Proceedings of the Edinburgh Mathematical Society (2018) **61**, 401–412
doi:10.1017/S0013091517000232

SOME REMARKS ON THE WIMAN–EDGE PENCIL

ALEXIS G. ZAMORA
U. A. Matemáticas, U.de Zacatecas, Camino a la Bufa y Calzada Solidaridad, C.P. 98000, Zacatecas, Zac. Mexico (alexiszamora06@gmail.com)

(Received 20 September 2016)

Abstract We rewrite in modern language a classical construction by W. E. Edge showing a pencil of sextic nodal curves admitting A_5 as its group of automorphism. Next, we discuss some other aspects of this pencil, such as the associated fibration and its connection to the singularities of the moduli of six-dimensional abelian varieties.

Keywords: group actions; fibred surfaces; abelian varieties

2010 *Mathematics subject classification:* Primary 14H40; 14H37; 14D06; 14H50

1. Introduction

In 1895 [**14**], A. Wiman exhibited the equation of a genus 6 nodal plane sextic W admitting an S_5 action. The equation is:

$$2\sum_{i,j,l}(x_i^4x_jx_l + x_j^3x_l^3) - 2\sum_{i\neq j}(x_i^4x_j^2) + \sum(x_i^3x_j^2x_l) - 6x_0^2x_1^2x_2^2 = 0.$$

In 1981 [**6**], this sextic attracted the interest of W. L. Edge, who studied not only W itself, but also a pencil of nodal sextics admitting an action of A_5 and having as one of its members Wiman's sextic. Moreover, he found a more symmetric equation for W in a different system of coordinates:

$$W : x^6 + y^6 + z^6 + (x^2 + y^2 + z^2)(x^4 + y^4 + z^4) - 12x^2y^2z^2 = 0.$$

Later on, González-Aguilera and Rodríguez [**8**] rediscovered many of Edge's results and intended to construct, from the above-mentioned pencil, a semistable pencil with only three singular members.

Wiman's sextic and the associated Wiman–Edge pencil are also discussed in several other texts. In [**3**, **5**, **11**], its relationship with the automorphism group of the degree 5 Del Pezzo surface (i.e. the rational surface obtained as the blowing-up of four points in general position in $\mathbb{P}^2$) is explained. This relationship was also noted by Edge. In fact, the rational action on $\mathbb{P}^2$ described at the beginning of the next section can be regularized by passing to the induced action on the Del Pezzo surface. In these references, the regular

© 2018 The Edinburgh Mathematical Society

https://doi.org/10.1017/S0013091517000232 Published online by Cambridge University Press

action of A_5 (which is the complete group of automorphism of the Del Pezzo surface) is studied in detail, its character is computed, and it is shown how we can obtain from this explicit equations for Wiman's sextic. In [2] it is proved that suitable $125:1$ covers of the curves in the Wiman–Edge pencil parametrize families of lines in the Dwork pencil, the pencil of quintic threefolds given by:

$$\sum_{i=0}^{4} x_i^5 - 5tx_0x_1x_2x_3x_4 = 0.$$

In [15], another pencil of sextics invariant under A_5 is studied, of which Wiman's sextic is also a member (see also [5, Remark 9.5.4]).

The aim of this note is to rewrite Edge's results in more modern terms and style, and to restate more accurately, in the light of fibration theory, the statement of Theorem 1 in [8]. We also include an application of Edge's pencil to the study of the singular locus of $\mathcal{A}_6$ (moduli of principally polarized six-dimensional abelian varieties).

In contrast to [6, 11] (another attempt to present Wiman's sextic in a modern way), our discussion about Edge's pencil avoids the use of cross ratios and is entirely based on analytical projective geometry (i.e. linear algebra).

Little originality can be claimed for this paper. Our only contribution is rewriting in a (hopefully) more modern language the original and beautiful geometric construction by Edge, and showing how this pencil provides interesting examples in different problems of algebraic geometry, besides its own importance for the theory of curves admitting large automorphism groups.

2. The Wiman–Edge pencil

Fix four points e_i, $i = 1, \ldots, 4$ in $\mathbb{P}^2(\mathbb{C})$ in general position (we can assume once and for all that they are the standard frame of reference). Consider the subgroup of birational automorphisms of $\mathbb{P}^2$ generated by linear automorphisms fixing one of the e_i and quadratic transformations Q_i fixing one e_i, and having a fundamental triangle determined by the remaining points. This group turns out to be the symmetric group S_5. Indeed, to this configuration we can associate five pencils of curves: the pencils α_i, $i = 1, \ldots, 4$ of lines passing through e_i and the pencil α_5 of conics having $\{e_i\}_{i=1,\ldots,4}$ as base locus. Then our group acts as the complete set of permutations of this set of five elements.

In this way, if we denote by L_i the linear transformation fixing e_i and permuting cyclically the other three points, then L_i represents the 3-cycle (lkm), $l, k, m \neq i$, and Q_i represents the transposition $(i5)$.

Each linear automorphism L_i determines two distinguished directions (and thus two lines m_i and m_i' through e_i) given by the eigenvectors of L_i associated with eigenvalues different from 1. If X is a curve having a node at e_i and invariant under L_i, then, because of the invariance, its tangent lines at e_i must be precisely m_i and m_i'.

Lemma 2.1. *Two reducible curves, C and C', exist, invariant under the above-described action of A_5. Both curves are the product of an irreducible conic and four lines.*

https://doi.org/10.1017/S0013091517000232 Published online by Cambridge University Press

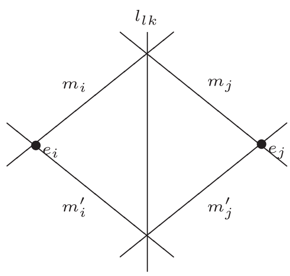
Proof. Consider the only conic Ω in α_5 having tangent m_4 at e_4. A simple computation, using explicit coordinates and associated matrices, shows that its tangents at e_i, $i = 1, 2, 3$ are some of the distinguished lines that will be accordingly denoted by m_1, m_2 and m_3. Then $C := \Omega \prod_{i=1}^{4} m'_i$ are A_5-invariant as can be checked by computing the action of 3-cycles. We can construct similarly $C' = \Omega' \prod_{i=1}^{4} m_i$ starting from the unique conic Ω' in α_5 having tangent m'_4 at e_4. □

Definition 2.2. The pencil $\mathcal{E} : \lambda C + \lambda' C'$ is called the Wiman–Edge Pencil.

Thus, the general element of $\mathcal{E}$ is a plane sextic having nodes at e_i with fixed tangents m_i, m'_i. In particular, the nodes at e_i are part of the base locus of $\mathcal{E}$. The remaining 12 base points, being determined just like the intersections of C with C', clearly correspond to the intersections of m_i with m'_j. In order to determine other reducible members of $\mathcal{E}$, denote by l_{ij} the line joining e_i and e_j. Then we have the following lemma.

Lemma 2.3. *The product* $\Pi := \prod_{i,j} l_{ij}$ *is a member of* $\mathcal{E}$.

Proof. We need to prove that Π contains the base locus of $\mathcal{E}$. For this, it would be sufficient to check that the lines m_i and m'_j intersect in a point on l_{lk}, with $\{i, j\}$ complementary to $\{l, k\}$ in $\{1, 2, 3, 4\}$, and that the same holds for m'_i and m'_j.



Indeed, if we fix coordinates such that $e_1 = (1 : 0 : 0)$, $e_2 = (0 : 1 : 0)$, $e_3 = (0 : 0 : 1)$ and $e_4 = (1 : 1 : 1)$, then the matrix representing L_4 is:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The lines m_4, m'_4 are given by the lines joining e_4 and the eigenvectors of L_4 not corresponding to the eigenvalue 1, and are in accordance expressed as:

$$m_4 : x + \omega y + \omega^2 z = 0 \qquad m'_4 : x + \omega^2 y + \omega z = 0.$$

A similar computation shows that

$$m_1 : \omega y + z = 0 \quad \text{and} \quad m'_1 : \omega^2 y + z = 0.$$

Thus, the intersections of m_4 and m'_1 and m'_4 and m_1 occur in points on the line:

$$l_{23} : x = 0.$$ □

https://doi.org/10.1017/S0013091517000232 Published online by Cambridge University Press

Note that Π is invariant not only under the action of A_5 (as C and C' are), but in fact under the action of S_5. This can be proved readily by considering the action of transpositions.

Now, we are in a position of proving the following.

Proposition 2.4. (i) *All the curves in $\mathcal{E}$ are invariant under the A_5 action.*

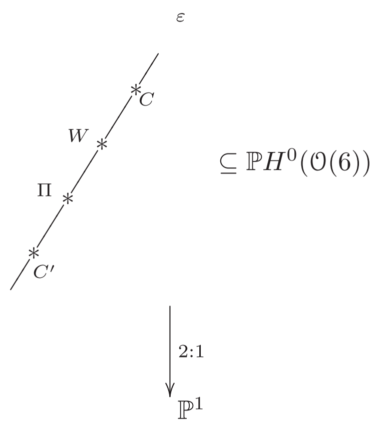
(ii) *$\langle\phi\rangle = \mathbb{Z}_2 \simeq S_5/A_5$ acts on $\mathcal{E}$ interchanging C and C'.*

(iii) *There exists another sextic W in $\mathcal{E}$ that is invariant under the action of S_5.*

Proof. (i) The actions being linear on C and C', it follows from the fact that these two curves generate the pencil.

(ii) This is just an explicit computation of the action of (12).

(iii) Consider the line determined by $\mathcal{E}$ in $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(6)))$. On this line $\langle\phi\rangle$ acts interchanging C and C'. The quotient by the action determines a $2:1$ cover of $\mathbb{P}^1$:



Thus, we must have two ramification points, one of them corresponding to Π, and a second one that is the desired sextic W. □

Definition 2.5. The sextic W, invariant under the action of S_5, is the Wiman sextic.

So far, we have determined the existence of three reducible members of $\mathcal{E}$. Probably more subtle is the existence of a pair of irreducible rational curves R, R' in $\mathcal{E}$ having 10 nodes (see also [**8**, Lemma 2] and [**4**, Theorem 6.2.9]).

Proposition 2.6. *Two elements R, R' exist in $\mathcal{E}$ admitting at least* 10 *nodes.*

Proof. Denote by p the intersection point of l_{23} and l_{14} and by q the intersection of l_{13} and l_{24}. The line L determined by p and q is invariant under the involution $\sigma := (14)(23)$, and the points p and q are fixed by σ.

https://doi.org/10.1017/S0013091517000232 Published online by Cambridge University Press

Intersections with elements of $\mathcal{E}$ give a $6:1$ covering of $\mathbb{P}^1$, which factorizes through the quotient by σ:

$$\begin{array}{ccc} \gamma : L & \xrightarrow{6:1} & \mathbb{P}^1 \\ \downarrow & \overset{3:1}{\nearrow_{\tau}} & \\ \mathbb{P}^1 & & \end{array}$$

The ramification points of γ correspond to elements of $\mathcal{E}$ having non-transversal intersection with L.

Given a point $x \in L$, let E be the element of $\mathcal{E}$ intersecting L at x. If E is non-singular at x, then the tangent line T_xE is sent under σ to $T_{\sigma(x)}E$. Thus, σ induces a rational involution (undefined if x is a singular point of E) on the set of tangent lines

$$\{T_xE \mid E \in \mathcal{E}, x \in L\}.$$

This determines an invariant subspace for the action of σ on $(\mathbb{P}^2)^\vee$. Computing the eigenspaces of σ, this subspace is readily identified with the set of lines passing through the point r given by the intersection of l_{12} and l_{34}. In other words, if T_xE is well defined, it must be a line passing through r.

We conclude that all the ramification points of γ are given by intersections of L with singular points of elements of $\mathcal{E}$. This set of singular points is invariant under the action of σ, and therefore they occur as pairs of singular points of the same member of $\mathcal{E}$ (except for the fixed points p and q). Moreover, the same construction can be done interchanging the roles of p, q and r. Therefore, and owing to the invariance under A_4, if a curve $E \in \mathcal{E}$ has singular points on L, it must have six singular points apart from the four nodes corresponding to the base locus of $\mathcal{E}$.

We can now determine the ramification points of γ: C, C' and Π contribute with six simple ramification points. There remain four ramification points to be counted with multiplicity. By the previous discussion, they must correspond to the intersection with L of a pair R, R' of curves in $\mathcal{E}$ having each two singular points of multiplicity two on L, and thus six singular points aside from the four on the base locus of $\mathcal{E}$. Using the invariance under A_4, we can see that R and R' are irreducible and the singular points are actually nodes. □

In the next section, it will be shown that C, C', Π, R and R' are the only singular members of $\mathcal{E}$.

Note that the rational action described in this section gives rise to the total automorphism group of the Del Pezzo degree 5 surface obtained by blowing up the points e_i, $i = 1, \dots, 4$. Almost all the results we have explained here are proved by a different method in [**3**, Theorem 6.2.9]. See also [**5**, § 8.5.4] and [**8**]. We have tried here to translate Edge's original arguments into the language of linear projective algebra.

3. The associated fibration

We can construct a fibration $f : X \to \mathbb{P}^1$ obtained from $\mathcal{E}$ by the standard procedure of resolving the base locus of $\mathcal{E}$. This will be a genus 6 fibration in the sense that its general fibre will be a non-singular genus 6 curve.

https://doi.org/10.1017/S0013091517000232 Published online by Cambridge University Press

For this, we need to perform a blowing-up centred at each of the 12 simple points of the base locus, the situation being more complicated for the points e_i. In fact, as the general curve in $\mathcal{E}$ has nodes at e_i with fixed tangents m_i, m_i', after blowing up e_i we obtain an induced pencil $\tilde{\mathcal{E}}$ with two base points, one in each point of the exceptional divisor corresponding to the directions m_i and m_i'.

Thus, we need to perform a new blow-up centred at each of these points. In this way, X is obtained after a total of $12 + 3 \times 4 = 24$ blow-ups.

We conclude that X is a rational surface with

$$K_X^2 = 9 - 24 = -15 \qquad e(X) = 3 + 24 = 27.$$

Now, being f a genus 6 fibration, the total number δ of nodes on the fibres of f must satisfy:

$$e(X) + 4(g-1) = 27 + 20 = 47 \geq \delta.$$

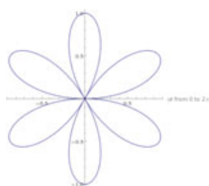
Denoting by $\tilde{F}$ the proper transform of a sextic F in $\mathcal{E}$, the bound 47 is achieved by the following.

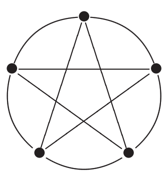
- $\tilde{C}$ contributes with 10 nodes. The same is valid for $\tilde{C}'$. The total contribution of these two fibres is 20.
- $\tilde{R}$ and $\tilde{R}'$ contribute with at least six nodes each, giving a total of at least 12.
- $\tilde{\Pi}$ contributes with 15 nodes.

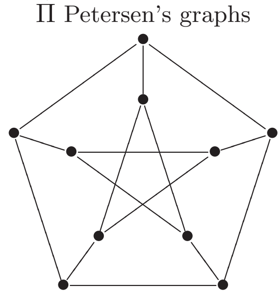
We conclude that these nodes are all the singular points in the fibres of f. Thus, f is a semistable fibration and has exactly five singular fibres.

The dual graphs of the singular fibres are the following:

$R, \tilde{R}'$ the six petal flower $\quad$ $\tilde{C}, \tilde{C}'$ The Pentacle or $K5$ $\quad$ $\tilde{\Pi}$ Petersen's graphs







So far, the conclusions are as follows.

Theorem 3.1 ([6, 8]). *Let $\mathcal{E}$ be the Wiman–Edge pencil and $f : X \to \mathbb{P}^1$ its associated fibration. Then f is a semistable fibration with exactly five singular fibres. All the irreducible components of the singular fibres are rational curves, and their associated dual graphs can be described as: two six-petal flowers (corresponding to irreducible rational curves), two pentacles (corresponding to the union of five (-4)-rational curves) and one Petersen's graph (corresponding to the union of 10 (-3)-rational curves).*

https://doi.org/10.1017/S0013091517000232 Published online by Cambridge University Press

Theorem 3.1 is important in the context of the following problem. Let $f : X \to \mathbb{P}^1$ be a semistable non-trivial fibration; what is the minimal number of singular fibres? If s denotes the number of singular fibres of f, then we have the following theorem.

Theorem 3.2. *Let $f : X \to \mathbb{P}^1$ be a semistable non-trivial fibred surface, and assume the general fibre of f is a genus g curve. Denote by s the number of singular fibres of f. Then we have the following*:

(i) $s \geq 4$ ([**1**]),

(ii) *if $g \geq 2$, then $s \geq 5$* ([**12**]).

Thus, the Wiman–Edge fibration gives (another) example of a fibration with the minimal possible number of singular fibres (see [**12**]). For other results and examples related to this problem, see [**13**].

The example provided by the pencil $\mathcal{E}$ seems to corroborate that having a minimal possible s could be related to the fact that the singular fibres are the union of rational curves.

The title of the paper [**8**] suggests the existence of a semistable fibration with only three singular fibres. This is, in fact, the assertion of Theorem 1.

The argument used in that paper is the following: using the notation in § 1, consider the involution (12); this involution acts on the curves in $\mathcal{E}$, leaving fixed W and Π and interchanging C and C' and R and R'. Thus, making 'the quotient' for the action of (12) on the set of curves in the pencil, we obtain a pencil with only three singular members: the classes of C, R and Π. The problem here is that if we want to consider the alluded quotient as a subvariety of the moduli of semistable 6 genus curves, it must be constructed from an action on an algebraic variety, and not only on the set of fibres of the pencil.

Thus, in order to construct this quotient, we need to consider the automorphism σ induced in X by (12), make the global quotient Y of X by this automorphism and consider the resulting fibration $\bar{f}$:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \simeq X/\sigma \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

The image of $\tilde{W}$ under π is therefore the quotient $W' := \tilde{W}/\sigma$. Now, the ramifications of

$$\pi : \tilde{W} \to W'$$

are induced by the intersections of W with the line $L_{12} : x = y$. These give in principle a total of six ramification points, but L_{12} contains the points e_1 and e_4 that transform in $\tilde{W}$ to non-fixed points of σ. In other words, the involution (12) interchanges the tangent lines m_1 and m_1' and m_4 and m_4', as simple computations show. In this way, $\pi : \tilde{W} \to W'$ has only two ramification points and Riemann–Hurwitz gives $g_{W'} = 3$. As $\bar{f}$ is a genus 6 fibration, we conclude that W' appears with multiplicity 2 as a fibre of $\bar{f}$.

The conclusion is as follows.

https://doi.org/10.1017/S0013091517000232 Published online by Cambridge University Press

Proposition 3.3. *Let σ be a the automorphism of X induced by (12). The quotient $Y \simeq X/\sigma$ is a non-singular surface, and the associated fibration $\bar{f} : Y \to \mathbb{P}^1$ is non-semistable and has exactly four singular fibres, one of them (corresponding to the quotient of the Wiman sextic W), is a 2-multiple of a non-singular genus 3 curve.*

That Y is non-singular follows from the fact that σ is a pseudo-reflection (it is indeed induced by a reflection!) [4]. In [7] a classification of non-semistable pencils admitting only three singular fibres can be found.

4. Singular locus of $\mathcal{A}_6$

Let $\mathcal{A}_g$ be the moduli space of g-dimensional principally polarized abelian varieties (PPAV). It is well known that the singular locus of this variety corresponds to abelian varieties with non-trivial automorphisms. In a series of papers [9, 10], the following result was proved.

Theorem 4.1. *Irreducible algebraic subvarieties $\mathcal{A}_g(p, \alpha) \subset \mathcal{A}_g$ exist, parametrizing abelian varieties X admitting an order p automorphism α with a fixed-type action α_Λ on the lattice Λ defining X.*

Much more information can be given about the structure of these subvarieties. For instance, if

$$d\alpha : T_0X \to T_0X$$

denotes the analytical representation of the automorphism α, then in suitable basis

$$d\alpha = diag(I_{n_0}, \xi I_{n_1}, \ldots, \xi^{p-1} I_{n_{p-1}}),$$

and

$$\dim \mathcal{A}_g(p, \alpha) = \frac{n_0(n_0 + 1)}{2} + \sum_{i=1}^{(p-1)/2} n_i n_{p-i}.$$

Returning to the pencil $\mathcal{E}$, we have naturally associated the pencil $J\mathcal{E}$ of PPAV consisting of the Jacobians of elements of $\mathcal{E}$ (it is better to say the closure of the Jacobian locus of general elements in $\mathcal{E}$). For a general curve $C \in \mathcal{E}$, the total group of automorphisms of C is $Aut(C) = A_5$, which coincides with $Aut_{\pm}(JC)$, the quotient of $Aut(JC)$ by $\{\pm 1\}$.

Consider the irreducible varieties $\mathcal{A}_6(3, \alpha)$ and $\mathcal{A}_6(5, \beta)$ associated with the actions of the 3-cycle $\alpha = (123)$ and the 5-cycle $\beta = (14532)$, respectively. The analytical action $d\alpha$ on an element of $J\mathcal{E}$, and thus on every element of $\mathcal{A}_6(3, \alpha)$, can be computed as the action of α on the tangent space of a general $JC \in J\mathcal{E}$, and analogously for β. On the other hand, this tangent space is identified with $H^0(C, \omega_C)$. According to the classical theory of adjoint linear systems, $H^0(C, \omega_C)$ is given by the space of plane cubics passing

https://doi.org/10.1017/S0013091517000232 Published online by Cambridge University Press

through the nodes e_i. A basis for this space is given by:

$$\{B_{ij} : x_o x_1 x_2 - x_i^2 x_j = 0\}_{i<j, 0\leq i,j\leq 2}.$$

An explicit computation for this action gives that the diagonal form is:

$$d\alpha = diag(I_2, \omega I_2, \omega^2 I_2),$$

with ω a primitive 3-root of 1. From this follows that:

$$\dim \mathcal{A}_6(3,\alpha) = 7.$$

Analogously we obtain:

$$d\beta = diag(I_2, \xi, \xi^2, \xi^3, \xi^4),$$

with ξ a primitive 5-root of unity and from this

$$\dim \mathcal{A}_6(5,\beta) = 5.$$

Theorem 4.2. *Assume the previous notation. Then:*

(i) $\mathcal{A}_6(3,\alpha)$ *and* $\mathcal{A}_6(5,\beta)$ *are irreducible components of* $Sing\mathcal{A}_6$*,*

(ii) $\mathcal{A}_6(3,\alpha)$ *and* $\mathcal{A}_6(5,\beta)$ *intersect exactly on the Edge–Wiman locus* $J\mathcal{E}$*,*

(iii) *the general element of* $\mathcal{A}_6(5,\beta)$ *is not a Jacobian variety.*

Proof. The proofs of the first two assertions are computations on the local complete algebras $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ pro-representing the local deformation functor associated with the automorphisms α and β (for general facts and notation see [**9**, **10**]).

Let $[X] \in \mathcal{A}_6(3,\alpha)$ (respectively $\mathcal{A}_6(5,\beta)$) be a general element. Fix in T_0X the basis induced by the cubics B_{ij}. Let T_α and T_β be the matrices determined by the relations:

$$d\alpha T d\alpha^t = T \text{ and } d\beta T d\beta^t = T$$

in $\mathcal{O} = \mathbb{C}[[t_{ij}]]$. Then:

$$T_\alpha = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{14} & t_{16} \\ t_{12} & t_{22} & t_{23} & t_{16} & t_{13} & t_{23} \\ t_{13} & t_{23} & t_{22} & t_{12} & t_{16} & t_{23} \\ t_{14} & t_{16} & t_{12} & t_{11} & t_{14} & t_{13} \\ t_{14} & t_{13} & t_{16} & t_{14} & t_{11} & t_{12} \\ t_{16} & t_{23} & t_{23} & t_{13} & t_{12} & t_{22} \end{pmatrix}$$

and

$$T_\beta = \begin{pmatrix} t_{11} & t_{11} & t_{11} & t_{14} & t_{14} & t_{16} \\ t_{11} & -t_{11}-t_{25} & -t_{11}-t_{14} & 0 & t_{25} & -(t_{14}+t_{16}+t_{25}) \\ t_{11} & -t_{11}-t_{14} & -t_{11}-t_{25} & t_{25} & 0 & -(t_{14}+t_{16}+t_{25}) \\ t_{14} & 0 & t_{25} & -2t_{25} & t_{14} & t_{25} \\ t_{14} & t_{25} & 0 & t_{14} & -2t_{25} & t_{25} \\ t_{16} & -(t_{14}+t_{16}+t_{25}) & -(t_{14}+t_{16}+t_{25}) & t_{25} & t_{25} & t_{66} \end{pmatrix}.$$

https://doi.org/10.1017/S0013091517000232 Published online by Cambridge University Press

First of all, note that the normalizer of α in A_5 is the subgroup of order 6, isomorphic to S_3 and generated by α and $\tau := (23)(45)$.

Now, denote by $G = Aut_+(X)$ the automorphism group of X modulo $\{\pm 1\}$ for $X \in \mathcal{A}_g(3, \alpha)$ a general element. It follows from [10], Lemma 2.2, that $|G| = 2^k 3^l$, but given $G \leq A_5$ the only possibilities are $|G| = 12$, 6 or 3. We want to prove that $|G|$ is in fact equal to 3.

Assume $|G| = 6$. In this case, $\langle\alpha\rangle$ is the only order 3 subgroup of G and $\langle\alpha\rangle \lhd G$. Therefore

$$\langle\alpha\rangle = N_G(\langle\alpha\rangle) = N_{A_5}(G) = \langle\alpha, \tau\rangle,$$

τ denoting the order 2 element (23)(45).

In this case, we must have an inclusion $\mathcal{A}_6(3, \alpha) \subseteq \mathcal{A}_6(2, \tau)$.

On the other hand, τ is represented, in the fixed basis, by the matrix

$$d\tau = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $\tau \in G$, then the identity $d\tau T_\alpha d\tau^t = T_\alpha$ must hold in $\mathcal{O}_\alpha$. But a simple computation shows that this is not the case, the equation given by the relation being $t_{12} - t_{13} = 0$. The geometric interpretation of this fact is that $\mathcal{A}_6(2, \tau)$ intersects $\mathcal{A}_6(3, \alpha)$ along a co-dimension 1 subvariety.

We must check now that $|G| = 12$ is also impossible. Note first that the number s_3 of 3-Sylow subgroups of G must be 4. Indeed, $s_3 = 1$ must imply that

$$12 = |N_G(\langle\alpha\rangle)| \mid |N_{A_5}(G)| = 6.$$

In this way, G admits a unique subgroup of order 4, and we have $G \simeq A_4$, with $A_4 \leq A_5$ realized as the subgroup fixing either 4 or 5. Thus, we must have either $(14)(32) \in G$ or $(15)(23) \in G$. A new simple computation in the local algebra deformation shows that both cases are impossible. For instance, assume $\sigma = (15)(34) \in G$. Then:

$$\mathcal{A}_6(3, \alpha) \subseteq \mathcal{A}_6(2, \sigma).$$

We have, always in the fixed basis:

$$d\sigma = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}.$$

A computation shows that the entry $(1, 1)$ of $d\sigma T_\alpha d\sigma^t$ is t_{22}; this is different from t_{11}, which turns out to be the $(1, 1)$ entry of T_α. We conclude that

$$d\sigma T_\alpha d\sigma^t \neq T_\alpha,$$

https://doi.org/10.1017/S0013091517000232 Published online by Cambridge University Press

and the assumed inclusion is impossible. The case $\sigma = (14)(32) \in G$ can be discharged by a similar calculation, the matrix of $d\sigma$ being:

$$d\sigma = \begin{pmatrix} 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

We now proceed under similar lines in order to prove that if $G = Aut_+(X)$ is the automorphism group (modulo $\{\pm 1\}$) of a general $[X] \in \mathcal{A}_6(5,\beta)$ then, G is cyclic of order 5. In this case, the order of G must be a divisor of $|A_5| = 60$ and must be divisible by 5. Moreover, by (ii) (whose proof is independent of these arguments), $G \neq A_5$. The only possibility then is $|G| = 10$ and G isomorphic to the dihedral group D_{10}. In this case, we must have $\sigma = (14)(23) \in G$, and once again the computation of the relation

$$d\sigma T_\beta d\sigma = T_\beta$$

implies that the inclusion $\mathcal{A}_6(5,\beta) \subseteq \mathcal{A}_6(2,\sigma)$ is impossible.

For (ii), in order to compute the local equations of the intersection $\mathcal{A}_6(3,\alpha) \cap \mathcal{A}_6(5,\beta)$, we must find the ideal generated by

$$d\alpha T_\beta d\alpha^t = T_\beta$$

in $\mathcal{O}_\beta$. The matrix representing $d\alpha$ is:

$$d\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, after a simple computation we obtain that the ideal of the intersection is $I_{\alpha\beta} =< t_{ij} >_{i,j\neq 1,4}$. Therefore, the dimension of the intersection is 1, and given that the Wiman–Edge locus $J\mathcal{E}$ is contained in this intersection we conclude part (ii).

The proof of (iii) is a computation on dimensions of moduli spaces. We have previously observed that $\dim \mathcal{A}_6(5,\beta) = 5$. Assume that $[X] \in \mathcal{A}_6(5,\beta)$ represents a Jacobian variety, say $X = JC$, and consider the quotient of C by the action of β:

$$\pi : C \to C_0.$$

Then the genus of C_0 coincides with the multiplicity of 1 as an eigenvalue in the induced action on $H^0(C,\omega_C)$. Thus, by the choice of β, $g_{C_0} = 2$, the Riemmann–Hurwitz formula implies that π is unramified.

Thus, 6-genus curves admitting an $\mathbb{Z}_5$-action equivalent to β and contained in $\mathcal{A}_6(5,\beta)$ are parameterized by pairs (Y,η) with Y a genus 2 curve and η a 5-torsion point in JY. Therefore, they have moduli $3 \times 2 - 3 = 3$. □

https://doi.org/10.1017/S0013091517000232 Published online by Cambridge University Press

Acknowledgements. This paper was partially supported by CONACyT Grant CB-257079.

The paper is a further elaboration of notes I wrote in preparing a talk for the Conference 'Celebration of the Chilean Complex Geometry Group, a homage to V. Gónzalez-Aguilera, R. Rodríguez and G. Riera'. I want to dedicate this article to this group of Chilean geometers as a testimony to their friendship and long-term collaborations.

I am indebted to Alejandra Huitrado-Mora for assisting me in producing the figures, particularly the dual graphs in § 3, and making explicit computations for § 4. An anonymous referee has contributed with many suggestion to the improvement of the paper, in particular, providing valuable references.

References

1. A. Beauville, Le nombre minimum de fibers singulieres d'un courbe stable sur $\mathbb{P}^1$, *Asterisque* **86** (1981), 97–108.
2. P. Candelas, X. De La Ossa, B. Van Geemen and D. Van Straten, Lines on the Dwork pencil of quintic threefolds, *Adv. Theor. Math. Phys.* **16** (2012), 1779–1836.
3. I. Cheltsov and C. Shramov, *Cremona groups and the icosahedron*, Monographs and Research Notes in Mathematics (CRC Press, Boca Raton, 2016).
4. C. Chevalley, Invariants of finite groups generated by reflections, *Amer. J. Math.* **77**(4) (1955), 778–782.
5. I. Dolgachev, *Classical algebraic geometry: a modern view* (Cambridge University Press, 2012).
6. W. L. Edge, A pencil of four-nodal plane sextics, *Math. Proc. Cambridge Philos. Soc.* **89** (1981), 413–421.
7. G. Gong, X. Lu and S.-L. Tan, Families of curves over $\mathbb{P}^1$ with 3 singular fibres, *C. R. Math. Acad. Sci. Paris* **351**(9–10) (2013), 375–380.
8. V. González-Aguilera and R. E. Rodríguez, A pencil in $\bar{\mathcal{M}}_6$ with three points at the boundary, *Geom. Dedicata* **42** (1992), 255–265.
9. V. González-Aguilera, Muñoz-Porras and A. G. Zamora, On the irreducible components of the singular locus of A_g, *J. Algebra* **240** (2001), 230–250.
10. V. González-Aguilera, Muñoz-Porras and A. G. Zamora, On the irreducible components of the singular locus of A_g II, *Proc. Amer. Math. Soc.* **140**(2) (2012), 479–492.
11. N. Inoue and F. Kato, On the geometry of Wiman's sextic, *J. Math. Kyoto Univ.* **45**(4) (2005), 743–757.
12. S.-L. Tan, The minimal number of singular fibers of a semi stable curve over $\mathbb{P}^1$, *J. Algebraic Geom.* **4** (1995), 591–596.
13. S.-L. Tan, Y. Tu and A. G. Zamora, On complex surfaces with 5 or 6 semistable singular fibers over $\mathbb{P}^1$, *Math. Z.* **249** (2005), 427–438.
14. A. Wiman, Zur Theorie del endlichen Gruppen von birationalen Transformationen in der Ebene, *Math. Ann.* **48** (1895), 195–240.
15. R. Winger, On the invariants of the ternary icosahedral group, *Math. Ann.* **93** (1925), 210–216.

https://doi.org/10.1017/S0013091517000232 Published online by Cambridge University Press