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WEIGHTED NORM INEQUALITIES FOR FRACTIONAL INTEGRAL OPERATORS WITH ROUGH KERNEL

YONG DING AND SHANZHEN LU

ABSTRACT. Given function Ω on $\mathbb{R}^n$, we define the fractional maximal operator and the fractional integral operator by

$$M_{\Omega,\alpha} f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|<r} |\Omega(y)|\,|f(x-y)|\,dy$$

and

$$T_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^{n-\alpha}} f(x-y)\,dy$$

respectively, where $0 < \alpha < n$. In this paper we study the weighted norm inequalities of $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$ for appropriate α, s and $A(p,q)$ weights in the case that $\Omega \in L^s(S^{n-1})(s > 1)$, homogeneous of degree zero.

1**. Introduction.** Suppose that $0 \le \alpha < n$, Ω is homogeneous of degree zero, and $\Omega \in L^s(S^{n-1})$,where S^{n-1} denotes the sphere of $\mathbb{R}^n$ and $s > 1$. Then we will consider the fractional maximal operator $M_{\Omega,\alpha}$ defined by

$$M_{\Omega,\alpha} f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|<r} |\Omega(y) f(x-y)|\,dy$$

and the fractional integral operator defined by

$$T_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^{n-\alpha}} f(x-y)\,dy.$$

When $\alpha = 0$, we denote $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$ by M_Ω and T_α respectively, where the integration is taken by the Cauchy principal value.

It is well known that Kurtz and Wheeden [KW] had proven certain weighted norm inequalities for T_Ω under the assumption that $\Omega \in L^s(S^{n-1})$ and Ω satisfies an $L^s(S^{n-1})$-Dini smoothness condition. Using Fourier transform methods, Watson [W] and Duoandikoetexea [Du] showed that the smoothness requirement in [KW] was in fact unnecessary. However, the corresponding results for fractional maximal and singular integral operators have not been proven even for smooth Ω. This paper aims to establish weighted norm inequalities for $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ with $0 < \alpha < n$ and $\Omega \in L^s(S^{n-1})$. To do this, we require some techniques related to weights from a (p,p) setting to a (p,q) setting.

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A locally integrable nonnegative function ω on $\mathbb{R}^n$ is said to belong to $A(p,q)(1 < p,q < \infty)$ if there exists C such that

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)^q\,dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q \omega(x)^{-p'}\,dx\right)^{1/p'} \le C < \infty, \tag{1.1}$$

where $p' = p/(p-1)$, Q denotes a cube in $\mathbb{R}^n$ with its sides parallel to the coordinate axes and the supremum is taken over all cubes. In 1971, Muckenhoupt and Wheeden [MW1] studied the weighted norm inequalities for $T_{\Omega,\alpha}$ with the weight $\omega(x) = |x|^\beta$. Recently, weak type inequalities with power weights for $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ have been obtained by one of the authors of this paper [D]. Moreover, Muckenhoupt and Wheeden [MW2] gave the following weighted results for $M_{1,\alpha}$ and $T_{1,\alpha}$ with $\Omega \equiv 1$.

THEOREM A. *If $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $\omega(x) \in A(p,q)$, then there is a constant C, independent of f, such that*

$$\left(\int_{\mathbb{R}^n}\left[M_{1,\alpha}f(x)\omega(x)\right]^q dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n}|f(x)\omega(x)|^p\,dx\right)^{1/p}$$

and

$$\left(\int_{\mathbb{R}^n}|T_{1,\alpha}f(x)\omega(x)|^q\,dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n}|f(x)\omega(x)|^p\,dx\right)^{1/p}.$$

On the other hand, Duoandikoetxea [Du] obtained the weighted norm inequalities for M_Ω and T_Ω with the weight $\omega(x) \in A_p$. Moreover, as usual, A_p denotes the Muckenhoupt's class.

In this paper we shall study the weighted norm inequalities for $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$ with more general weights, that is, we will look for some appropriate indices p,q,α,s such that for $\omega(x) \in A(p,q)$, $\Omega \in L^s(S^{n-1})$,

$$\left(\int_{\mathbb{R}^n}\left[T_{\Omega,\alpha}f(x)\omega(x)\right]^q dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n}|f(x)\omega(x)|^p\,dx\right)^{1/p} \tag{1.2}$$

holds,where C is independent of f. The same conclusion is true for $M_{\Omega,\alpha}$. Precisely, we obtain the following

THEOREM 1. *Let $0 < \alpha < n$, $s' < p < n/\alpha$, and $1/q = 1/p - \alpha/n$. If $\Omega \in L^s(S^{n-1})$ and $\omega(x)^{s'} \in A(p/s', q/s')$, then there is a constant C, independent of f, such that*

$$\left(\int_{\mathbb{R}^n}\left[T_{\Omega,\alpha}f(x)\omega(x)\right]^q dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n}|f(x)\omega(x)|^p\,dx\right)^{1/p}.$$

THEOREM 2. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $s > q$. If $\Omega \in L^s(S^{n-1})$ and $\omega(x)^{-s'} \in A(q'/s', p'/s')$, then there is a constant C, independent of f, such that*

$$\left(\int_{\mathbb{R}^n}\left[T_{\Omega,\alpha}f(x)\omega(x)\right]^q dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n}|f(x)\omega(x)|^p\,dx\right)^{1/p}.$$

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THEOREM 3. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, and $1/q = 1/p - \alpha/n$. If Ω is homogeneous of degree zero, $\Omega \in L^s(S^{n-1})$ for some $s > 1$ with $\alpha/n + 1/s < 1/p < 1/s'$, and there exists an r, $1 < r < s/(\frac{n}{\alpha})'$, such that $\omega^{r'} \in A(p,q)$, then there is a constant C, independent of f, such that*

$$\left(\int_{\mathbb{R}^n} \left[T_{\Omega,\alpha} f(x)\omega(x)\right]^q dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} |f(x)\omega(x)|^p dx\right)^{1/p}.$$

To prove Theorem 1, let us first set up the following Proposition 1.

PROPOSITION 1. *Let $0 < \alpha < n$, $s' < p < n/\alpha$, and $1/q = 1/p - \alpha/n$. If $\Omega \in L^s(S^{n-1})$ and $\omega(x)^{s'} \in A(p/s', q/s')$, then there is a constant C, independent of f, such that*

$$\left(\int_{\mathbb{R}^n} |M_{\Omega,\alpha} f(x)\omega(x)|^q dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} |f(x)\omega(x)|^p dx\right)^{1/p}.$$

As a corollary of Theorem 2, we have the following

PROPOSITION 2. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $s > q$. If $\Omega \in L^s(S^{n-1})$ and $\omega(x)^{-s'} \in A(q'/s', p'/s')$, then there is a constant C, independent of f, such that*

$$\left(\int_{\mathbb{R}^n} |M_{\Omega,\alpha} f(x)\omega(x)|^q dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} |f(x)\omega(x)|^p dx\right)^{1/p}.$$

As a direct corollary of Theorem 3, we also have

PROPOSITION 3. *Under the assumption of Theorem 3, $M_{\Omega,\alpha}$ is also a bounded operator from $L^p(\omega^p)$ to $L^q(\omega^q)$.*

2. **Some properties of $A(p,q)$ weights and proof of Proposition 1.** Some elementary properties of $A(p,q)$ weights will be first given in this section. Then we shall give the proof of Proposition 1. Let us recall the elementary properties of A_p weight. A locally integrable nonnegative function ν on $\mathbb{R}^n$ is said to belong to $A_p (1 < p < \infty)$ if there exists C such that

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \nu(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \nu(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C < \infty, \tag{2.1}$$

where Q denotes a cube in $\mathbb{R}^n$ with the sides parallel to the coordinate axes and the supremum is taken over all cubes. When $p = 1$, a nonnegative measurable function ν is said to belong to A_1, if there exists C such that for any cube Q,

$$\frac{1}{|Q|}\int_Q \nu(y)\,dy \le C\nu(x), \quad \text{a.e. } x \in Q.$$

THE ELEMENTARY PROPERTIES OF A_p (SEE [GR]).

(2.2) $A_{p_1} \subset A_{p_2}$ if $1 < p_1 \le p_2 < \infty$.

(2.3) $\nu(x) \in A_p$ if and only if $\nu(x)^{1-p'} \in A_{p'}$.

(2.4) If $\nu(x) \in A_p$,then there exists an $\varepsilon > 0$ such that $p - \varepsilon > 1$ and $\nu(x) \in A_{p-\varepsilon}$.

(2.5) If $\nu(x) \in A_p$,then there exists an $\varepsilon > 0$ such that $\nu(x)^{1+\varepsilon} \in A_p$.

(2.6) $\nu \in A_p (1 < p < \infty)$ if and only if there exist $u(x), v(x) \in A_1$ such that $\nu(x) = u(x) \cdot v(x)^{1-p}$.

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THE ELEMENTARY PROPERTIES OF $A(p,q)$. Suppose that $0<\alpha<n$, $1<p<n/\alpha$, and $1/q=1/p-\alpha/n$. Then

$$\begin{aligned}\omega(x)\in A(p,q) &\iff \omega(x)^q\in A_{q(n-\alpha)/n}\\ &\iff \omega(x)^q\in A_{1+q/p'}\\ &\iff \omega(x)^{-p'}\in A_{1+p'/q}\end{aligned}\tag{2.7}$$

$$\omega(x)\in A(p,q)\Longrightarrow \omega(x)^q\in A_q \quad\text{and}\quad \omega(x)^p\in A_p.\tag{2.8}$$

PROOF. (2.7) can be deduced from the definitions of A_p and $A(p,q)$. Let us now prove (2.8) by (2.2), (2.3) and (2.7). Since $q(n-\alpha)/n<q$, we have $\omega(x)^q\in A_{q(n-\alpha)/n}\subset A_q$. From $1/q=1/p-\alpha/n$, it follows that $1/q<1/p=(p'-1)/p'$, *i.e.* $1+p'/q<p'$. Using (2.7) and (2.2), we have $\omega(x)^{-p'}\in A_{1+p'/q}\subset A_{p'}$. However, this is equivalent to $\omega(x)^p\in A_p$ by (2.3).

We shall give the proof of Proposition 1 in the following. The proof is based on the following observation.

LEMMA 1. *If $0<\alpha<n$, $s'>1$, $1<p/s'<n/\alpha$, $1/(q/s')=1/(p/s')-\alpha/n$, and $\omega(x)^{s'}\in A(p/s',q/s')$, then*

$$\left(\int_{\mathbb{R}^n}\left[N_{\alpha,s'}f(x)\omega(x)\right]^q dx\right)^{1/q}\le C\left(\int_{\mathbb{R}^n}|f(x)\omega(x)|^p dx\right)^{1/p},\tag{2.9}$$

where $N_{\alpha,s'}$ is the fractional maximal operator of order s' defined by

$$N_{\alpha,s'}f(x)=\sup_{r>0}\left(\frac{1}{r^{n-\alpha}}\int_{|y|<r}|f(x-y)|^{s'}dy\right)^{1/s'}.$$

PROOF. Since $N_{\alpha,s'}f(x)=\left(M_{1,\alpha}(|f|^{s'})(x)\right)^{1/s'}$, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\left[N_{\alpha,s'}f(x)\omega(x)\right]^q dx\right)^{1/q} &= \left(\int_{\mathbb{R}^n}\left[M_{1,\alpha}(|f|^{s'})(x)\right]^{q/s'}\omega(x)^q dx\right)^{1/q}\\ &= \left\{\left(\int_{\mathbb{R}^n}\left[M_{1,\alpha}(|f|^{s'})(x)\nu(x)\right]^{q/s'}dx\right)^{s'/q}\right\}^{1/s'},\end{aligned}$$

where $\nu(x)=\omega(x)^{s'}$ and $\nu(x)\in A(p/s',q/s')$. By Theorem A, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\left[M_{1,\alpha}(|f|^{s'})(x)\nu(x)\right]^{q/s'}dx\right)^{s'/q} &\le C\left(\int_{\mathbb{R}^n}\left[|f(x)|^{s'}\nu(x)\right]^{p/s'}dx\right)^{s'/p}\\ &= C\left(\int_{\mathbb{R}^n}|f(x)|^p\nu(x)^{p/s'}dx\right)^{s'/p}.\end{aligned}$$

Thus,

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\left[N_{\alpha,s'}f(x)\omega(x)\right]^q dx\right)^{1/q} &\le C\left(\int_{\mathbb{R}^n}|f(x)|^p\nu(x)^{p/s'}dx\right)^{1/p}\\ &= C\left(\int_{\mathbb{R}^n}|f(x)\omega(x)|^p dx\right)^{1/p}.\end{aligned}$$

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This proves (2.9).

Let us now give the proof of Proposition 1. By $s > 1$, $\Omega(x') \in L^s(S^{n-1})$, we have

$$\begin{aligned} M_{\Omega,\alpha} f(x) &= \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|<r} |\Omega(y) f(x-y)|\, dy \\ &\le \sup_{r>0} \frac{1}{r^{n-\alpha}} \left(\int_{|y|<r} |\Omega(y)|^s\, dy \right)^{1/s} \left(\int_{|y|<r} |f(x-y)|^{s'}\, dy \right)^{1/s'}. \end{aligned}$$

Since $\left(\int_{|y|<r} |\Omega(y)|^s\, dy\right)^{1/s} \le C r^{n/s} \|\Omega\|_s$, where $\|\Omega\|_s = \left(\int_{S^{n-1}} |\Omega(y')|^s\, d\sigma(y')\right)^{1/s}$, then we have

$$\begin{aligned} M_{\Omega,\alpha} f(x) &\le C \sup_{r>0} \frac{1}{r^{n-\alpha}} \cdot r^{n/s} \left(\int_{|y|<r} |f(x-y)|^{s'}\, dy \right)^{1/s'}. \\ &= C \sup_{r>0} \left(\frac{1}{r^{n-\alpha s'}} \int_{|y|<r} |f(x-y)|^{s'}\, dy \right)^{1/s'} \\ &= C \cdot N_{\alpha s', s'} f(x). \end{aligned}$$

From $1 < s' < p < n/\alpha$ and $1/q = 1/p - \alpha/n$, it follows that $0 < \alpha s' < n$, $1 < p/s' < n/\alpha s'$ and $1/(q/s') = 1/(p/s') - \alpha s'/n$. Therefore, by Lemma 1, we get

$$\begin{aligned} \left(\int_{\mathbb{R}^n} \left[M_{\Omega,\alpha} f(x) \omega(x) \right]^q dx \right)^{1/q} &\le C \left(\int_{\mathbb{R}^n} \left[N_{\alpha s', s'} f(x) \omega(x) \right]^q dx \right)^{1/q} \\ &\le C \left(\int_{\mathbb{R}^n} |f(x)\omega(x)|^p\, dx \right)^{1/p}. \end{aligned}$$

This completes the proof of Proposition 1.

3**. The proofs of Theorem 1 and Theorem 2.** In this section we will prove Theorems 1 and 2. At first we give some lemmas related to $A(p,q)$ weights.

LEMMA 2. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $\omega \in A(p,q)$, then there exists an $\varepsilon > 0$ such that*

(i) $\varepsilon < \alpha < \alpha + \varepsilon < n$;

(ii) $1/p > (\alpha+\varepsilon)/n, 1/q < (n-\varepsilon)/n$,

$\omega \in A(p, q_\varepsilon)$ and $\omega \in A(p, \tilde{q}_\varepsilon)$, where $1/q_\varepsilon = 1/p - (\alpha+\varepsilon)/n$ and $1/\tilde{q}_\varepsilon = 1/p - (\alpha-\varepsilon)/n$.

PROOF. Since $\alpha > 0$, $1/q < 1$, we can take $\varepsilon_1 > 0$ such that $\varepsilon_1 < \alpha$ and $1/q + \varepsilon_1/n < 1$. Denote $1/q_{\varepsilon_1} = 1/p - (\alpha - \varepsilon_1)/n = 1/q + \varepsilon_1/n$, then $q > q_{\varepsilon_1} > 1$ and $1 + p'/q < 1 + p'/q_{\varepsilon_1}$. Thus, from (2.7) and (2.2), we have $\omega^{-p'} \in A_{1+p'/q} \subset A_{1+p'/q_{\varepsilon_1}}$, which is equivalent to

$$\omega \in A(p, q_{\varepsilon_1}) \tag{3.1}$$

by (2.7).

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On the other hand, there exists an η, $0<\eta<1/q$, such that $\omega^{-p'}\in A_{1+p'(1/q-\eta)}$ by (2.4). Of course, we also choose $\varepsilon_2>0$ small enough such that $\varepsilon_2<\min\{\alpha,n-\alpha\}$, $1/p>(\alpha+\varepsilon_2)/n$ and $\varepsilon_2/n<\eta$ hold at same time. Denote $1/q_{\varepsilon_2}=1/p-(\alpha+\varepsilon_2)/n$, then by $1/p>(\alpha+\varepsilon_2)/n$ and $\varepsilon_2/n<\eta$ we have $0<1/q_{\varepsilon_2}<1$ and $1/q_{\varepsilon_2}=1/q-\varepsilon_2/n>1/q-\eta$. Hence,we get $\omega^{-p'}\in A_{1+p'(1/q-\eta)}\subset A_{1+p'/q_{\varepsilon_2}}$, which is equivalent to

$$\omega\in A(p,q_{\varepsilon_2}) \tag{3.2}$$

Now let $\varepsilon=\min\{\varepsilon_1,\varepsilon_2\}$, then ε satisfies all conditions satisfied by ε_1 and ε_2. Therefore, if we denote $1/q_\varepsilon=1/p-(\alpha+\varepsilon)/n$ and $1/\tilde{q}_\varepsilon=1/p-(\alpha-\varepsilon)/n$, then by (3.1) and (3.2) we have $\omega\in A(p,q_\varepsilon)$ and $\omega\in A(p,\tilde{q}_\varepsilon)$. This is the desired conclusion.

LEMMA 3. *Let $0<\alpha<n$, $1\le s'<p<n/\alpha$, $1/q=1/p-\alpha/n$ and $\omega^{s'}\in A(p/s',q/s')$, then there exists an $\varepsilon>0$ such that*

(i) $\varepsilon<\alpha<\alpha+\varepsilon<n$,

(ii) $1/p>(\alpha+\varepsilon)/n, 1/q<(n-\varepsilon)/n$,

$\omega^{s'}\in A(p/s',q_\varepsilon/s')$ and $\omega^{s'}\in A(p/s',\tilde{q}_\varepsilon/s')$, where $1/q_\varepsilon=1/p-(\alpha+\varepsilon)/n$ and $1/\tilde{q}_\varepsilon=1/p-(\alpha-\varepsilon)/n$.

PROOF. Since $1/(q/s')=1/(p/s')-\alpha s'/n$, by Lemma 2, there exists an $\eta>0$ such that $\eta<\alpha s'<\alpha s'+\eta<n$, $1/(p/s')>(\alpha s'+\eta)/n$, $1/(q/s')<(n-\eta)/n$, $\omega^{s'}\in A(p/s',q_\eta)$ and $\omega^{s'}\in A(p/s',\tilde{q}_\eta)$, where $1/q_\eta=1/(p/s')-(\alpha s'+\eta)/n$, $1/\tilde{q}_\eta=1/(p/s')-(\alpha s'-\eta)/n$. Now let $\varepsilon=\eta/s'$, $q_\varepsilon=s'q_\eta$ and $\tilde{q}_\varepsilon=s'\tilde{q}_\eta$, then ε satisfies $0<\varepsilon<\alpha<\alpha+\varepsilon<n$, $1/p>(\alpha+\varepsilon)/n$ and $1/q<(n-\varepsilon)/n$. Obviously, we have $\omega^{s'}\in A(p/s',q_\varepsilon/s')$ and $\omega^{s'}\in A(p/s',\tilde{q}_\varepsilon/s')$, where $1/q_\varepsilon=1/p-(\alpha+\varepsilon)/n$ and $1/\tilde{q}_\varepsilon=1/p-(\alpha-\varepsilon)/n$.

In order to finish the proof of Theorem 1, we also need the following lemma which shows $T_{\Omega,\alpha}$ is controlled pointwise by $M_{\Omega,\alpha}$.

LEMMA 4. *For any $\varepsilon>0$ with $0<\alpha-\varepsilon<\alpha+\varepsilon<n$, we have*

$$|T_{\Omega,\alpha}f(x)|\le C\big[M_{\Omega,\alpha+\varepsilon}f(x)\big]^{1/2}\cdot\big[M_{\Omega,\alpha-\varepsilon}f(x)\big]^{1/2},\quad x\in\mathbb{R}^n,$$

where C depends only on ε, α, n.

PROOF. The proof will follow after [We]. Given $x\in\mathbb{R}^n$ and $\varepsilon>0$ with $0<\alpha-\varepsilon<\alpha+\varepsilon<n$, we choose a $\delta>0$ such that

$$\delta^{2\varepsilon}=M_{\Omega,\alpha+\varepsilon}f(x)/M_{\Omega,\alpha-\varepsilon}f(x).$$

Now we put

$$\begin{aligned} T_{\Omega,\alpha-\varepsilon}f(x) &= \int_{|x-y|<\delta}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)\,dy+\int_{|x-y|\ge\delta}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)\,dy\\ &:= I_1+I_2. \end{aligned}$$

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Thus

$$\begin{aligned}|I_1| &\le \sum_{j=0}^{\infty}\int_{2^{-j-1}\delta\le|x-y|<2^{-j}\delta}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|\,dy\\ &\le \sum_{j=0}^{\infty}(2^{-j-1}\delta)^{-(n-\alpha)}\int_{|x-y|<2^{-j}\delta}|\Omega(x-y)|\,|f(y)|\,dy\\ &= 2^{n-\alpha}\sum_{j=0}^{\infty}(2^{-j}\delta)^{\varepsilon}\frac{1}{(2^{-j}\delta)^{n-\alpha+\varepsilon}}\int_{|x-y|<2^{-j}\delta}|\Omega(x-y)|\,|f(y)|\,dy\\ &\le C\cdot\delta^{\varepsilon}\cdot M_{\Omega,\alpha-\varepsilon}f(x).\end{aligned}$$

Similarly,

$$\begin{aligned}|I_2| &\le \sum_{j=1}^{\infty}\int_{2^{j-1}\delta\le|x-y|<2^{j}\delta}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|\,dy\\ &\le C\sum_{j=1}^{\infty}(2^{j}\delta)^{-\varepsilon}\frac{1}{(2^{j}\delta)^{n-\alpha-\varepsilon}}\int_{|x-y|<2^{j}\delta}|\Omega(x-y)|\,|f(y)|\,dy\\ &\le C\cdot\delta^{-\varepsilon}\cdot M_{\Omega,\alpha+\varepsilon}f(x).\end{aligned}$$

Therefore, we get

$$|T_{\Omega,\alpha}f(x)|\le C\left[\delta^{\varepsilon}M_{\Omega,\alpha-\varepsilon}f(x)+\delta^{-\varepsilon}M_{\Omega,\alpha+\varepsilon}f(x)\right]$$

and so with the above election of δ the lemma is proved.

THE PROOF OF THEOREM 1. Under the conditions of Theorem 1, by Lemma 3, there exists an $\varepsilon>0$ such that $0<\varepsilon<\alpha<\alpha+\varepsilon<n$, $1/p>(\alpha+\varepsilon)/n$, $\omega^{s'}\in A(p/s',q_\varepsilon/s')$ and $\omega^{s'}\in A(p/s',\tilde q_\varepsilon/s')$, where $1/q_\varepsilon=1/p-(\alpha+\varepsilon)/n$ and $1/\tilde q_\varepsilon=1/p-(\alpha-\varepsilon)/n$. Now let $l_1=2q_\varepsilon/q$, $l_2=2\tilde q_\varepsilon/q$, then $1/l_1+1/l_2=1$. For above given $\varepsilon>0$, using Lemma 4 and Hölder's inequality, we have

$$\begin{aligned}\|T_{\Omega,\alpha}f\|_{q,\omega^q} &\le C\left(\int_{\mathbb{R}^n}\left[M_{\Omega,\alpha+\varepsilon}f(x)\omega(x)\right]^{q/2}\cdot\left[M_{\Omega,\alpha-\varepsilon}f(x)\omega(x)\right]^{q/2}dx\right)^{1/q}\\ &\le C\left(\int_{\mathbb{R}^n}\left[M_{\Omega,\alpha+\varepsilon}f(x)\omega(x)\right]^{ql_1/2}dx\right)^{1/ql_1}\left(\int_{\mathbb{R}^n}\left[M_{\Omega,\alpha-\varepsilon}f(x)\omega(x)\right]^{ql_2/2}dx\right)^{1/ql_2}\\ &= C\left(\int_{\mathbb{R}^n}\left[M_{\Omega,\alpha+\varepsilon}f(x)\omega(x)\right]^{q_\varepsilon}dx\right)^{1/2q_\varepsilon}\left(\int_{\mathbb{R}^n}\left[M_{\Omega,\alpha-\varepsilon}f(x)\omega(x)\right]^{\tilde q_\varepsilon}dx\right)^{1/2\tilde q_\varepsilon}.\end{aligned}$$

Therefore, from Lemma 3 and Proposition 1, it follows that

$$\left(\int_{\mathbb{R}^n}\left[M_{\Omega,\alpha+\varepsilon}f(x)\omega(x)\right]^{q_\varepsilon}dx\right)^{1/2q_\varepsilon}\le C\|f\|_{p,\omega^p}^{1/2}$$

and

$$\left(\int_{\mathbb{R}^n}\left[M_{\Omega,\alpha-\varepsilon}f(x)\omega(x)\right]^{\tilde q_\varepsilon}dx\right)^{1/2\tilde q_\varepsilon}\le C\|f\|_{p,\omega^p}^{1/2}.$$

Hence, we obtain

$$\|T_{\Omega,\alpha}f\|_{q,\omega^q}\le C\|f\|_{p,\omega^p}.$$

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Let us now turn to the proof of Theorem 2. In fact, Theorem 2 is a consequence of Theorem 1 by duality. To see this, let $\tilde{T} := \widetilde{T_{\Omega,\alpha}}$ be the adjoint operator of $T_{\Omega,\alpha}$, that means $\widetilde{T_{\Omega,\alpha}} = T_{\tilde{\Omega},\alpha}$ with $\tilde{\Omega}(x) = \overline{\Omega(x)}$. Obviously, $\tilde{\Omega}$ is also homogeneous of degree zero and satisfies the same essential inequalities as Ω. Thus, we have

$$\|T_{\Omega,\alpha}f\|_{q,\omega^q} = \sup_g \left| \int_{\mathbb{R}^n} T_{\Omega,\alpha}f(x)g(x)\,dx \right|,$$

where the supremum is taken over all $g(x)$ with $\|g\|_{q',\omega^{-q'}} \leq 1$. Since $\tilde{T}$ is the adjoint operator of $T_{\Omega,\alpha}$, then

$$\int_{\mathbb{R}^n} T_{\Omega,\alpha}f(x)g(x) = \int_{\mathbb{R}^n} f(x)\cdot \tilde{T}g(x)\,dx.$$

Hence,

$$\begin{aligned}\|T_{\Omega,\alpha}f\|_{q,\omega^q} &= \sup_q \left| \int_{\mathbb{R}^n} f(x)\cdot\tilde{T}g(x)\,dx \right| \\ &\leq \|f\|_{p,\omega^p}\cdot \sup_g \|\tilde{T}g\|_{p',\omega^{-p'}}.\end{aligned}$$

By the condition of Theorem 2, we see that $1/q = 1/p - \alpha/n$ and $1 < p < q < s$. Thus, $1/p' = 1/q' - \alpha/n$ and $s' < q' < n/\alpha$. From $(\omega^{-1})^{s'} \in A(q'/s', p'/s')$ and Theorem 1, it follows that

$$\|\tilde{T}g\|_{p',\omega^{-p'}} \leq C\|g\|_{q',\omega^{-q'}}.$$

Therefore,

$$\|T_{\Omega,\alpha}f\|_{q,\omega^q} \leq \|f\|_{p,\omega^p}\cdot \sup_g \|\tilde{T}g\|_{p',\omega^{-p'}} \leq C\|f\|_{p,\omega^p}.$$

This finishes the proof of Theorem 2.

Finally, let us point out that Proposition 2 is a direct consequence of Theorem 2 and the following lemma, which shows that $M_{\Omega,\alpha}(f)(x)$ can be controlled pointwise by $T_{|\Omega|,\alpha}(|f|)(x)$ for any $f(x)$.

LEMMA 5. *Let $0 < \alpha < n$, $\Omega \in L^1(S^{n-1})$. Then we have*

$$M_{\Omega,\alpha}(f)(x) \leq T_{|\Omega|,\alpha}(|f|)(x).$$

In fact, fix $r > 0$, we have

$$\begin{aligned}T_{|\Omega|,\alpha}(|f|)(x) &\geq \int_{|x-y|<r} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|\,dy \\ &\geq \frac{1}{r^{n-\alpha}} \int_{|x-y|<r} |\Omega(x-y)|\,|f(y)|\,dy.\end{aligned}$$

Taking the supremum for $r > 0$ on two sides of the inequality above, we get

$$T_{|\Omega|,\alpha}(|f|)(x) \geq \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|x-y|<r} |\Omega(x-y)|\,|f(y)|\,dy.$$

This is just our desired conclusion.

4. **The proof of Theorem 3.** Let us first state a lemma which is easily deduced from the Stein-Weiss interpolation theorem with change of measures (see [BL], p. 120).

LEMMA 6. *Let $0 < \alpha < n$, $1 < p_0 < p_1 < n/\alpha$, $1/q_0 = 1/p_0 - \alpha/n$, and $1/q_1 = 1/p_1 - \alpha/n$. If linear operator T is a bounded operator from $L^{p_0}(\omega_0^{p_0})$ to $L^{q_0}(\omega_0^{q_0})$ and from $L^{p_1}(\omega_1^{p_1})$ to $L^{q_1}(\omega_1^{q_1})$ with norms C_0 and C_1 respectively, then T is also bounded operator from $L^p(\omega^p)$ to $L^q(\omega^q)$ with norm C, where $0 < \theta < 1$, $C \le C_0^{1-\theta}C_1^{\theta}$, $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = 1/p - \alpha/n$, and $\omega = \omega_0^{1-\theta}\omega_1^{\theta}$.*

Let us now turn to prove Theorem 3. If we can prove that there exist $\theta (0 < \theta < 1)$, p_0, p_1, q_0, and q_1 satisfying

$$1 \le s' < p_0 < p < p_1 < n/\alpha, \tag{4.1}$$

$$n/(n-\alpha) < q_0 < q < q_1 < s, \tag{4.2}$$

$$1/q_0 = 1/p_0 - \alpha/n, 1/q_1 = 1/p_1 - \alpha/n, 1/p = (1-\theta)/p_0 - \alpha/n, \tag{4.3}$$

$$\omega = \omega_0^{1-\theta} \cdot \omega_1^{\theta}, \tag{4.4}$$

and

$$\omega_0^{s'} \in A(p_0/s', q_0/s'), \omega_1^{-s'} \in A(q_1'/s', p_1'/s'), \tag{4.5}$$

then the conclusion of Theorem 3 will be deduced from Theorem 1, Theorem 2 and Lemma 6. Therefore, it suffices to seek above θ, p_0, p_1, q_0, q_1, ω_0 and ω_1 such that (4.1)–(4.5) hold.

Since there is an r, $1 < r < s/(\frac{n}{\alpha})'$, such that $\omega^{r'} \in A(p,q)$, it follows from (2.7) that $\omega^{r'q} \in A_{q(n-\alpha)/n}$. However, it follows from (2.6) that there exist $u(x), v(x) \in A_1$ such that

$$\omega(x)^{r'q} = u(x) \cdot v(x)^{1-q(n-\alpha)/n},$$

or

$$\omega(x) = u(x)^{1/r'q} \cdot v(x)^{1/r'q-(n-\alpha)/r'n}. \tag{4.6}$$

By (4.6), we can write $\omega(x)$ as

$$\omega = (u^{\tau}v^{\beta})^{1-\theta}(u^{\gamma}v^{\delta})^{\theta}, \tag{4.7}$$

where

$$\tau(1-\theta) + \gamma\theta = 1/r'q, \quad \beta(1-\theta) + \delta\theta = 1/r'q - (n-\alpha)/r'n. \tag{4.8}$$

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Now we denote $\omega_0(x) = u(x)^\tau v(x)^\beta$ and $\omega_1(x) = u(x)^\gamma v(x)^\delta$. We shall see that if $1 \le s' < p_0 < p < n/\alpha$ and $1/q_0 = 1/p_0 - \alpha/n$, then when $\tau = 1/q_0$ and $\beta = -1/s'(\frac{p_0}{s'})'$, we have $\omega_0^{s'} \in A(p_0/s', q_0/s')$. In fact, since $u(x), v(x) \in A_1$, we have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left[\omega_0(x)^{s'}\right]^{q_0/s'} dx\right)^{s'/q_0} \left(\frac{1}{|Q|}\int_Q \left[\omega_0(x)^{s'}\right]^{-(p_0/s')'} dx\right)^{1/(p_0/s')'} \\
&\quad = \left(\frac{1}{|Q|}\int_Q u(x)^{q_0\tau} v(x)^{q_0\beta}\, dx\right)^{s'/q_0} \left(\frac{1}{|Q|}\int_Q u(x)^{-\tau s'(p_0/s')'} v(x)^{-\beta s'(p_0/s')'}\, dx\right)^{1/(p_0/s')'} \\
&\quad \le C\left(\frac{1}{|Q|}\int_Q v(x)\, dx\right)^{s'\beta} \left(\frac{1}{|Q|}\int_Q u(x)^{q_0\tau}\, dx\right)^{s'/q_0} \left(\frac{1}{|Q|}\int_Q u(x)\, dx\right)^{-s'\tau} \\
&\qquad \cdot \left(\frac{1}{|Q|}\int_Q v(x)^{-\beta s'(p_0/s')'}\, dx\right)^{1/(p_0/s')'} \\
&\quad \le C,
\end{aligned}$$

where C is independent of Q. By the same method, we can prove that if $n/(n-\alpha) < q < q_1 < s$ and $1/q_1 = 1/p_1 - \alpha/n$, then when $\gamma = -1/p'$, $\delta = 1/s'(\frac{q_1'}{s'})'$, we have $\omega_1^{-s'} \in A(q_1'/s', p_1'/s')$.

Let us now figure θ out by (4.8). Note that

$$\beta = -\left\{s'\left(\frac{p_0}{s'}\right)'\right\}^{-1} = 1/p_0 - 1/s'$$

and

$$\delta = \left\{s'\left(\frac{q_1'}{s'}\right)'\right\}^{-1}.$$

Thus, it follows from (4.8) that

$$\begin{aligned}
\theta &= \frac{\tau - \beta - (n-\alpha)/r'n}{\delta - \gamma - \beta + \tau} \\
&= \frac{1/s' - \alpha/n - (n-\alpha)/r'n}{2(1/s' - \alpha/n)}.
\end{aligned} \tag{4.9}$$

Since $1 < r < s/(\frac{n}{\alpha})'$, we may write $\frac{1}{r} = \frac{n}{n-\alpha}(\frac{1}{s} + \varepsilon)$, where $\varepsilon > 0$. Thus,

$$1/s' - \alpha/n - (n-\alpha)/r'n = \frac{1}{s'} - \frac{\alpha}{n} - \frac{n-\alpha}{n}\left[1 - \frac{n}{n-\alpha}\left(\frac{1}{s} + \varepsilon\right)\right] = \varepsilon,$$

and then $\theta = \varepsilon/2(\frac{1}{s'} - \frac{\alpha}{n})$. Since $s' < n/\alpha$, we have $\theta > 0$. On the other hand, we easily see that $\theta < 1$ by (4.9). Therefore, $0 < \theta < 1$ and (4.4), (4.5) hold by the above estimates. It remains to prove that we can choose proper p_0, p_1, q_0 and q_1 such that (4.1)–(4.3) hold. Since $1/p > \alpha/n + 1/s$ and $\theta > 0$, we have

$$\frac{1}{p(1-\theta)} - \frac{\alpha\theta}{n(1-\theta)} - \frac{\theta}{s(1-\theta)} > \frac{1}{p}. \tag{4.10}$$

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By (4.10) and $1/p < 1/s'$, we can choose p_0 such that

$$\frac{1}{p} < \frac{1}{p_0} < \min\left\{\frac{1}{s'}, \frac{1}{p(1-\theta)} - \frac{\alpha\theta}{n(1-\theta)} - \frac{\theta}{s(1-\theta)}\right\}. \tag{4.11}$$

Thus, we have $s' < p_0 < p$ and $1/p > (1-\theta)/p_0 + \alpha\theta/n$. Therefore, there exists a $\sigma > 0$ such that

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \left(\frac{\alpha}{n} + \sigma\right)\theta. \tag{4.12}$$

Let us denote $\frac{1}{p_1} = \frac{\alpha}{n} + \sigma$. Then it follows from $1/p_1 > \alpha/n$ and $1/p < 1/p_0$ that $s' < p_0 < p < p_1 < n/\alpha$. This proves (4.1). Also (4.3) holds by (4.12). Now let us denote $1/q_0 = 1/p_0 - \alpha/n$ and $1/q_1 = 1/p_1 - \alpha/n$. Obviously, by (4.11), we have

$$\frac{1}{p\theta} - \frac{1-\theta}{p_0\theta} - \frac{\alpha}{n} > \frac{1}{s}.$$

However, the above is equivalent to $1/p_1 - \alpha/n > 1/s$. Thus, $q_1 < s$, and therefore (4.2) holds. Hence, we finish the proof of Theorem 3.

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Department of Mathematics
Beijing Normal University
Beijing 100875
China
email: lusz@sun.ihep.ac.cn