) there need not be equality.

**Proof.** \( G \) is a large \( l \)-subgroup of \((G^d)^W\) which is divisible. Thus \( G^W \) is large in \((G^d)^W\) and so \((G^W)^d \leq (G^d)^W\). Clearly \((G^v)^d = (G^d)^v = G^v\). If \( 0 < g \in (G^p)^d \) then \( ng \in G^p \) for some \( n > 0 \) and hence \( G^p = (ng)^+ \oplus (ng)^- \). Thus \((G^p)^d = ((ng)^+)d \oplus ((ng)^-)d = (ng)^+ \oplus (ng)^-\), where \( \oplus \) is the polar operation in \((G^p)^d\). Thus \((G^p)^d\) is a \( P \)-group and hence \((G^p)^d = (G^d)^p\).

If \( C \) is a polar in \((G^SP)^d\) then \( C \cap G^SP \) is a polar in \( G^SP \) and so \( G^SP = (C \cap G^SP) \oplus (C \cap G^SP)' \). Thus
\[
(G^SP)^d = (C \cap G^SP)^d \oplus ((C \cap G^SP)^d)' = C \oplus C^*.
\]
Therefore \((G^SP)^d\) is an \( SP \)-group and so \((G^SP)^d = (G^d)^SP\).

If \( G = Z \) then \((G^v)^d = Z^d = Q \subset R = Q^\circ = (G^d)^e\). If \( G = \sum_{i=1}^\infty Z_i \) then \((G^d)^L = (G^d)^O = \prod_{i=1}^\infty Q_i \) and \( G^L = G^O = \prod_{i=1}^\infty Z_i \). Thus \( a = (1, 1/2, 1/3, \ldots) \) belongs to \((G^d)^L \setminus (G^L)^d\) since no multiple of \( a \) belongs to \( G^L \).

From the above computation we have.

3) For an abelian \( l \)-group \( G, (G^X)^d \leq (G^d)^X \). For \( X = P \) or \( SP \) there is equality, but for \( X = L \) or \( O \) there need not be equality.

*For the remainder of this section \( G \) will denote an archimedean \( l \)-group.*

4) \((G^W)^v \leq (G^v)^W\). For \( W = d, e \) or \( SP \) we have equality, but for \( W = c, P, O \) or \( L \) there need not be equality.

**Proof.** \((G^v)^W\) is a vector lattice. This is clear except for \((G^p)^L\), but if \( \{a_\lambda, \lambda \in \Lambda\} \) is a disjoint subset of \( G^v \) and \( 0 < r \in R \) then \( r(\vee a_\lambda) = \vee ra_\lambda \) since \( x \rightarrow rx \) is a \( p \)-endomorphism of \( G^v \) and hence has a unique extension to \((G^p)^L\). Thus it follows that \((G^p)^L\) is also a vector lattice. Now since \( G^W \) is large in the vector lattice \((G^v)^W\) we have \((G^W)^v \leq (G^v)^W\).

Now let \( G = \prod \limits_\Lambda Z_\lambda \), where \( \Lambda \) is an infinite set. Then
\[
G^v = \{r_1g_1 + \cdots + r_tg_t \mid r_i \in R, g_i \in G \text{ and } t > 0\} = T.
\]
For clearly \( T \) is a subspace of \( \prod R_\lambda \) and hence it suffices to prove that
\[
(r_1g_1 + \cdots + r_tg_t) \vee 0 \in T.
\]
Consider the \( \lambda \)-th component
\[
(r_1g_1 + \cdots + r_tg_t)_\lambda = (r_1g_1)_\lambda + \cdots + (r_tg_t)_\lambda.
\]
If this is negative then replace \((g_i)_\lambda \) by 0 in each of the \( g_i \). Do this for each \( \lambda \) and call the new element \( g^\cdash \). Then \((r_1g_1 + \cdots + r_tg_t) \vee 0 = r_1g^\cdash_1 + \cdots + r_tg^\cdash_t \in T \) and hence \((G^v)^e = G^e \subset \prod R_\lambda = (G^e)^e\). Now let \( H = \Sigma Z_\lambda \). Then \( H^L = H^O = \prod Z_\lambda \), \( H^p = \Sigma R_\lambda \) and \((H^p)^L = (H^p)^O = \prod R_\lambda \). Thus
Next let $G$ be the subgroup of $\prod_{i=1}^{n} R_i$ generated by $\Sigma R_i$, $a = (1, 1, \ldots)$ and $b = (\pi + 1/2, \pi - 1/3, \pi + 1/4, \pi - 1/5, \ldots)$. Then $G$ is the direct sum of $\Sigma R_i$ and the cyclic groups generated by $a$ and $b$. It is reasonably easy to check that $G$ is a $P$-group but $G^v$ is not a $P$-group.

Finally we show that $(G^{\text{SP}})^v$ is an $SP$-group and hence $(G^{\text{SP}})^v = (G^v)^{\text{SP}}$. For let $C$ be a polar in $(G^{\text{SP}})^v$. Then $C \cap G^{\text{SP}}$ is a polar in $G^{\text{SP}}$ and hence

$$G^{\text{SP}} = (C \cap G^{\text{SP}}) \oplus (C \cap G^{\text{SP}})^v,$$

and so since the operators $^v$ and $^\vee$ preserve summands we have

$$(G^{\text{SP}})^v = (C \cap G^{\text{SP}})^v \oplus ((C \cap G^{\text{SP}})^v)^v.$$

But $(C \cap G^{\text{SP}})^v = C$ and so $(G^{\text{SP}})^v$ is an $SP$-group. For clearly $(C \cap G^{\text{SP}})^v \subseteq C$ and if $0 < c \in C$ then $c = x + y \in (C \cap G^{\text{SP}})^v \oplus ((C \cap G^{\text{SP}})^v)^v$. Thus $y \in C$ and so if $y \neq 0$ then $ny > g > 0$ for some $g \in G^{\text{SP}}$. Then $g \in C \cap G^{\text{SP}}$ and so $g \wedge y = 0$ a contradiction.

An element $s > 0$ in an $l$-group $H$ is called singular if for each $a \in H$

$$0 \leq a < s \text{ implies } a \wedge (s - a) = 0.$$

The following proposition is essentially due to Iwasawa, see [12] for a proof.

**Proposition.** If $G$ is an archimedean $l$-group then $G^c$ is a vector lattice if and only if $G$ contains no singular elements.

**Corollary.** If $G$ is an archimedean $l$-group with no singular elements then $(G^v)^c = (G^c)^v = G^c$.

5) $(G^X)^c = (G^c)^X = G^c$ for $X = P$ or $SP$.

**Proof.** This follows from the fact that $G^c$ is an $SP$-group (see [14] p. 91 for a proof).

6) $(G^L)^c \subseteq (G^c)^L = (G^\circ)^c = (G^\circ)^c \subseteq G^c$.

**Proof.** Since $G^c$ is a $P$-group it follows from Theorem 2.9 that $(G^\circ)^L = (G^\circ)^\circ$. Now $G^L \subseteq G^\circ \subseteq (G^\circ)^c$ and since $G^L$ is dense in $G^\circ$ we have $(G^\circ)^c \subseteq (G^\circ)^c$. So we need to prove $(G^\circ)^c = (G^\circ)^\circ$.

We first show that $(G^\circ)^c$ is laterally complete and hence $(G^\circ)^c \subseteq (G^\circ)^\circ$. Let $\{a_\lambda \mid \lambda \in \Lambda\}$ be a disjoint subset of $(G^\circ)^c$. Now for each $\lambda \in \Lambda$, $(G^\circ)^c = a_\lambda^{**} \oplus a_\lambda^*$, and since $G^\circ$ is a large $P$-subgroup of $(G^\circ)^c$ we have

$$G^\circ = (a_\lambda^{**} \cap G^\circ) \oplus (a_\lambda^* \cap G^\circ).$$

Now for each $\lambda \in \Lambda$ let $b_\lambda$ be an upper bound for $a_\lambda$ in $G^\circ$. Then without loss of generality $b_\lambda \in a_\lambda^{**} \cap G^\circ$ and hence the $b_\lambda$ are disjoint in $G^\circ$ and so $\vee b_\lambda = \cdots$
exists. Thus $\forall b_\lambda$ is an upper bound for the $a_\lambda$ in $G^O$ and so since $(G^O)^C$ is complete, $\forall a_\lambda$ exists.

We now show that $H = \mathcal{C}(G^C)$ is complete and so $(G^O)^C \subseteq (G^C)^O$. If $C \in P(G^C)$ then $G^C = C \oplus C'$ and so $G/C' \cong C$ is complete. Thus the groups $G^C_\mathfrak{a}$ used in the construction of $\mathcal{C}(G^C)$ are complete. Also the map $\pi_{\mathfrak{a}, x}$ of $G^C_\mathfrak{a}$ into $G^C_\mathfrak{a}$ is onto a large subgroup of $G^C_\mathfrak{a}$ and hence preserves all joins and intersections.

Thus without loss of generality, $H$ is the set join of a directed set of complete $l$-groups $G^C_\mathfrak{a}$ and if $\mathfrak{a} \leq \mathfrak{c}$ then $G^C_\mathfrak{a}$ is a complete $l$-subgroup of $G^C_\mathfrak{c}$. Now let $\{a_\lambda \mid \lambda \in \Lambda\}$ be a subset of $H$ that is bounded from above by $a \in H$. Then $a \in G^C_\mathfrak{a}$ for some partition $\mathfrak{a}$. By Theorem 2.9 each $a_\lambda$ is the join of disjoint elements from $G^C$ and of course each of these elements belongs to the complete $l$ group $G^C_\mathfrak{a}$ and they are bounded by $a$ in $G^C_\mathfrak{a}$. It follows that each $a_\lambda \in G^C_\mathfrak{a}$ and so $\forall a_\lambda \in G^C_\mathfrak{a} \subseteq H$.

7) $(G^O)^C = G^C$ if and only if $G$ contains no singular elements.

Proof. If $G$ contains no singular elements then $G^C$ is a vector lattice. Thus $(G^L)^C = ((G^L)^C)^C = G^C$ (see [10]). If $G^C = (G^O)^C$ then $(G^O)^O$ is a vector lattice and hence contains no singular element. If $0 < g \in G^C$ is singular in $G^C$ and $C \in P(G^C)$ then $C' + g$ is singular in $G^C/C'$ (see [10]). It follows that $\overline{g}$ is singular in $\mathcal{C}(G)$. Thus $G^C$ contains no singular elements and hence is a vector lattice. Thus $G$ contains no singular elements.

Remarks. If $G$ has a basis then in [10] it is shown that $(G^L)^C = (G^C)^L$. whether or not this is always the case is an open question. In Section 2 we showed that $(G^L)^{SP} \subseteq G^O$ and equality need not hold. If $G$ is archimedean then do we have equality? If so then $G^L \subseteq (G^L)^C \rightarrow (G^L)^{SP} \subseteq (G^L)^C \rightarrow (G^O)^C = ((G^L)^{SP})^C \subseteq (G^L)^C$ and hence $(G^L)^C = (G^L)^C$, since by (6) $(G^L)^C \subseteq (G^C)^L \subseteq (G^O)^C$.

References


Department of Mathematics
University of Kansas
Lawrence, Kansas 66044
U. S. A.