WEIGHTED NORM INEQUALITIES FOR FRACTIONAL INTEGRAL OPERATORS WITH ROUGH KERNEL

YONG DING AND SHANZHEN LU

ABSTRACT. Given function $\Omega$ on $\mathbb{R}^n$, we define the fractional maximal operator and the fractional integral operator by

$$M_{\Omega, \alpha} f(x) = \sup_{r > 0} \frac{1}{r^{n-\alpha}} \int_{|y| < r} |\Omega(y)||f(x-y)| \, dy$$

and

$$T_{\Omega, \alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^{n-\alpha}} f(x-y) \, dy$$

respectively, where $0 < \alpha < n$. In this paper we study the weighted norm inequalities of $M_{\Omega, \alpha}$ and $T_{\Omega, \alpha}$ for appropriate $\alpha$s and $A(p, q)$ weights in the case that $\Omega \in L^s(\mathbb{S}^{n-1})$ ($s > 1$), homogeneous of degree zero.

1. Introduction. Suppose that $0 \leq \alpha < n$, $\Omega$ is homogeneous of degree zero, and $\Omega \in L^1(\mathbb{S}^{n-1})$, where $\mathbb{S}^{n-1}$ denotes the sphere of $\mathbb{R}^n$ and $s > 1$. Then we will consider the fractional maximal operator $M_{\Omega, \alpha}$ defined by

$$M_{\Omega, \alpha} f(x) = \sup_{r > 0} \frac{1}{r^{n-\alpha}} \int_{|y| < r} \Omega(y) |f(x-y)| \, dy$$

and the fractional integral operator defined by

$$T_{\Omega, \alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^{n-\alpha}} f(x-y) \, dy.$$

When $\alpha = 0$, we denote $M_{\Omega, \alpha}$ and $T_{\Omega, \alpha}$ by $M_\Omega$ and $T_\alpha$ respectively, where the integration is taken by the Cauchy principal value.

It is well known that Kurtz and Wheeden [KW] had proven certain weighted norm inequalities for $T_\Omega$ under the assumption that $\Omega \in L^1(\mathbb{S}^{n-1})$ and $\Omega$ satisfies an $L^1(\mathbb{S}^{n-1})$-Dini smoothness condition. Using Fourier transform methods, Watson [W] and Duoandikoetxea [Du] showed that the smoothness requirement in [KW] was in fact unnecessary. However, the corresponding results for fractional maximal and singular integral operators have not been proven even for smooth $\Omega$. This paper aims to establish weighted norm inequalities for $T_{\Omega, \alpha}$ and $M_{\Omega, \alpha}$ with $0 < \alpha < n$ and $\Omega \in L^1(\mathbb{S}^{n-1})$. To do this, we require some techniques related to weights from a $(p, p)$ setting to a $(p, q)$ setting.

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A locally integrable nonnegative function \( \omega \) on \( \mathbb{R}^n \) is said to belong to \( A(p, q)(1 < p, q < \infty) \) if there exists \( C \) such that
\[
(1.1) \quad \sup_Q \left( \frac{1}{|Q|} \int_Q \omega(x)^p \, dx \right)^{1/q} \left( \frac{1}{|Q|} \int_Q \omega(x)^{-p'} \, dx \right)^{1/p'} \leq C < \infty,
\]
where \( p' = p/(p - 1) \), \( Q \) denotes a cube in \( \mathbb{R}^n \) with its sides parallel to the coordinate axes and the supremum is taken over all cubes. In 1971, Muckenhoupt and Wheeden [MW1] studied the weighted norm inequalities for \( T_{\Omega, \sigma} \) with the weight \( \omega(x) = |x|^\beta \). Recently, weak type inequalities with power weights for \( T_{\Omega, \sigma} \) and \( M_{\Omega, \sigma} \) have been obtained by one of the authors of this paper [D]. Moreover, Muckenhoupt and Wheeden [MW2] gave the following weighted results for \( M_{1, \sigma} \) and \( T_{1, \sigma} \) with \( \Omega = 1 \).

**Theorem A.** If \( 0 < \alpha < n \), \( 1 < p < n/\alpha \), \( 1/q = 1/p - \alpha/n \) and \( \omega(x) \in A(p, q) \), then there is a constant \( C \), independent of \( f \), such that
\[
\left( \int_{\mathbb{R}^n} \left[ M_{1, \sigma} f(x) \omega(x) \right]^q \, dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p},
\]
and
\[
\left( \int_{\mathbb{R}^n} \left[ T_{1, \sigma} f(x) \omega(x) \right]^q \, dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}.
\]

On the other hand, Duanandikotxea [Du] obtained the weighted norm inequalities for \( M_{\Omega} \) and \( T_{\Omega} \) with the weight \( \omega(x) \in A_p \). Moreover, as usual, \( A_p \) denotes the Muckenhoupt’s class.

In this paper we shall study the weighted norm inequalities for \( M_{\Omega, \sigma} \) and \( T_{\Omega, \sigma} \) with more general weights, that is, we will look for some appropriate indices \( p, q, \alpha, s \) such that for \( \omega(x) \in A(p, q), \Omega \in L^s(S^{n - 1}) \),
\[
(1.2) \quad \left( \int_{\mathbb{R}^n} \left[ T_{\Omega, \sigma} f(x) \omega(x) \right]^q \, dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p},
\]
holds, where \( C \) is independent of \( f \). The same conclusion is true for \( M_{\Omega, \sigma} \). Precisely, we obtain the following

**Theorem 1.** Let \( 0 < \alpha < n \), \( s' < p < n/\alpha \), and \( 1/q = 1/p - \alpha/n \). If \( \Omega \in L^s(S^{n - 1}) \) and \( \omega(x)^s \in A(p/s', q/s') \), then there is a constant \( C \), independent of \( f \), such that
\[
\left( \int_{\mathbb{R}^n} \left[ T_{\Omega, \sigma} f(x) \omega(x) \right]^q \, dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}.
\]

**Theorem 2.** Let \( 0 < \alpha < n \), \( 1 < p < n/\alpha \), \( 1/q = 1/p - \alpha/n \), and \( s > q \). If \( \Omega \in L^s(S^{n - 1}) \) and \( \omega(x)^{-s} \in A(p'/s', q'/s') \), then there is a constant \( C \), independent of \( f \), such that
\[
\left( \int_{\mathbb{R}^n} \left[ T_{\Omega, \sigma} f(x) \omega(x) \right]^q \, dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}.
\]
THEOREM 3. Let $0 < \alpha < n$, $1 < p < n/\alpha$, and $1/q = 1/p - \alpha/n$. If $\Omega$ is homogeneous of degree zero, $\Omega \in L^s(S^{n-1})$ for some $s > 1$ with $\alpha/n + 1/s < 1/p < 1/s'$, and there exists an $r$, $1 < r < s/(n\alpha)'$, such that $\omega \in A(p, q)$, then there is a constant $C$, independent of $f$, such that

$$\left( \int_{\mathbb{R}^n} [T_{\Omega, \omega} f(x)]^q dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}.$$ 

To prove Theorem 1, let us first set up the following Proposition 1.

PROPOSITION 1. Let $0 < \alpha < n$, $s' < p < n/\alpha$, and $1/q = 1/p - \alpha/n$. If $\Omega \in L^s(S^{n-1})$ and $\omega(x)' \in A(p/s', q/s')$, then there is a constant $C$, independent of $f$, such that

$$\left( \int_{\mathbb{R}^n} |M_{\Omega, \omega} f(x)^q dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}.$$ 

As a corollary of Theorem 2, we have the following.

PROPOSITION 2. Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $s > q$. If $\Omega \in L^s(S^{n-1})$ and $\omega(x)^- \in A(q/s', p/s')$, then there is a constant $C$, independent of $f$, such that

$$\left( \int_{\mathbb{R}^n} |M_{\Omega, \omega} f(x)^q dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)^p dx \right)^{1/p}.$$ 

As a direct corollary of Theorem 3, we also have.

PROPOSITION 3. Under the assumption of Theorem 3, $M_{\Omega, \omega}$ is also a bounded operator from $L^p(\omega^p)$ to $L^q(\omega^q)$.

2. Some properties of $A(p, q)$ weights and proof of Proposition 1. Some elementary properties of $A(p, q)$ weights will be first given in this section. Then we shall give the proof of Proposition 1. Let us recall the elementary properties of $A_p$ weight. A locally integrable nonnegative function $\nu$ on $\mathbb{R}^n$ is said to belong to $A_p(1 < p < \infty)$ if there exists $C$ such that

$$\sup_Q \left( \frac{1}{|Q|} \int_Q \nu(x) dx \right) \left( \frac{1}{|Q|} \int_Q \nu(x)^{-1/(p-1)} dx \right)^{p-1} \leq C < \infty,$$

where $Q$ denotes a cube in $\mathbb{R}^n$ with the sides parallel to the coordinate axes and the supremum is taken over all cubes. When $p = 1$, a nonnegative measurable function $\nu$ is said to belong to $A_1$, if there exists $C$ such that for any cube $Q$,

$$\frac{1}{|Q|} \int_Q \nu(y) dy \leq C \nu(x), \quad a.e. \ x \in Q.$$

THE ELEMENTARY PROPERTIES OF $A_p$ (SEE [GR]).

(2.2) $A_{p_1} \subset A_{p_2}$ if $1 < p_1 \leq p_2 < \infty$.

(2.3) $\nu \in A_p$ if and only if $\nu^{1-p'} \in A_{p'}$.

(2.4) If $\nu \in A_p$, then there exists an $\varepsilon > 0$ such that $p - \varepsilon > 1$ and $\nu \in A_{p-\varepsilon}$.

(2.5) If $\nu \in A_p$, then there exists an $\varepsilon > 0$ such that $\nu^{1+\varepsilon} \in A_p$.

(2.6) $\nu \in A_p(1 < p < \infty)$ if and only if there exist $u, v \in A_1$ such that $\nu(x) = u(x) \cdot v(x)^{1-p}$.
Thus, 

\[
\omega(x) \in A(p, q) \iff \omega(x)^p \in A_{q(n-\alpha)/n}.
\] (2.7)

\[
\iff \omega(x)^p \in A_{q(n-\alpha)/n}
\]

\[
\iff \omega(x)^{-\beta} \in A_{1+\beta/p}
\]

(2.8) \quad \omega(x) \in A(p, q) \implies \omega(x)^p \in A_q \quad \text{and} \quad \omega(x)^p \in A_p.

**Proof.** (2.7) can be deduced from the definitions of \(A_p\) and \(A(p, q)\). Let us now prove (2.8) by (2.2), (2.3) and (2.7). Since \(q(n-\alpha)/n < q\), we have \(\omega(x)^p \in A_{q(n-\alpha)/n} \subset A_q\). From \(1/q = 1/p - \alpha/n\), it follows that \(1/q < 1/p = (p'-1)/p', \) i.e. \(1+p'/q < p'\).

Using (2.7) and (2.2), we have \(\omega(x)^{-\beta} \in A_{1+\beta/p} \subset A_p\). However, this is equivalent to \(\omega(x)^p \in A_p\) by (2.3).

We shall give the proof of Proposition 1 in the following. The proof is based on the following observation.

**Lemma 1.** If \(0 < \alpha < n, s' > 1, 1 < p / s' < n / \alpha, 1 / (q / s') = 1 / (p / s') - \alpha / n, \) and \(\omega(x)^p \in A(p / s', q / s')\), then

\[
\left( \int_{\mathbb{R}^n} \left[ N_{\sigma, s} f(x) \omega(x) \right]^q dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)\omega(x)|^p dx \right)^{1/p},
\] (2.9)

where \(N_{\sigma, s} f(x)\) is the fractional maximal operator of order \(s'\) defined by

\[
N_{\sigma, s} f(x) = \sup_{r > 0} \left( \frac{1}{r^{n-\alpha}} \int_{|y| < r} |f(x-y)|^{s'} dy \right)^{1/s'}. 
\]

**Proof.** Since \(N_{\sigma, s} f(x) = (M_{1, \sigma}(|f|^{s'})(x))^{1/s'}\), we have

\[
\left( \int_{\mathbb{R}^n} \left[ N_{\sigma, s} f(x) \omega(x) \right]^q dx \right)^{1/q} = \left( \int_{\mathbb{R}^n} \left[ M_{1, \sigma}(|f|^{s'})(x) \omega(x)^q \right] dx \right)^{1/q}
\]

\[
= \left( \left( \int_{\mathbb{R}^n} \left[ M_{1, \sigma}(|f|^{s'})(x) \nu(x) \omega(x)^q \right] dx \right)^{s'/q} \right)^{1/s'},
\]

where \(\nu(x) = \omega(x)^{s'}\) and \(\nu(x) \in A(p / s', q / s')\). By Theorem A, we have

\[
\left( \int_{\mathbb{R}^n} \left[ M_{1, \sigma}(|f|^{s'})(x) \nu(x) \omega(x)^q \right] dx \right)^{s'/q} \leq C \left( \int_{\mathbb{R}^n} \left[ |f(x)|^{s'} \nu(x)^{p/s'} dx \right]^{s'/p} \right)^{1/s'}
\]

\[
= C \left( \int_{\mathbb{R}^n} |f(x)|^p \nu(x)^{\rho/s} dx \right)^{s'/p}.
\]

Thus,

\[
\left( \int_{\mathbb{R}^n} \left[ N_{\sigma, s} f(x) \omega(x) \right]^q dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p \nu(x)^{\rho/s} dx \right)^{1/p}
\]

\[
= C \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x)^{s} dx \right)^{1/p}.
\]
This proves (2.9).

Let us now give the proof of Proposition 1. By $s > 1$, $\Omega(x') \in L'(S^{n-1})$, we have

$$M_{\Omega,\sigma} f(x) = \sup_{r > 0} \frac{1}{r^{n-\alpha}} \int_{|y| < r} |\Omega(y)f(x-y)| \, dy$$

$$\leq \sup_{r > 0} \frac{1}{r^{n-\alpha}} \left( \int_{|y| < r} |\Omega(y)|^s \, dy \right)^{1/s} \left( \int_{|y| < r} |f(x-y)|^{s'} \, dy \right)^{1/s'}.$$ 

Since $\left( \int_{|y| < r} |\Omega(y)|^s \, dy \right)^{1/s} \leq Cr'^s ||\Omega||_s$, where $||\Omega||_s = \left( \int_{S^{n-1}} |\Omega(y'y')^s \, d\sigma(y') \right)^{1/s}$, then we have

$$M_{\Omega,\sigma} f(x) \leq C \sup_{r > 0} \frac{1}{r^{n-\alpha}} \cdot r^{n/s} \left( \int_{|y| < r} |f(x-y)|^{s'} \, dy \right)^{1/s'}.$$

$$= C \sup_{r > 0} \left( \frac{1}{r^{n-\alpha'}} \int_{|y| < r} |f(x-y)|^{s'} \, dy \right)^{1/s'}$$

$$= C : N_{\alpha'} : f(x).$$

From $1 < s' < p < n/\alpha$ and $1/q = 1/p - \alpha/n$, it follows that $0 < \alpha s' < n$, $1 < p/s' < n/\alpha s'$ and $1/(q/s') = 1/(p/s') - \alpha s'/n$. Therefore, by Lemma 1, we get

$$\left( \int_{\mathbb{R}^n} [M_{\alpha,\sigma} f(x) \omega(x)]^q \, dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} [N_{\alpha',\omega} f(x) \omega(x)]^q \, dx \right)^{1/q}$$

$$\leq C \left( \int_{\mathbb{R}^n} |f(x)\omega(x)|^p \, dx \right)^{1/p}.$$

This completes the proof of Proposition 1.

3. The proofs of Theorem 1 and Theorem 2. In this section we will prove Theorems 1 and 2. At first we give some lemmas related to $A(p, q)$ weights.

**Lemma 2.** Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $\omega \in A(p, q)$, then there exists an $\varepsilon > 0$ such that

(i) $\varepsilon < \alpha < \alpha + \varepsilon < n$;

(ii) $1/p > (\alpha + \varepsilon)/n$, $1/q < (n - \varepsilon)/n$.

$\omega \in A(p, q_{\varepsilon})$ and $\omega \in \hat{A}(p, \hat{q}_{\varepsilon})$, where $1/q_{\varepsilon} = 1/p - (\alpha + \varepsilon)/n$ and $1/\hat{q}_{\varepsilon} = 1/p - (\alpha - \varepsilon)/n$.

**Proof.** Since $\alpha > 0$, $1/q < 1$, we can take $\varepsilon_1 > 0$ such that $\varepsilon_1 < \alpha$ and $1/q + \varepsilon_1/n < 1$. Denote $1/q_{\varepsilon_1} = 1/p - (\alpha - \varepsilon_1)/n = 1/q + \varepsilon_1/n$, then $q > q_{\varepsilon_1} > 1$ and $1 + p'/q < 1 + p'/q_{\varepsilon_1}$. Thus, from (2.7) and (2.2), we have $\omega^{-p'} \in A_{1+p'/q} \subset A_{1+p'/q_{\varepsilon_1}}$, which is equivalent to

$$\omega \in A(p, q_{\varepsilon_1})$$

by (2.7).
On the other hand, there exists an \( \eta, 0 < \eta < 1/q \), such that \( \omega^{-p'} \in A_{1+p'(1/q-\eta)} \) by (2.4). Of course, we also choose \( \varepsilon_2 > 0 \) small enough such that \( \varepsilon_2 < \min \{ \alpha, n - \alpha \} \), \( 1/p > (\alpha + \varepsilon_2)/n \) and \( \varepsilon_2/n < \eta \) hold at same time. Denote \( 1/q_{\varepsilon_2} = 1/p - (\alpha + \varepsilon_2)/n \), then by \( 1/p > (\alpha + \varepsilon_2)/n \) and \( \varepsilon_2/n < \eta \) we have \( 0 < 1/q_{\varepsilon_2} < 1 \) and \( 1/q_{\varepsilon_2} = 1/q - \varepsilon_2/n > 1/q - \eta \). Hence, we get \( \omega^{-p'} \in A_{1+p'(1/q-\eta)} \subset A_{1+p'/q_{\varepsilon_2}} \), which is equivalent to

\[
\omega \in A(p, q_{\varepsilon_2})
\]

Now let \( \varepsilon = \min \{ \varepsilon_1, \varepsilon_2 \} \), then \( \varepsilon \) satisfies all conditions satisfied by \( \varepsilon_1 \) and \( \varepsilon_2 \). Therefore, if we denote \( 1/q_{\varepsilon} = 1/p - (\alpha + \varepsilon)/n \) and \( 1/q_{\varepsilon} = 1/p - (\alpha - \varepsilon)/n \), then by (3.1) and (3.2) we have \( \omega \in A(p, q_{\varepsilon}) \) and \( \omega \in A(p, q_{\varepsilon}) \). This is the desired conclusion.

**Lemma 3.** Let \( 0 < \alpha < n, 1 \leq s' < p < n/\alpha, 1/q = 1/p - \alpha/n \) and \( \omega' \in A(p/s', q/s') \), then there exists an \( \varepsilon > 0 \) such that

(i) \( \varepsilon < \alpha < \alpha + \varepsilon < n \),

(ii) \( 1/p > (\alpha + \varepsilon)/n, 1/q < (n - \varepsilon)/n \),

\( \omega' \in A(p/s', q_{\varepsilon}/s') \) and \( \omega' \in A(p/s', \bar{q}_{\varepsilon}/s') \), where \( 1/q_{\varepsilon} = 1/p - (\alpha - \varepsilon)/n \) and \( 1/\bar{q}_{\varepsilon} = 1/p - (\alpha + \varepsilon)/n \).

**Proof.** Since \( 1/(q/s') = 1/(p/s') - \alpha s'/n \), by Lemma 2, there exists an \( \eta > 0 \) such that \( \eta < \alpha < \alpha + \varepsilon < n \), \( 1/(p/s') > (\alpha s' + \eta)/n \), \( 1/(q/s') < (n - \eta)/n \), \( \omega' \in A(p/s', q_{\eta}) \) and \( \omega' \in A(p/s', \bar{q}_{\eta}) \), where \( 1/q_{\eta} = 1/(p/s') - (\alpha s' + \eta)/n \) and \( 1/\bar{q}_{\eta} = 1/(p/s') - (\alpha s' + \eta)/n \). Now let \( \varepsilon = \eta/s', q_{\varepsilon} = s'q_{\eta} \) and \( \bar{q}_{\varepsilon} = s'\bar{q}_{\eta} \), then \( \varepsilon \) satisfies \( 0 < \varepsilon < \alpha < \alpha + \varepsilon < n, 1/p > (\alpha + \varepsilon)/n \) and \( 1/q < (n - \varepsilon)/n \). Obviously, we have \( \omega' \in A(p/s', q_{\varepsilon}/s') \) and \( \omega' \in A(p/s', \bar{q}_{\varepsilon}/s') \), where \( 1/q_{\varepsilon} = 1/p - (\alpha + \varepsilon)/n \) and \( 1/\bar{q}_{\varepsilon} = 1/p - (\alpha - \varepsilon)/n \).

In order to finish the proof of Theorem 1, we also need the following lemma which shows \( T_{\Omega, \alpha} \) is controlled pointwise by \( M_{\Omega, \alpha} \).

**Lemma 4.** For any \( \varepsilon > 0 \) with \( 0 < \alpha - \varepsilon < \alpha + \varepsilon < n \), we have

\[
|T_{\Omega, \alpha}f(x)| \leq C \left[ M_{\Omega, \alpha + \varepsilon}f(x) \right]^{1/2} \cdot \left[ M_{\Omega, \alpha - \varepsilon}f(x) \right]^{1/2}, \quad x \in \mathbb{R}^n,
\]

where \( C \) depends only on \( \varepsilon, \alpha, n \).

**Proof.** The proof will follow after [We]. Given \( x \in \mathbb{R}^n \) and \( \varepsilon > 0 \) with \( 0 < \alpha - \varepsilon < \alpha + \varepsilon < n \), we choose a \( \delta > 0 \) such that

\[
\delta^2 = M_{\Omega, \alpha + \varepsilon}f(x)/M_{\Omega, \alpha - \varepsilon}f(x).
\]

Now we put

\[
T_{\Omega, \alpha - \varepsilon}f(x) = \int_{|x-y|<\delta} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y) dy + \int_{|x-y|\geq \delta} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y) dy
\]

\[
:= I_1 + I_2.
\]
Thus
\[
|I_1| \leq \sum_{j=0}^{\infty} \int_{2^{j-1} \delta \leq |x-y| < 2^j \delta} \frac{|\Omega(x-y)|}{|x-y|^{\alpha+\varepsilon}} |f(y)| \, dy
\]
\[
\leq \sum_{j=0}^{\infty} (2^{j-1} \delta)^{-(\alpha+\varepsilon)} \int_{|x-y| < 2^j \delta} |\Omega(x-y)| |f(y)| \, dy
\]
\[
= 2^n \sum_{j=0}^{\infty} (2^{j-1} \delta)^{-\varepsilon} \int_{|x-y| < 2^j \delta} |\Omega(x-y)| |f(y)| \, dy
\]
\[
\leq C \cdot \delta^{-\varepsilon} \cdot M_{\Omega, \alpha+\varepsilon} f(x).
\]
Similarly,
\[
|I_2| \leq \sum_{j=0}^{\infty} \int_{2^{j-1} \delta \leq |x-y| < 2^j \delta} \frac{|\Omega(x-y)|}{|x-y|^{\alpha+\varepsilon}} |f(y)| \, dy
\]
\[
\leq C \sum_{j=1}^{\infty} (2^{j-1} \delta)^{-\varepsilon} \int_{|x-y| < 2^j \delta} |\Omega(x-y)| |f(y)| \, dy
\]
\[
\leq C \cdot \delta^{-\varepsilon} \cdot M_{\Omega, \alpha+\varepsilon} f(x).
\]
Therefore, we get
\[
|T_{\Omega, \alpha} f(x)| \leq C \left[ \delta^\varepsilon M_{\Omega, \alpha+\varepsilon} f(x) + \delta^{-\varepsilon} M_{\Omega, \alpha+\varepsilon} f(x) \right]
\]
and so with the above election of \( \delta \) the lemma is proved.

**The Proof of Theorem 1.** Under the conditions of Theorem 1, by Lemma 3, there exists an \( \varepsilon > 0 \) such that \( 0 < \varepsilon < \alpha < \alpha+\varepsilon < n, 1/p > (\alpha+\varepsilon)/n, \omega^\alpha \in A(p, q, \alpha+\varepsilon) \) and \( \omega \in A(p, q, \alpha+\varepsilon) \), where \( 1/q = 1/p - (\alpha+\varepsilon)/n \) and \( 1/q \geq 1/p - (\alpha+\varepsilon)/n \). Now let \( l_1 = 2 \alpha+\varepsilon, l_2 = 2 \alpha+\varepsilon, \) then \( 1/l_1 + 1/l_2 = 1 \). For above given \( \varepsilon > 0 \), using Lemma 4 and Hölder’s inequality, we have
\[
||T_{\Omega, \alpha} f||_{L^{q, \omega}} \leq C \left( \int_{R^d} |M_{\Omega, \alpha+\varepsilon} f(x) \omega(x)|^{q_1/2} \left[ M_{\Omega, \alpha-\varepsilon} f(x) \omega(x) \right]^{q_2/2} \, dx \right)^{1/q_1}
\]
\[
\leq C \left( \int_{R^d} |M_{\Omega, \alpha+\varepsilon} f(x) \omega(x)|^{q_1/2} \, dx \right)^{1/q_1} \left( \int_{R^d} \left[ M_{\Omega, \alpha-\varepsilon} f(x) \omega(x) \right]^{q_2/2} \, dx \right)^{1/q_2}
\]
\[
= C \left( \int_{R^d} |M_{\Omega, \alpha+\varepsilon} f(x) \omega(x)|^{q_1} \, dx \right)^{1/2q_1} \left( \int_{R^d} \left[ M_{\Omega, \alpha-\varepsilon} f(x) \omega(x) \right]^{q_2} \, dx \right)^{1/2q_2}.
\]
Therefore, from Lemma 3 and Proposition 1, it follows that
\[
\left( \int_{R^d} |M_{\Omega, \alpha+\varepsilon} f(x) \omega(x)|^{q_1} \, dx \right)^{1/2q_1} \leq C ||f||_{L^{q_1, \omega}}^{1/2q_1}
\]
and
\[
\left( \int_{R^d} \left[ M_{\Omega, \alpha-\varepsilon} f(x) \omega(x) \right]^{q_2} \, dx \right)^{1/2q_2} \leq C ||f||_{L^{q_2, \omega}}^{1/2q_2}.
\]
Hence, we obtain
\[
||T_{\Omega, \alpha} f||_{L^{q, \omega}} \leq C ||f||_{L^{q_1, \omega}}.
\]
Let us now turn to the proof of Theorem 2. In fact, Theorem 2 is a consequence of Theorem 1 by duality. To see this, let \( \mathcal{T} := \mathcal{T}_{\Omega, \sigma} \) be the adjoint operator of \( \mathcal{T}_{\Omega, \sigma} \), that means \( \mathcal{T}_{\Omega, \sigma} = \mathcal{T}_{\Omega, \sigma}^* \) with \( \mathcal{\Omega}(x) = \overline{\Omega}(x) \). Obviously, \( \mathcal{\Omega} \) is also homogeneous of degree zero and satisfies the same essential inequalities as \( \Omega \). Thus, we have

\[
\| \mathcal{T}_{\Omega, \sigma} f \|_{\Omega, \omega} = \sup_{g} \left| \int_{\mathbb{R}^n} \mathcal{T}_{\Omega, \sigma} f(x) g(x) \, dx \right|
\]

where the supremum is taken over all \( g(x) \) with \( \|g\|_{\omega'} \leq 1 \). Since \( \mathcal{T} \) is the adjoint operator of \( \mathcal{T}_{\Omega, \sigma} \), then

\[
\int_{\mathbb{R}^n} \mathcal{T}_{\Omega, \sigma} f(x) g(x) = \int_{\mathbb{R}^n} f(x) \cdot \mathcal{T} g(x) \, dx.
\]

Hence,

\[
\| \mathcal{T}_{\Omega, \sigma} f \|_{\Omega, \omega} = \sup_{q} \left| \int_{\mathbb{R}^n} f(x) \cdot \mathcal{T} g(x) \, dx \right|
\]

\[
\leq \|f\|_{\omega, \omega'} \cdot \sup_{g} \|g\|_{\omega'}.
\]

By the condition of Theorem 2, we see that \( 1/q = 1/p - \alpha/n \) and \( 1 \leq p < q < s \). Thus, \( 1/p' = 1/q' - \alpha/n \) and \( s' < q' < n'/\alpha \). From \( q^{-1} = A(q'/s', p'/s') \) and Theorem 1, it follows that

\[
\| \mathcal{T} g \|_{\omega', \omega'} \leq C\|g\|_{\omega, \omega'}.
\]

Therefore,

\[
\| \mathcal{T}_{\Omega, \sigma} f \|_{\Omega, \omega} \leq \|f\|_{\omega, \omega'} \cdot \sup_{g} \|g\|_{\omega'} \leq C\|f\|_{\omega, \omega'}.
\]

This finishes the proof of Theorem 2.

Finally, let us point out that Proposition 2 is a direct consequence of Theorem 2 and the following lemma, which shows that \( M_{\Omega, \sigma}(f)(x) \) can be controlled pointwise by \( T_{[\Omega, \sigma]}(|f|)(x) \) for any \( f(x) \).

**LEMMA 5.** Let \( 0 < \alpha < n \), \( \Omega \in L^1(S^{n-1}) \). Then we have

\[
M_{\Omega, \sigma}(f)(x) \leq T_{[\Omega, \sigma]}(|f|)(x).
\]

In fact, fix \( r > 0 \), we have

\[
T_{[\Omega, \sigma]}(|f|)(x) \geq \int_{|x-y| < r} \frac{\Omega(x-y)}{|x-y|^{n-\sigma}} |f(y)| \, dy
\]

\[
\geq \frac{1}{r^{n-\sigma}} \int_{|x-y| < r} \Omega(x-y) |f(y)| \, dy.
\]

Taking the supremum for \( r > 0 \) on two sides of the inequality above, we get

\[
T_{[\Omega, \sigma]}(|f|)(x) \geq \sup_{r > 0} \frac{1}{r^{n-\sigma}} \int_{|x-y| < r} \Omega(x-y) |f(y)| \, dy.
\]

This is just our desired conclusion.
4. The proof of Theorem 3. Let us first state a lemma which is easily deduced from the Stein-Weiss interpolation theorem with change of measures (see [BL], p. 120).

**Lemma 6.** Let \(0 < \alpha < n, 1 < p_0 < p_1 < n/\alpha, 1/q_0 = 1/p_0 - \alpha/n, and 1/q_1 = 1/p_1 - \alpha/n\). If linear operator \(T\) is a bounded operator from \(L^{p_0}(\omega_0^n)\) to \(L^{p_0}(\omega_0^n)\) and from \(L^{p_1}(\omega_1^n)\) to \(L^{p_1}(\omega_1^n)\) with norms \(C_0\) and \(C_1\) respectively, then \(T\) is also a bounded operator from \(L^{p}(\omega^n)\) to \(L^{q}(\omega^n)\) with norm \(C\), where \(0 < \theta < 1, C \leq C_0^{-\theta} C_1^\theta, 1/p = (1 - \theta)/p_0 + \theta/p_1, 1/q = 1/p - \alpha/n, and \omega = \omega_0^{-\theta} \omega_1^\theta\).

Let us now turn to prove Theorem 3. If we can prove that there exist \(\theta(0 < \theta < 1), p_0, p_1, q_0, q_1, \theta_0\) and \(\theta_1\) such that
\[
1 \leq s' < p < p_1 < n/\alpha, \\
\frac{n}{n - \alpha} < q_0 < q < q_1 < s, \\
1/q_0 = 1/p_0 - \alpha/n, 1/q_1 = 1/p_1 - \alpha/n, 1/p = (1 - \theta)/p_0 + \theta/p_1, \\
\omega = \omega_0^{1-\theta} \cdot \omega_1^\theta,
\]
and
\[
\omega_0^{s'} \in A(p_0/s', q_0/s'), \omega_1^{-s'} \in A(q_1/s', p_1/s'),
\]
then the conclusion of Theorem 3 will be deduced from Theorem 1, Theorem 2 and Lemma 6. Therefore, it suffices to seek above \(\theta, p_0, p_1, q_0, q_1, \omega_0\) and \(\omega_1\) such that (4.1)–(4.5) hold.

Since there is an \(r, 1 < r < s/(\frac{n-\alpha}{n})\), such that \(\omega^{r/q} \in A(p, q)\), it follows from (2.7) that \(\omega^{r/q} \in A_{q(n-\alpha)/n}\). However, it follows from (2.6) that there exist \(u(x), v(x) \in A_1\) such that
\[
\omega(x)^{r/q} = u(x) \cdot v(x)^{1-(q(n-\alpha)/n)},
\]
or
\[
\omega(x) = u(x)^{1/r} \cdot v(x)^{1/r' - (n-\alpha)/r' n}.
\]
By (4.6), we can write \(\omega(x)\) as
\[
\omega = (u^riv^bj)^{1-\theta} (u^riv^b)^{\theta},
\]
where
\[
\tau(1 - \theta) + \gamma\theta = 1/r' q, \quad \beta(1 - \theta) + \delta\theta = 1/r' q - (n-\alpha)/r' n.
\]
Now we denote $\omega_0(x) = u(x)^q v(x)^{s_0}$ and $\omega_1(x) = u(x)^q v(x)^{s_1}$. We shall see that if $1 \leq s' < p_0 < p < n'$ and $1/q_0 = 1/p_0 - \alpha/n$, then when $\tau = 1/q_0$ and $\beta = -1/s'(p_0')$, we have $\omega_0' \in A(p_0'/s', q_0'/s')$. In fact, since $u(x), v(x) \in A_1$, we have

\[
\left(\frac{1}{|Q|} \int_Q [\omega_0(x)^q]^{p_0/s'} \, dx\right)^{1/p_0} \leq C\left(\frac{1}{|Q|} \int_Q u(x)^{p_0} v(x)^{p_0') \, dx\right)^{1/p_0} \leq C, \tag{4.8}
\]

where $C$ is independent of $Q$. By the same method, we can prove that if $n/(n-\alpha) < q < q_1 < s$ and $1/q_1 = 1/p_1 - \alpha/n$, then when $\gamma = -1/p', \delta = 1/s'(q_1')$, we have $\omega_1' \in A(q_1'/s', p_1'/s')$.

Let us now figure $\theta$ out by (4.8). Note that

\[
\beta = \left\{ s' \left(\frac{p_0}{s'}\right)\right\}^{-1} = 1/p_0 - 1/s'
\]

and

\[
\delta = \left\{ s' \left(\frac{q_1'}{s'}\right)\right\}^{-1}.
\]

Thus, it follows from (4.8) that

\[
\theta = \frac{\tau - \beta - (n-\alpha)/\alpha' n}{\delta - \gamma - \beta + \tau} = \frac{1/s' - \alpha/n - (n-\alpha)/\alpha' n}{2(1/s' - \alpha/n)}. \tag{4.9}
\]

Since $1 < r < s/(\frac{\alpha'}{n})$, we may write $\frac{1}{r} = \frac{1}{n} - \frac{1}{n} \left(\frac{1}{s'} + \varepsilon\right)$, where $\varepsilon > 0$. Thus,

\[
1/s' - \alpha/n - (n-\alpha)/\alpha' n = \frac{1}{s'} - \frac{1}{n} - \frac{n-\alpha}{n} \left[1 - \frac{n}{n-\alpha} \left(\frac{1}{s'} + \varepsilon\right)\right] = \varepsilon,
\]

and then $\theta = \varepsilon/2(\frac{1}{s'} - \frac{1}{n})$. Since $s' < n/\alpha$, we have $\theta > 0$. On the other hand, we easily see that $\theta < 1$ by (4.9). Therefore, $0 < \theta < 1$ and (4.4), (4.5) hold by the above estimates. It remains to prove that we can choose proper $p_0, p_1, q_0$ and $q_1$ such that (4.1)–(4.3) hold. Since $1/p > \alpha/n + 1/s$ and $\theta > 0$, we have

\[
\frac{1}{p(1-\theta)} - \frac{\alpha \theta}{n(1-\theta)} - \frac{\theta}{s(1-\theta)} > \frac{1}{p}. \tag{4.10}
\]
By (4.10) and $1/p < 1/s'$, we can choose $p_0$ such that

$$\frac{1}{p} < \frac{1}{p_0} < \min \left\{ \frac{1}{s'}, \frac{1}{p(1-\theta)} - \frac{\alpha\theta}{n(1-\theta)} - \frac{\theta}{s(1-\theta)} \right\}.$$

Thus, we have $s' < p_0 < p$ and $1/p > (1-\theta)/p_0 + \alpha\theta/n$. Therefore, there exists a $\sigma > 0$ such that

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \left( \frac{\alpha}{n} + \sigma \right) \theta.$$

Let us denote $q_1 = \frac{1}{p_1} = \frac{q_0}{p_0} + \sigma$. Then it follows from $1/p_1 > \alpha/n$ and $1/p < 1/p_0$ that $s' < p_0 < p < p_1 < n/\alpha$. This proves (4.1). Also (4.3) holds by (4.12). Now let us denote $1/q_0 = 1/p_0 - \alpha/n$ and $1/q_1 = 1/p_1 - \alpha/n$. Obviously, by (4.11), we have

$$\frac{1}{p_0} - \frac{1-\theta}{p_0\theta} - \frac{\alpha}{n} > \frac{1}{s}.$$

However, the above is equivalent to $1/p_1 - \alpha/n > 1/s$. Thus, $q_1 < s$, and therefore (4.2) holds. Hence, we finish the proof of Theorem 3.

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