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OPENNESS OF FID-LOCI

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Abstract. Let R be a commutative Noetherian ring and M a finite R-module. In this paper, we consider Zariski-openness of the FID-locus of M, namely, the subset of $\operatorname{Spec} R$ consisting of all prime ideals $\mathfrak{p}$ such that $M_{\mathfrak{p}}$ has finite injective dimension as an $R_{\mathfrak{p}}$-module. We prove that the FID-locus of M is an open subset of $\operatorname{Spec} R$ whenever R is excellent.

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1. Introduction. Throughout the present paper, we assume that all rings are commutative and Noetherian.

Let $\mathbb{P}$ be a property of local rings. The $\mathbb{P}$-locus of a ring R is the set of prime ideals $\mathfrak{p}$ of R such that the local ring $R_{\mathfrak{p}}$ satisfies the property $\mathbb{P}$. It is a natural question to ask whether the $\mathbb{P}$-locus of R is an open subset of $\operatorname{Spec} R$ in the Zariski topology, and it has been considered for a long time. For example, it is known that the $\mathbb{P}$-locus of an excellent ring is open if $\mathbb{P}$ is any of the regular property, the complete intersection property, the Gorenstein property, and the Cohen-Macaulay property. As to the details of openness of loci for properties of local rings, see [**3**], [**4**, §6–7], [**6**], [**7**, §24], [**8**], and [**9**].

On the other hand, let $\mathbb{P}$ be a property of modules over a local ring. The $\mathbb{P}$-locus of a module M over a ring R is defined to be the subset of $\operatorname{Spec} R$ consisting of all prime ideals $\mathfrak{p}$ such that the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ satisfies $\mathbb{P}$. The locus of a finite module for the property of finite projective dimension is known to be an open subset [**1**, Corollary 9.4.7], and so is the locus of a finite module for the Gorenstein property if the base ring is acceptable, and therefore if it is excellent [**5**, Corollaries 4.6 and 4.7].

In this paper, we will consider openness of the locus of a finite module for the property of finite injective dimension, which we call the FID-locus. We shall prove that the FID-locus of a finite module satisfying certain conditions is an open subset. Using this result, we will show the following:

THEOREM. *Let R be an excellent ring and M a finite R-module. Then the FID-locus*

$$\mathrm{FID}_R(M) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{id}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} < \infty\}$$

of M is an open subset of $\operatorname{Spec} R$ *in the Zariski topology.*

Of course, this theorem implies the result of Greco and Marinari [**3**, Corollary 1.5] asserting that the Gorenstein locus of an excellent ring is open.

2. The results. Throughout this section, let R be a commutative Noetherian ring. Recall that a subset U of $\operatorname{Spec} R$ is called *stable under generalization* provided that if $\mathfrak{p} \in U$ and $\mathfrak{q} \in \operatorname{Spec} R$ with $\mathfrak{q} \subseteq \mathfrak{p}$ then $\mathfrak{q} \in U$. We begin by stating two lemmas. The former is called the "topological Nagata criterion"; it is a criterion for Zariski-openness which is due to Nagata.

LEMMA 2.1. [**7**, Theorem 24.2] *The following are equivalent for a subset U of* $\operatorname{Spec} R$*:*
(1) *U is an open subset of* $\operatorname{Spec} R$*;*
(2) *U is stable under generalization, and contains a nonempty open subset of $V(\mathfrak{p})$ for any $\mathfrak{p} \in U$.*

LEMMA 2.2. [**3**, Lemma 1.1] *Let $\mathfrak{p}$ be a minimal prime of a finite R-module M. Then there exist an element $f \in R \setminus \mathfrak{p}$ and a chain*

$$0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_n = M_f$$

of R_f-submodules of M_f such that $N_i/N_{i-1} \cong R_f/\mathfrak{p}R_f$ for $1 \le i \le n$.

Next, we study an easy lemma.

LEMMA 2.3. *Let $\mathfrak{p}$ be a prime ideal of R and M a finite R-module. If $M_{\mathfrak{p}} = 0$, then $M_f = 0$ for some $f \in R \setminus \mathfrak{p}$.*

Proof. If $M_{\mathfrak{p}} = 0$, then $\mathfrak{p}$ is not in the support of the R-module M, hence $\mathfrak{p}$ does not contain the annihilator ideal $\operatorname{Ann}_R M$. Therefore there is an element $f \in \operatorname{Ann}_R M \setminus \mathfrak{p}$. We easily obtain $M_f = 0$. □

We define the *FID-locus* of an R-module M to be the set of prime ideals $\mathfrak{p}$ of R such that the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ has finite injective dimension, and denote it by $\operatorname{FID}_R(M)$. Now, we can prove the following proposition, which will play a key role in the proof of our main result.

PROPOSITION 2.4. *Let M be a finite R-module, and let $\mathfrak{p} \in \operatorname{FID}_R(M)$. Suppose that the FID-locus $\operatorname{FID}_{R/\mathfrak{p}}(\operatorname{Ext}^j_R(R/\mathfrak{p}, M))$ contains a nonempty open subset of $\operatorname{Spec} R/\mathfrak{p}$ for each integer j with $0 \le j \le \operatorname{ht}\mathfrak{p}$. Then there exists an element $f \in R \setminus \mathfrak{p}$ such that the FID-locus $\operatorname{FID}_R(M)$ contains $V(\mathfrak{p}) \cap D(f)$.*

Proof. First of all, we note that to prove the proposition we can freely replace our ring R with its localization R_g for an element $g \in R \setminus \mathfrak{p}$. In fact, we have $\mathfrak{p}R_g \in \operatorname{FID}_{R_g}(M_g)$ and $\operatorname{ht}\mathfrak{p}R_g = \operatorname{ht}\mathfrak{p}$. Let U_j be a nonempty open subset of $\operatorname{Spec} R/\mathfrak{p}$ which is contained in $\operatorname{FID}_{R/\mathfrak{p}}(\operatorname{Ext}^j_R(R/\mathfrak{p}, M))$ for $0 \le j \le \operatorname{ht}\mathfrak{p}$. Write $U_j = D(I_j/\mathfrak{p})$ for some ideal I_j of R containing $\mathfrak{p}$, and we see that $D(I_jR_g/\mathfrak{p}R_g)$ is a nonempty open subset of $\operatorname{Spec} R_g/\mathfrak{p}R_g$ which is contained in $\operatorname{FID}_{R_g/\mathfrak{p}R_g}(\operatorname{Ext}^j_{R_g}(R_g/\mathfrak{p}R_g, M_g))$. If there exists an element $\frac{h}{g^n} \in R_g \setminus \mathfrak{p}R_g$ with $h \in R$ and $n \ge 0$ such that $V(\mathfrak{p}R_g) \cap D(\frac{h}{g^n})$ is contained in $\operatorname{FID}_{R_g}(M_g)$, then h is an element of $R \setminus \mathfrak{p}$ and $V(\mathfrak{p}) \cap D(gh)$ is contained in $\operatorname{FID}_R(M)$.

Suppose that $M_{\mathfrak{p}} = 0$. Then we have $M_f = 0$ for some $f \in R \setminus \mathfrak{p}$ by Lemma 2.3. Hence the set $D(f)$ is itself contained in the locus $\operatorname{FID}_R(M)$, and there is nothing more to prove. Therefore in what follows we consider the case where $M_{\mathfrak{p}} \neq 0$. Since $M_{\mathfrak{p}}$ is a finite $R_{\mathfrak{p}}$-module of finite injective dimension, $R_{\mathfrak{p}}$ is a Cohen-Macaulay local ring by virtue of [**2**, Corollary 9.6.2, Remark 9.6.4(a)]. Put $n = \dim R_{\mathfrak{p}}$, and take a sequence $\boldsymbol{x} = x_1, x_2, \ldots, x_n$ of elements in $\mathfrak{p}$ which forms an $R_{\mathfrak{p}}$-regular sequence. Then, putting $H_i = (0 :_{R/(x_1, x_2, \ldots, x_{i-1})} x_i)$, we have $(H_i)_{\mathfrak{p}} = 0$ for $1 \le i \le n$. Hence Lemma 2.3 implies

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that $(H_i)_{f_i} = 0$ for some $f_i \in R \setminus \mathfrak{p}$. Setting $f = f_1 f_2 \cdots f_n$, we see that f is in $R \setminus \mathfrak{p}$ and that $\boldsymbol{x}$ is an R_f-regular sequence. Replacing R with R_f, we may assume that $\boldsymbol{x}$ is an R-regular sequence.

Set $\overline{R} = R/(\boldsymbol{x})$ and $\overline{\mathfrak{p}} = \mathfrak{p}/(\boldsymbol{x})$. Then $\overline{\mathfrak{p}}$ is a minimal prime of $\overline{R}$, hence is an associated prime of $\overline{R}$. Let $\mathfrak{P}_1 = \overline{\mathfrak{p}}, \mathfrak{P}_2, \dots, \mathfrak{P}_s$ be the associated primes of $\overline{R}$. Taking an element of the set $\bigcap_{i=2}^{s} \mathfrak{P}_i \setminus \mathfrak{P}_1$, we easily see that there is an element $f \in R \setminus \mathfrak{p}$ such that $\operatorname{Ass} \overline{R}_{\overline{f}} = \{\overline{\mathfrak{p}}\, \overline{R}_{\overline{f}}\}$, where $\overline{f}$ denotes the residue class of f in $\overline{R}$. Replacing R with R_f, we may assume that $\operatorname{Ass} \overline{R} = \{\overline{\mathfrak{p}}\}$.

On the other hand, since $\operatorname{id}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$, we have $\operatorname{id}_{R_\mathfrak{p}} M_\mathfrak{p} = n$ by [**2**, Theorem 3.1.17] and hence $\operatorname{Ext}_{R_\mathfrak{p}}^{n+1}(\kappa(\mathfrak{p}), M_\mathfrak{p}) = 0$, where $\kappa(\mathfrak{p})$ denotes the residue field of $R_\mathfrak{p}$. Therefore it follows from Lemma 2.3 that $\operatorname{Ext}_{R_f}^{n+1}(R_f/\mathfrak{p}R_f, M_f) = 0$ for some $f \in R \setminus \mathfrak{p}$. Replacing R with R_f, we may assume that

$$\operatorname{Ext}_R^{n+1}(R/\mathfrak{p}, M) = 0. \tag{2.4.1}$$

Here, we establish a claim.

Claim. *One may assume that* $\operatorname{Ext}_R^j(R/\mathfrak{p}, M) = 0$ *for all integers* $j > n$.

Proof of Claim. If $\overline{\mathfrak{p}} = 0$, then $\mathfrak{p} = (\boldsymbol{x})$ and the R-module $R/\mathfrak{p}$ has projective dimension n since $\boldsymbol{x}$ is an R-regular sequence of length n. Hence $\operatorname{Ext}_R^j(R/\mathfrak{p}, M) = 0$ for $j > n$, as desired. Assume $\overline{\mathfrak{p}} \neq 0$. Then we have $\emptyset \neq \operatorname{Min}_{\overline{R}}(\overline{\mathfrak{p}}) \subseteq \operatorname{Ass}_{\overline{R}}(\overline{\mathfrak{p}}) \subseteq \operatorname{Ass} \overline{R} = \{\overline{\mathfrak{p}}\}$, and therefore $\operatorname{Min}_{\overline{R}}(\overline{\mathfrak{p}}) = \{\overline{\mathfrak{p}}\}$. According to Lemma 2.2, for some element $f \in R \setminus \mathfrak{p}$ there is a chain

$$0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_l = \overline{\mathfrak{p}}\overline{R}_{\overline{f}}$$

of $\overline{R}_{\overline{f}}$-modules such that $N_i/N_{i-1} \cong \overline{R}_{\overline{f}}/\overline{\mathfrak{p}}\overline{R}_{\overline{f}} \cong R_f/\mathfrak{p}R_f$ for any $1 \leq i \leq l$. Replacing R with R_f, we may assume that there is a chain $0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_l = \overline{\mathfrak{p}}$ of $\overline{R}$-modules such that each N_i/N_{i-1} is isomorphic to $R/\mathfrak{p}$.

We have obtained a series of exact sequences of R-modules

$$0 \to N_{i-1} \to N_i \to R/\mathfrak{p} \to 0 \quad (1 \leq i \leq l). \tag{2.4.2}$$

Using these sequences and 2.4.1, we can get $\operatorname{Ext}_R^{n+1}(\overline{\mathfrak{p}}, M) = 0$. The natural exact sequence $0 \to \overline{\mathfrak{p}} \to \overline{R} \to R/\mathfrak{p} \to 0$ induces an exact sequence of Ext modules: $0 = \operatorname{Ext}_R^{n+1}(\overline{\mathfrak{p}}, M) \to \operatorname{Ext}_R^{n+2}(R/\mathfrak{p}, M) \to \operatorname{Ext}_R^{n+2}(\overline{R}, M)$. Noting that $\overline{R}$ has projective dimension n as an R-module, we have $\operatorname{Ext}_R^i(\overline{R}, M) = 0$ for every $i > n$, and $\operatorname{Ext}_R^{n+2}(R/\mathfrak{p}, M) = 0$. Using the sequences 2.4.2 again, we get $\operatorname{Ext}_R^{n+2}(\overline{\mathfrak{p}}, M) = 0$. Iterating this procedure shows the claim. □

The assumption of the proposition yields a nonempty open subset U_j of $\operatorname{Spec} R/\mathfrak{p}$ contained in the locus $\operatorname{FID}_{R/\mathfrak{p}}(\operatorname{Ext}_R^j(R/\mathfrak{p}, M))$ for $0 \leq j \leq n$. We can write $U_j = D(I_j/\mathfrak{p})$ for some ideal I_j of R which strictly contains $\mathfrak{p}$. Hence there exists an element $f_j \in I_j \setminus \mathfrak{p}$, and setting $f = f_0 f_1 \cdots f_n$, we see that the set $D(f)$ is contained in $D(I_j)$ for any $0 \leq j \leq n$.

Fix a prime ideal $\mathfrak{q} \in V(\mathfrak{p}) \cap D(f)$. Then $\mathfrak{q}/\mathfrak{p}$ belongs to $D(I_j/\mathfrak{p}) = U_j$, which is contained in $\operatorname{FID}_{R/\mathfrak{p}}(\operatorname{Ext}_R^j(R/\mathfrak{p}, M))$ for $0 \leq j \leq n$. Hence $\operatorname{Ext}_{R_\mathfrak{q}}^j(R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}, M_\mathfrak{q})$ has finite injective dimension as an $R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}$-module for any integer j with $0 \leq j \leq n$. Put $m = \max\{\operatorname{id}_{R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}}(\operatorname{Ext}_{R_\mathfrak{q}}^j(R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}, M_\mathfrak{q})) \mid 0 \leq j \leq n\}$. Consider the following spectral sequence:

$$E_2^{i,j} = \mathrm{Ext}^i_{R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}}(\kappa(\mathfrak{q}), \mathrm{Ext}^j_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}, M_\mathfrak{q})) \Longrightarrow \mathrm{Ext}^{i+j}_{R_\mathfrak{q}}(\kappa(\mathfrak{q}), M_\mathfrak{q}).$$

We have $E_2^{i,j} = 0$ if $i > m$, and the above claim shows that $E_2^{i,j} = 0$ if $j > n$. From this spectral sequence, we see that $\mathrm{Ext}^i_{R_\mathfrak{q}}(\kappa(\mathfrak{q}), M_\mathfrak{q}) = 0$ for $i > m + n$. This implies that the $R_\mathfrak{q}$-module $M_\mathfrak{q}$ has finite injective dimension (cf. [**2**, Proposition 3.1.14]), that is $\mathfrak{q} \in \mathrm{FID}_R(M)$. It follows that $V(\mathfrak{p}) \cap D(f)$ is contained in $\mathrm{FID}_R(M)$, which completes the proof of the proposition. □

Now we state and prove our main result of this paper.

THEOREM 2.5. *Let M be a finite R-module. Suppose that* $\mathrm{FID}_{R/\mathfrak{p}}(\mathrm{Ext}^j_R(R/\mathfrak{p}, M))$ *contains a nonempty open subset of* $\mathrm{Spec}\, R/\mathfrak{p}$ *for any prime ideal* $\mathfrak{p} \in \mathrm{FID}_R(M)$ *and any integer* $0 \le j \le \mathrm{ht}\,\mathfrak{p}$. *Then* $\mathrm{FID}_R(M)$ *is an open subset of* $\mathrm{Spec}\, R$.

Proof. Proposition 2.4 shows that for any $\mathfrak{p} \in \mathrm{FID}_R(M)$ there exists $f \in R \setminus \mathfrak{p}$ such that $\mathrm{FID}_R(M)$ contains $V(\mathfrak{p}) \cap D(f)$. Note that $V(\mathfrak{p}) \cap D(f)$ is not an empty set since $\mathfrak{p}$ belongs to it. On the other hand, it is easy to see from [**2**, Proposition 3.1.9] that $\mathrm{FID}_R(M)$ is stable under generalization. Thus the theorem follows from Lemma 2.1. □

We denote by $\mathrm{Reg}(R)$ the *regular locus* of R, namely, the set of prime ideals $\mathfrak{p}$ of R such that the local ring $R_\mathfrak{p}$ is regular. The following result can be obtained from the above theorem.

COROLLARY 2.6. *Let R be an excellent ring. Then* $\mathrm{FID}_R(M)$ *is an open subset of* $\mathrm{Spec}\, R$ *for any finite R-module M.*

Proof. Fix a prime ideal $\mathfrak{p} \in \mathrm{FID}_R(M)$ and an integer j with $0 \le j \le \mathrm{ht}\,\mathfrak{p}$. By the definition of an excellent ring, the regular locus $\mathrm{Reg}(R/\mathfrak{p})$ is an open subset of $\mathrm{Spec}\, R/\mathfrak{p}$. The zero ideal of $R/\mathfrak{p}$ belongs to $\mathrm{Reg}(R/\mathfrak{p})$, hence it is nonempty. Noting that any module over a regular local ring has finite injective dimension, we see that $\mathrm{Reg}(R/\mathfrak{p})$ is contained in the locus $\mathrm{FID}_{R/\mathfrak{p}}(\mathrm{Ext}^j_R(R/\mathfrak{p}, M))$. Thus all the assumptions of Theorem 2.5 are satisfied, and it follows that $\mathrm{FID}_R(M)$ is open in $\mathrm{Spec}\, R$. □

We denote by $\mathrm{Gor}(R)$ the *Gorenstein locus* of R, that is, the subset of $\mathrm{Spec}\, R$ consisting of all prime ideals $\mathfrak{p}$ of R such that $R_\mathfrak{p}$ is a Gorenstein local ring. Since $\mathrm{Gor}(R)$ coincides with $\mathrm{FID}_R(R)$, the above corollary yields a result of Greco and Marinari [**3**, Corollary 1.5]:

COROLLARY 2.7 (Greco-Marinari). *Let R be an excellent ring. Then the Gorenstein locus* $\mathrm{Gor}(R)$ *is open in* $\mathrm{Spec}\, R$.

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