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ON HOMEOMORPHIC EMBEDDINGS OF $K_{m,n}$ IN THE CUBE

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1. Introduction. Homeomorphic embeddings of K_n in the m-cube were investigated in [**6**]. In particular, it was proved that any homeomorph of K_{n+1} embedded in the m-cube has at least n^2 edges. Furthermore, homeomorphic embeddings of K_{n+1} having exactly n^2 edges are unique up to isomorphism. In this paper a similar problem for the complete bipartite graph is considered.

We adopt the notation and terminology of [**5**].

All graphs considered are without loops and multiple edges.

Let $x = uv$ be an edge of a graph G; x will be called *subdivided* if it is replaced by a vertex w and by edges uw and wv. A graph G' is called a *subdivision* of G if it is obtained from G by a subdivision of an edge of G. A *refinement* $\hat{G}$ of G, is a graph isomorphic to a graph obtained from G by a finite sequence of subdivisions. The vertices of $\hat{G}$ corresponding to vertices of G are called *essential* vertices, whereas the vertices of G which are not essential are called *false* vertices. Two graphs are said to be *homeomorphic* if both can be obtained from the same graph by a sequence of subdivisions of edges. Note that if $m, n > 2$, then the homeomorphs of $K_{m,n}$ are refinements of $K_{m,n}$. A graph G' is defined to be *homeomorphically embeddable*, or simply *embeddable* in a graph G, if there exists a homeomorph of G' which is isomorphic to a subgraph of G.

Let Q^l denote the graph of the l-dimensional cube. Q^l has 2^l vertices, which may be labeled by binary vectors of length l. Two vertices of Q^l are adjacent if their binary representations differ at exactly one coordinate. The infinite graph Q is defined as a graph whose vertices are infinite binary sequences with a finite number of ones, and two vertices are adjacent in Q if their binary representations differ at exactly one place. Clearly, a finite graph G is a subgraph of Q if and only if there exists a finite l such that $G \subset Q^l$.

Since K_{m+n} is embeddable in Q^{m+n-1} [**6**] and $K_{m,n} \subset K_{m+n}$, $K_{m,n}$ is also embeddable in Q^{m+n-1} and therefore in Q.

Denote by $e(G)$ the number of edges of G and for $1 \leqq m, n < \infty$ define

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the function $\bar{e}(m, n)$ as follows:

$$\bar{e}(m, n) = \text{Min}_\Gamma\{e(\Gamma): \Gamma \text{ is a refinement of } K_{m,n},\ \Gamma \subset Q\}.$$

Our aim in this paper is to calculate $\bar{e}(m, n)$ for $1 \leqq m, n < \infty$ and to characterize refinements of $K_{m,n}$ embedded in Q having exactly $\bar{e}(m, n)$ edges (i.e., the minimal embedding of $K_{m,n}$ in Q).

2. Bounds on $\bar{e}(m, n)$. In this section we introduce more notation and derive lower and upper bounds for $\bar{e}(m, n)$.

$K_{m,n}$ has two sets of vertices, which shall be denoted by $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$. Let $\hat{K}_{m,n}$ be a refinement of $K_{m,n}$. Denote by $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_m, \hat{y}_1, \hat{y}_2, \ldots, \hat{y}_n$ the essential vertices of $\hat{K}_{m,n}$ corresponding to $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$, respectively. Let $\hat{p}_{ij}$ be the path in $\hat{K}_{m,n}$ connecting $\hat{x}_i$ and $\hat{y}_j$ and corresponding to the edge x_iy_j in $K_{m,n}$. We denote by $\hat{K}_{m,n} - \hat{x}_i$ the refinement of $K_{m-1,n}$ obtained from $\hat{K}_{m,n}$ by elimination of the vertex $\hat{x}_i$ and all the false vertices on the paths $\hat{p}_{ij}$ $(1 \leqq j \leqq n)$. $\hat{K}_{m,n} - \hat{y}_j$ is defined similarly.

By $\{x\}$ we mean the smallest integer $\geqq x$.

LEMMA 1. *For* $2 \leqq n$ *and* $1 \leqq m$,

$$(1) \qquad \bar{e}(m, n) \geqq \left\{\frac{n}{n-1} \cdot \bar{e}(m, n-1)\right\}.$$

Proof. Let $\hat{K}_{m,n} \subset Q$ be any refinement of $K_{m,n}$. Since $\hat{K}_{m,n} - \hat{y}_j$ is a refinement of $K_{m,n-1}$ $(1 \leqq j \leqq n)$, we have,

$$(2) \qquad e(\hat{K}_{m,n} - \hat{y}_j) \geqq \bar{e}(m, n-1).$$

Therefore

$$(3) \qquad \sum_{j=1}^{n} e(\hat{K}_{m,n} - \hat{y}_j) \geqq n \cdot \bar{e}(m, n-1).$$

On the other hand,

$$(4) \qquad \sum_{j=1}^{n} e(\hat{K}_{m,n} - \hat{y}_j) = (n-1)e(\hat{K}_{m,n}).$$

From (3) and (4)

$$(5) \qquad (n-1)e(\hat{K}_{m,n}) \geqq n \cdot \bar{e}(m, n-1).$$

By choosing $\hat{K}_{m,n}$ with $\bar{e}(m, n)$ edges, we obtain from (5)

$$(n-1)\bar{e}(m, n) \geqq n\bar{e}(m, n-1),$$

from which (1) follows.

Note that from the symmetry of $\bar{e}(m, n)$ we have

$$(6) \qquad \bar{e}(m, n) \geqq \left\{\frac{m}{m-1} \cdot \bar{e}(m-1, n)\right\} \quad \text{for } 2 \leqq m,\ 1 \leqq n.$$

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Since K_n is homeomorphically embeddable in Q^{n-1} and $K_{m,n} \subset K_{m+n}$, the methods of [**6**] can be used to obtain homeomorphic embeddings of $K_{m,n}$ in Q^{m+n-1} and consequently achieve an upper bound on $\bar{e}(m, n)$.

THEOREM 1. *There exists a subgraph* Γ *of* Q^{m+n-1} *which is a refinement of* $K_{m,n}$ *and*

$$e(\Gamma) = 2mn - \max(m, n).$$

Proof. Let $v_0 = (0, 0, \ldots, 0)$ and $v_i = (a_1, a_2, \ldots, a_{m+n-1})$, where $a_i = 1$, $a_j = 0\ \forall j \neq i$ and define v_{ij} as the vector sum of v_i and v_j. Assume $n \geqq m$ and construct Γ as follows. The vertices of Γ are $v_0, v_1, \ldots, v_{m+n-1}$, whereas the edges are v_0v_j $(m \leqq j < m + n)$ and v_iv_{ij}, $v_{ij}v_j$ $(m \leqq j < m + n,\ 1 \leqq i < m)$. Γ is obviously a refinement of $K_{m,n}$ and has $2mn - n$ edges.

If we denote $e^*(m, n) = 2mn - \max(m, n)$, then by Theorem 1,

$$(7) \qquad \bar{e}(m, n) \leqq e^*(m, n).$$

It will be shown that except for a finite number of cases, equality holds in (7).

A subgraph Γ of Q is defined to be *standard* if there exists an automorphism of Q transforming Γ to the graph described in Theorem 1. For a refinement $\hat{K}_{m,n}$ of $K_{m,n}$ we define a matrix $H_{m,n} = (h_{ij})$ $(1 \leqq i \leqq m$, $1 \leqq j \leqq n)$ where h_{ij} is the number of false vertices of $\hat{K}_{m,n}$ on the path $\hat{p}_{ij}$ (connecting $\hat{x}_i$ with $\hat{y}_j$ in $\hat{K}_{m,n}$). $H_{m,n}$ is called the *refinement matrix* of $\hat{K}_{m,n}$, and it characterizes $\hat{K}_{m,n}$ up to isomorphism.

Note that after an appropriate arrangement of the vertices of a standard refinement of $K_{m,n}$ $(n \geqq m)$ the corresponding refinement matrix will have the form,

$$H_{m,n} = \begin{bmatrix} 1 & 1 \ldots 1 \\ 1 & 1 \ldots 1 \\ \cdot & \cdot \quad\ \ \cdot \\ \cdot & \cdot \quad\ \ \cdot \\ \cdot & \cdot \quad\ \ \cdot \\ 1 & 1 \quad\ \ 1 \\ 0 & 0 \ldots 0 \end{bmatrix}.$$

On the other hand, it is clear that a matrix $H_{m,n}$ of this form represents a standard refinement of $K_{m,n}$. Let

$$r_i = \sum_{j=1}^{n} h_{ij} \quad 1 \leqq i \leqq m,$$

$$c_i = \sum_{i=1}^{m} h_{ij} \quad 1 \leqq j \leqq n$$

and

$$m_{ij} = h_{ij} + h_{i+1,j} + h_{i,j+1} + h_{i+1,j+1}, \quad 1 \leqq i < m, \quad 1 \leqq j < n.$$

m_{ij} is the number of false vertices on a cycle of $\hat{K}_{m,n}$. Since all cycles of Q are even, we have

LEMMA 2. *If $\hat{K}_{m,n}$ is a refinement of $K_{m,n}$ and $\hat{K}_{m,n} \subset Q$, then*

$$m_{ij} \equiv 0 (\text{mod } 2), 1 \leqq i < m, 1 \leqq j < n.$$

3. The minimal embeddings. Denote by $P(m, n)$ the following statement: If Γ is any subgraph of Q which is a refinement of $K_{m,n}$ then $e(\Gamma) \geqq e^*(m, n)$. In view of (7) if $P(m, n)$ then $\bar{e}(m, n) = e^*(m, n)$.

Denote by $P^*(m, n)$ the statement: $P(m, n)$ and if $e(\Gamma) = e^*(m, n)$, then Γ is standard. Obviously, $P(m, n) \leftrightarrow P(n, m)$ and the same holds for $P^*(m, n)$.

We shall prove $P(m, n)$ for $1 \leqq m \leqq n < \infty$ except for the pairs (2, 2), (2, 3) and (3, 3), where $P(m, n)$ is not true. $P^*(m, n)$ will be proved for $1 \leqq m \leqq n < \infty$ except for the pairs (2, 2), (2, 3), (2, 4), (3, 3), (3, 4) and (4, 4), where $P^*(m, n)$ does not hold. (Clearly, $P^*(1, n)$). The exceptional cases were investigated and the results will be stated without proofs.

THEOREM 2. *If $n \geqq m$, then*

$$P(m, n) \rightarrow P(m, n + 1).$$

Proof. Assume $P(m, n)$, i.e., $\bar{e}(m, n) = (2m - 1)n$. By Lemma 1,

$$\bar{e}(m, n + 1) \geqq \frac{n+1}{n} \bar{e}(m, n) = (2m - 1)(n + 1),$$

which proves $P(m, n + 1)$.

THEOREM 3. *If $n \geqq m > 2$, then*

$$P^*(m, n) \rightarrow P^*(m, n + 1).$$

Proof. We assume $P^*(m, n)$ and therefore $P(m, n)$. By Theorem 2 $P(m, n + 1)$ follows. Let $\hat{K}_{m,n+1}$ be any refinement of $K_{m,n+1}$ such that $K_{m,n+1} \subset Q$ and $e(\hat{K}_{m,n+1}) = \bar{e}(m, n + 1)$. Using (4) and (5), the graph $\hat{K}_{m,n+1} - \hat{y}_j$ must be minimal for $1 \leqq j \leqq n + 1$. By the assumption $P^*(m, n)$, $\hat{K}_{m,n+1} - \hat{y}_j$ is standard for $1 \leqq j \leqq n + 1$. In particular, $\hat{K}_{m,n+1} - \hat{y}_{n+1}$ is standard. Let $H_{m,n+1}$ be the refinement matrix of $\hat{K}_{m,n+1}$ and assume $n > m$. Since $\hat{K}_{m,n+1} - \hat{y}_{n+1}$ is standard, $H_{m,n+1}$ has

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the following form:

$$H_{m,n+1} = \begin{bmatrix} 1 & 1 \ldots 1 & h_{1,n+1} \\ 1 & 1 \ldots 1 & h_{2,n+1} \\ . & . \quad . & \\ . & . \quad . & \\ . & . \quad . & \\ 1 & 1 \ldots 1 & \\ 0 & 0 \quad 0 & h_{m,n+1} \end{bmatrix}$$

The elimination of the j-th column $(1 \leqq j \leqq n)$ from $H_{m,n+1}$ results in a refinement matrix of $\hat{K}_{m,n+1} - \hat{y}_j$ which is also standard; therefore

$$h_{i,n+1} = 0, 1 \text{ for } 1 \leqq i \leqq m.$$

From the minimality of $\hat{K}_{m,n+1} - \hat{y}_{n+1}$,

(8) $\quad c_{n+1} = (2m - 1)(n + 1) - m(n + 1) - n(m - 1) = m - 1.$

Furthermore, from Lemma 2,

$$h_{i,n+1} \equiv h_{j,n+1} (\text{mod } 2) \text{ and } h_{m,n+1} \not\equiv h_{i,n+1} (\text{mod } 2),\ 1 \leqq i, j < m.$$

Hence the $n + 1$-th column of $H_{m,n+1}$ is either $(0, 0, \ldots, 0, 1)^T$ or $(1, 1, \ldots, 1, 0)^T$. The first possibility contradicts (8) $(m > 2)$; the second possibility proves that $\hat{K}_{m,n+1}$ is standard. The case $n = m$ is treated similarly.

LEMMA 3. *If* $n \geqq 5$ *then*

$$P^*(n - 1, n) \rightarrow P(n, n).$$

Proof. By Lemma 1 and $P^*(n - 1, n)$,

(9) $$\bar{e}(n, n) \geqq \left\{\frac{n}{n-1} \cdot \bar{e}(n, n-1)\right\} = \left\{\frac{n}{n-1} \cdot (2n(n-1) - n)\right\}$$
$$= \left\{2n^2 - n - 1 - \frac{1}{n-1}\right\} = 2n^2 - n - 1.$$

By (7) and (9),

(10) $\quad 2n^2 - n \geqq \bar{e}(n, n) \geqq 2n^2 - n - 1.$

Assume $\bar{e}(n, n) = 2n^2 - n - 1$. Then there must exist a graph $\hat{K}_{n,n} \subset Q$, such that $\hat{K}_{n,n}$ is a refinement of $K_{n,n}$ and has $n^2 - n - 1$ false vertices.

If $H_{n,n}$ denotes the refinement matrix of $\hat{K}_{n,n}$ then,

$$\sum_{i=1}^{n} r_i = \sum_{j=1}^{n} c_j = n^2 - n - 1.$$

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Observe that

(11) $c_j, r_i \leqq n - 1, 1 \leqq i, j \leqq n.$

Otherwise there would exist a graph $\hat{K}_{n-1,n} \subset Q$ such that

$$e(\hat{K}_{n-1,n}) \leqq 2n^2 - n - 1 - 2n < e^*(n-1, n),$$

contradicting $P^*(n-1, n)$. On the other hand, there must be a natural number k $(1 \leqq k \leqq n)$, such that $c_k = n - 1$. Otherwise,

$$\sum_{j=1}^{n} c_j \leqq n(n-2) < n^2 - n - 1.$$

Assume without loss of generality

(12) $c_n = n - 1.$

Let $H_{n,n-1}$ be the matrix obtained from $H_{n,n}$ by omitting the n-th column. For the graph $\hat{K}_{n,n} - \hat{y}_n$ whose refinement matrix is $H_{n,n-1}$ we have

$$e(\hat{K}_{n,n} - \hat{y}_n) = 2n^2 - n - 1 - (2n-1) = e^*(n-1, n).$$

By $P^*(n-1, n)$, $\hat{K}_{n,n} - \hat{y}_n$ must be standard and hence we may assume for $H_{n,n}$ that $h_{i1} = 0$ $(1 \leqq i \leqq n)$, $h_{ij} = 1$ $(1 \leqq i \leqq n, 2 \leqq j \leqq n-1)$.

From (11) $h_{in} \leqq 1, 1 \leqq i \leqq n$. From (12) we may assume without loss of generality $h_{1n} = 0$ and consequently $h_{in} = 1$, $1 < i \leqq n$. But then $m_{1,n-1} \equiv 1 (\mathrm{mod}\ 2)$, contradicting Lemma 2.

Therefore from (10), $\bar{e}(n, n) = 2n^2 - n$, which proves the lemma.

THEOREM 4. *If* $n \geqq 5$, *then*

$$P^*(n-1, n-1) \to P^*(n, n).$$

Proof. $P^*(n-1, n-1) \to P^*(n-1, n)$ by Theorem 2 and $P^*(n-1, n) \to P(n, n)$ by Lemma 3. Let $\hat{K}_{n,n} \subset Q$ be any refinement of $K_{n,n}$ such that

(13) $e(\hat{K}_{n,n}) = 2n^2 - n.$

From (13) we have,

(14) $\sum_{i=1}^{n} r_i = \sum_{j=1}^{n} c_j = n^2 - n.$

Similarly to the proof of (11), the following can be shown:

(15) $c_k, r_k \leqq n, 1 \leqq k \leqq n.$

Now we show that there must exist an integer k $(1 \leqq k \leqq n)$, such that $r_k = n$ or $c_k = n$. For assume $r_k \leqq n - 1$ and $c_k \leqq n - 1$ $(1 \leqq k \leqq n)$. Then by (14),

(16) $r_k = c_k = n - 1, 1 \leqq k \leqq n.$

Since

$$\sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij} = n^2 - n,$$

there must exist integers i and j $(1 \leqq i, j \leqq n)$, such that $h_{ij} = 0$.

Without loss of generality we may assume $h_{n,n} = 0$.

Let $H_{n-1,n-1}$ be the matrix obtained from $H_{n,n}$ by omitting the nth row and nth column. $H_{n-1,n-1}$ is a refinement matrix of a graph $\hat{K}_{n-1,n-1}$, which is a refinement of $K_{n-1,n-1}$ and is obtained from $\hat{K}_{n,n}$ by omitting $\hat{x}_n$ and $\hat{y}_n$ and all the false vertices on the paths corresponding to the edges incident with x_n and y_n in $K_{n,n}$. Thus $\hat{K}_{n-1,n-1}$ has exactly $n^2 - n - 2(n-1)$ false vertices. Consequently,

$$e(\hat{K}_{n-1,n-1}) = (n-1)^2 + n^2 - n - 2(n-1) = e^*(n-1, n-1).$$

By $P^*(n-1, n-1)$, $\hat{K}_{n-1,n-1}$ is standard. We may therefore assume that $H_{n,n}$ has the form,

$$H_{n,n} = \begin{bmatrix} 0 & 1 & \dots & 1 & h_{1n} \\ 0 & 1 & \dots & 1 & h_{2n} \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & \cdot & \\ 0 & 1 & \dots & 1 & h_{n-1,n} \\ h_{n1} & h_{n2} & & h_{n,n-1} & 1 \end{bmatrix}$$

From (16),

$$\begin{aligned} & h_{n2} = h_{n3} = \dots = h_{n,n-1} = 0 \\ (17) \quad & h_{n1} = n - 1 \\ & h_{1n} = h_{2n} = \dots = h_{n-1,n} = 1 \end{aligned}$$

$\hat{y}_2$, $\hat{y}_3$, $\hat{y}_4$ are all adjacent to $\hat{y}_1$ in Q, since $\hat{K}_{n-1,n-1}$ is standard. From (17) $\hat{y}_2$, $\hat{y}_3$, $\hat{y}_4$ are also adjacent to $\hat{x}_n$. Thus, $\hat{K}_{n,n} \supset K_{2,3}$. However $K_{2,3}$ is not a subgraph of Q (see Proposition 1). This completes the proof that there exists an integer k $(1 \leqq k \leqq n)$, such that

$$r_k = n \text{ or } c_k = n.$$

Without loss of generality assume $c_n = n$ and let $H_{n,n-1}$ be the matrix obtained from $H_{n,n}$ by the elimination of the nth column. From (14),

$$(18) \quad \sum_{i=1}^{n} \sum_{j=1}^{n-1} h_{ij} = n^2 - 2n.$$

Therefore, if $\hat{K}_{n,n} - \hat{y}_n$ is the refinement of $K_{n,n-1}$, represented by $H_{n,n-1}$, then

$$(19) \quad e(\hat{K}_{n,n} - \hat{y}_n) = n^2 - 2n + n(n-1) = e^*(n, n-1).$$

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By Theorem 3 and (19), $\hat{K}_{n,n} - \hat{y}_n$ must be standard. Therefore, for $H_{n,n}$,

$$h_{11} = h_{21} = \ldots = h_{n1} = 0$$

and

$$h_{ij} = 1 \quad 1 \leqq i \leqq n, 1 < j < n.$$

But then $c_{n-1} = n$ and as before, $\hat{K}_{n,n} - \hat{y}_{n-1}$ is standard, which indicates $h_{in} \leqq 1$, $1 \leqq i \leqq n$. Since $c_n = n$ we have $h_{in} = 1$, $1 \leqq i \leqq n$. Therefore $\hat{K}_{n,n}$ is standard and $P^*(n, n)$ is proved.

From Theorems 3 and 4 we conclude the following.

COROLLARY 1. *If there exist $i, j \leqq 5$ such that $P^*(i, j)$ then*

$P^*(m, n)$ for $m \geqq i$, $n \geqq j$.

We now list some special cases. The proofs of the statements follow from similar methods used in the previous arguments. (Clearly, $\bar{e}(2, 2) = 4$).

PROPOSITION 1. (a) $\bar{e}(2, 3) = 8$.
(b) *Let Γ be any refinement of $K_{2,3}$ such that $\Gamma \subset Q$ and $e(\Gamma) = 8$. Then Γ is unique (up to automorphism of Q).*

PROPOSITION 2. (a) *$P(2, n)$ for $n \geqq 4$.*
(b) *There are exactly two isomorphism types of subgraphs of Q, having exactly $e^*(2, 4)$ edges, which are refinements of $K_{2,4}$.*
(c) *$P^*(2, n)$ for $n \geqq 5$.*

Note that, in a standard refinement of $K_{2,n}$, essential vertices of degree two may be exchanged by false vertices.

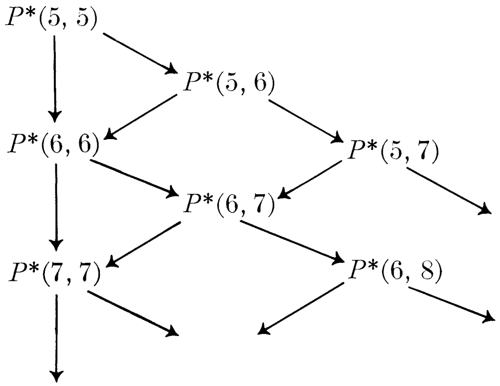
PROPOSITION 3. (a) $\bar{e}(3, 3) = 14$.
(b) *Let Γ be any refinement of $K_{3,3}$ such that $\Gamma \subset Q$ and $e(\Gamma) = 14$. Then Γ is unique (up to automorphism of Q).*

PROPOSITION 4. (a) *$P(3, n)$ for $n \geqq 4$.*
(b) *There are exactly four isomorphism types of subgraphs of Q, having $e^*(3, 4)$ edges, which are refinements of $K_{3,4}$.*
(c) *$P^*(3, n)$ for $n \geqq 5$.*

PROPOSITION 5. (a) *$P(4, n)$ for $n \geqq 4$.*
(b) *There are exactly three isomorphism types of subgraphs of Q, having $e^*(4, 4)$ edges, which are refinements of $K_{4,4}$.*
(c) *$P^*(4, n)$ for $n \geqq 5$.*

PROPOSITION 6. (a) $P^*(5, 5)$.

From Corollary 1 and Proposition 6, we get the following scheme for proving $P^*(m, n)$ for $m, n \geqq 5$.



Thus,

THEOREM 5. $P^*(m, n)$ *for* $m, n \geqq 5$.

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