COMPLEMENTS TO MOONS' LUNAR LIBRATION THEORY

Comparisons and fits to JPL numerical integrations

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Abstract. Analytical complements have been brought to Moons' lunar libration theory concerning tidal effects, direct perturbations due to the Earth's figure, and indirect non periodic perturbations. Comparisons to JPL numerical integrations DE245 and DE403 have been performed and the residuals treated by frequency analysis, allowing the determination of fitted free libration parameters and numerical complements.

1. Introduction

Elementary descriptions of lunar librations are given in (Danjon, 1959) or in (Hilton, 1992). Since 1975, several precise solutions have been developped in connection with the appearance of Lunar Laser Ranging observations. Moons' analytical theory (Moons, 1981, 1982, and 1984) includes series for the forced libration and series, denoted as "free libration series", which contain pure free libration terms and mixed terms depending on free libration and forced libration. Forced libration series involve :

- Main problem, in which the Earth is reduced to its mass center, the selenocentric position of the Earth is provided by the main problem of the lunar orbital motion, and the position of the Sun with respect to the Earth-Moon barycenter is approximated by a pseudo-keplerian motion in the mean ecliptic of date ;
- Indirect planetary perturbations, derived from the periodic part of the planetary perturbations of the lunar orbital motion ;
- Indirect perturbations due to the Earth's figure, derived from the perturbations of the lunar orbital motion by the Earth's figure ;
- Direct and indirect perturbations due to the ecliptic motion.

The coefficients of main problem series are literal with respect to increments to nominal values of the parameters $\beta = (C-A)/B$ and $\gamma = (B-A)/C$, and literal with respect to the ratio of third and fourth degree harmonic coefficients C_{ij} and S_{ij} ($i = 3,4,\ 0 \le j \le i$) to $C/m_L R_L^2$. A, B, C are the lunar principal moments of intertia, m_L and R_L the lunar mass and equatorial radius. The coefficients of the indirect planetary perturbation series are literal in a similar way as the main problem, while the coefficients of the other perturbation series are numerical. The arguments are combinations of Delaunay arguments D, F, l, l', and, for perturbations, of planetary longitudes and ζ, the lunar mean mean longitude referred to the mean equinox of date.

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The coefficients of free libration series are literal with respect to three free libration parameters $\sqrt{2P}$, $\sqrt{2Q}$, $\sqrt{2R}$ and are otherly similar to the main problem series, except that fourth degree harmonics are not taken into account. The arguments are combinations of Delaunay arguments and of three arguments p, q, r of the free libration.

The series yield p_1, p_2, and τ. p_1 and p_2 are the components of the unit vector pointing toward the pole of the mean ecliptic of date, in the inertial sense as defined by Standish (1981), on the two lunar equatorial principal axes of inertia ; τ is the libration in longitude referred to the inertial mean ecliptic of date.

Moons' theory takes into account a rigid body. The perturbations due to the deformation of the Moon by the Earth, the Sun and lunar rotation (tidal perturbations) are missing, except those derived from constant perturbations of β, γ and $C/m_L R_L^2$ which may be included. The direct perturbations due to the Earth's figure and to the planets are also missing. At last, the indirect perturbations derived from the part of the perturbations of the lunar orbital motion which contains the time as a factor (Poisson terms), or purely secular terms, have not been computed.

In Sect. 2 we give analytical expressions of some of those missing perturbations for the forced libration only:

- Tidal perturbations in the case of an elastic model and in two examples of an anelastic model ;
- Direct perturbations due to the Earth's figure ;
- A rough estimate of two Poisson terms among the indirect planetary perturbations of τ. They come from Poisson terms of the lunar orbital motion due to Venus action and to secular variation of the solar eccentricity.

Sect. 3 gives a comparison of the so-completed Moons' theory to the numerical integrations DE245 and DE403 of the Jet Propulsion Laboratory (JPL). The residuals are analyzed by means of a frequency analysis which puts into evidence the "free libration terms" and allows to derive fitted values of the free libration parameters.

The frequency analysis also allows to complete the analytical solution by a small number of trigonometric terms whose coefficients, frequencies, and phases are purely numerical. Sect. 4 shows the resulting improvements on residuals of Lunar laser ranging observations.

2. Analytical Complements to Moons' Theory

2.1. Method

The method is similar to Eckhardt's (1981). The column matrix X, whose elements are p_1, p_2, τ, is given by the differential equation:

$$TX = (T - R)X + Y + Y' + \Psi. \tag{1}$$

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R is a differential operator function of the components ω_i of ω, referred to the lunar principal axes of inertia ; ω is the angular rotational velocity vector of the Moon with respect to the usual reference frame of the lunar motion (mean ecliptic of date and departure point). Y results from the lunar potential and is expressed as a function of the selenocentric coordinates of the Earth y_i referred to the lunar principal axes of inertia. T is a linear differential operator such that $(T-R)X+Y$ does not contain any linear term with constant coefficient in p_1, p_2, τ and their derivatives at first order of the small parameters involved. The expressions of TX, RX, and Y used in this paper can be found in (Chapront-Touzé, 1990) except that ε is denoted as γ in the present paper and $\nu^2 a_0^3$ must be replaced by Gm_T, G being the constant of gravitation and m_T the terrestrial mass. Y' is obtained from Y by changing the selenographic coordinates of the Earth y_i to those of the Sun y_i' and m_T to the solar mass m_S. Ψ is a vectorial disturbing function.

The leading effect of Ψ is to add to the solution X_M of the main problem of the forced libration the correction ΔX given by:

$$T\Delta X = \left[\frac{\partial}{\partial X}(T-R)X\right]\Delta X + \Psi + \left[\frac{\partial}{\partial X}Y\right]\Delta X + \left[\frac{\partial}{\partial X}Y'\right]\Delta X. \quad (2)$$

Ψ and the jacobian matrices $[\partial\ /\partial X \cdots]$ are computed for X_M by disregarding the contribution of the free libration. Similarly, the contribution of Ψ to the free libration has been disregarded. Eq. (2) is solved by two iterations, ΔX being set to zero in the right hand member at the first iteration.

2.2. Direct perturbations due to tidal effects

The actions of the Earth, the Sun and lunar rotation induce distortions of the lunar surface which, in turn, induce an additional lunar potential (Lambeck, 1980). This additional potential is equivalent to time dependent corrections ΔC_{ij}, ΔS_{ij} to the constant harmonic coefficients C_{ij}, S_{ij} of the potential of the rigid Moon. Restricting ourselves to harmonics of degree 2 and disregarding the Sun effect, we have:

$$\begin{aligned}
\Delta C_{20} &= k_2 \frac{m_T}{m_L}\frac{R_L^3}{r^{*5}}\left(y_3^{*2} - \frac{1}{2}y_1^{*2} - \frac{1}{2}y_2^{*2}\right) + \frac{1}{6}\frac{k_2 R_L^3}{Gm_L}\left(\omega_1^{*2} + \omega_2^{*2} - 2\omega_3^{*2}\right) \\
\Delta C_{21} &= k_2 \frac{m_T}{m_L}\frac{R_L^3}{r^{*5}} y_3^* y_1^* - \frac{1}{3}\frac{k_2 R_L^3}{Gm_L}\omega_1^*\omega_3^* \\
\Delta S_{21} &= k_2 \frac{m_T}{m_L}\frac{R_L^3}{r^{*5}} y_2^* y_3^* - \frac{1}{3}\frac{k_2 R_L^3}{Gm_L}\omega_2^*\omega_3^* \qquad (3) \\
\Delta C_{22} &= \frac{k_2}{4}\frac{m_T}{m_L}\frac{R_L^3}{r^{*5}}\left(y_1^{*2} - y_2^{*2}\right) + \frac{1}{12}\frac{k_2 R_L^3}{Gm_L}\left(\omega_2^{*2} - \omega_1^{*2}\right) \\
\Delta S_{22} &= \frac{k_2}{2}\frac{m_T}{m_L}\frac{R_L^3}{r^{*5}} y_1^* y_2^* - \frac{1}{6}\frac{k_2 R_L^3}{Gm_L}\omega_1^*\omega_2^*.
\end{aligned}$$

TABLE I

Tidal perturbations of the lunar libration for three cases: no time delay, a constant time delay of 0.16485 day, a constant lag angle of 2°.1721. The value of the Love number is 0.02992. The coefficients of cosine and sine terms are given in arcsec. Values between parenthesis are derived from (Yoder, 1979) and given for comparison

	No time delay		Constant time delay		Constant lag angle	
Argument	cos	sin	cos	sin	cos	sin
Variable p_1						
F	0	0	0.2733	0.0049	0.2647	0.0045
		(−0.0009)	(0.2710)	(−0.0009)	(0.2710)	(−0.0009)
$F-l$	0	0.0587	−0.0153	0.0587	−0.0170	0.0586
		(0.0625)	(0.0030)	(0.0625)	(0.0015)	(0.0625)
0	−0.0240		−0.0240		−0.0240	
$2D-F-l$	0	−0.0012	−0.0003	−0.0012	−0.0004	−0.0012
$F-2l$	0	0	−0.0007	−0.0001	−0.0003	−0.0001
			(−0.0019)		(−0.0017)	
Variable p_2						
F	0.0134	0	0.0184	−0.2718	0.0179	−0.2632
	(0.0129)		(0.0129)	(−0.2710)	(0.0129)	(−0.2710)
$F-l$	0.0112	0	0.0112	0.0196	0.0111	0.0219
	(0.0284)		(0.0284)	(0.0314)	(0.0284)	(0.0351)
$F-2l$	0	0	0	−0.0008	0	−0.0003
				(−0.0019)		(−0.0017)
0	0		−0.0006		−0.0005	
	(0.0760)		(0.0760)		(0.0760)	
Variable r						
0	0		0.3971		0.3846	
			(0.3974)		(0.3974)	
$2F-2l$	0	−0.0103	−0.0475	−0.0104	−0.0263	−0.0099
		(0.0024)	(0.0045)	(0.0024)	(0.0192)	(0.0024)
l	0	−0.0059	−0.0011	−0.0059	−0.0011	−0.0059
		(−0.0057)		(−0.0057)		(−0.0057)
$2D-2l$	0	0.0004	−0.0058	0.0005	−0.0047	0.0004
			(0.0057)		(0.0067)	
$2D-l$	0	−0.0014	−0.0004	−0.0014	−0.0004	−0.0014
		(−0.0015)		(−0.0015)		(−0.0015)
l'	0	0	0.0003	0	0.0092	0.0001
					(0.0096)	

r is the Earth-Moon distance, k_2 is a Love number. The exponent $*$ means that, in the computation of ΔC_{ij} and ΔS_{ij} at time t, the function must be evaluated at time $t - t_0$; t_0 is a time delay, equal to zero for an elastic model of the Moon, and constant for a viscous model. Similarly to (Yoder, 1979) a third model, with t_0 inversely proportional to the absolute value of the frequency of each term in which it is involved, has also been considered. If the sign of the coefficient of each term is determined so that the frequency is always positive, this case corresponds to a constant lag angle. The terms due to solar action in Eq. (3) are obtained by changing m_T to m_S, y_i to y'_i, and r to r' (Sun-Moon distance).

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Corrections ΔC_{ij} induce a time dependent corrective tensor ΔI to the constant tensor of inertia of the rigid Moon I. Assuming that the trace of ΔI is zero, elements of ΔI are:

$$\begin{aligned}
\Delta I_{11} &= m_L R_L^2 \left(\tfrac{1}{3}\Delta C_{20} - 2\Delta C_{22}\right) & \Delta I_{12} &= -2 m_L R_L^2 \Delta S_{22} \\
\Delta I_{22} &= m_L R_L^2 \left(\tfrac{1}{3}\Delta C_{20} + 2\Delta C_{22}\right) & \Delta I_{23} &= -m_L R_L^2 \Delta S_{21} \\
\Delta I_{33} &= -\tfrac{2}{3} m_L R_L^2 \Delta C_{20} & \Delta I_{13} &= -m_L R_L^2 \Delta C_{21}.
\end{aligned}$$

ΔI induces in Eq. (1) the disturbing function Ψ whose components are given by:

$$\begin{aligned}
I_{ii}\Psi_i = &-\textstyle\sum_{j=1}^{3}\left[\Delta \dot{I}_{ij}\omega_j + \Delta I_{ij}\dot{\omega}_j\right] + \sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}\times \\
&\left[\Delta I_{jk}\left(\omega_k^2 - \tfrac{3Gm_T}{r^5}y_k^2 - \tfrac{3Gm_S}{r'^5}y_k'^2\right) + \Delta I_{ij}\left(\omega_k\omega_i - \tfrac{3Gm_T}{r^5}y_i y_k - \tfrac{3Gm_S}{r'^5}y_i' y_k'\right)\right. \\
&\left. + \Delta I_{jj}\left(\omega_j\omega_k - \tfrac{3Gm_T}{r^5}y_j y_k - \tfrac{3Gm_S}{r'^5}y_j' y_k'\right)\right]
\end{aligned}$$

with $\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj}$ and $\varepsilon_{123} = 1$. I_{ii} are respectively the principal moments of inertia A, B, C.

Table I gives the perturbations obtained in the three cases mentioned above. These perturbations involve the complete direct tidal effects by the Earth and the Sun, but the constant contribution of the lunar rotation to ΔC_{20} and ΔC_{22}, and consequently to ΔI_{ii}, has been removed. This contribution is supposed to be included in the parameters of the rigid Moon. Table I gives also, for comparison, the corresponding quantities derived from Yoder's results (1979) by converting his complex variable p to p_1 and p_2. Coefficients of terms whose amplitudes are smaller than $0''.001$ in both solutions are not given. The greatest difference concerns a constant term of $0''.0760$ in Yoder's results for p_2 which is much smaller in ours. In the opposite, we have a constant term in p_1 which does not exist in Yoder's results.

2.3. Direct Perturbations due to the Earth's Figure

In this section we have supposed that the Earth has a rotational symetry around its polar axis. Furthermore, in the conversion of terrestrial body fixed coordinates to lunar ones we have neglected the libration and the nutation since the corresponding quantities should be mutiplied by the Earth's J_2.

By expressing the elements of the Earth's tensor of inertia with respect to the lunar principal axes of inertia, and by subtituting the results in the expressions given by Schutz (1981), we obtain for the components of the disturbing function induced by the Earth's figure in Eq. (1):

$$\Psi_i = -\frac{3Gm_T\alpha_i J_2 R_T^2}{r^5}\left[\frac{35}{2}\frac{y_j y_k}{r^4}D^2 - \frac{5}{2}\frac{y_j y_k}{r^2} - \frac{5}{r^2}(\xi_k y_j + \xi_j y_k)D + \xi_j\xi_k\right].$$

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TABLE II

Direct perturbations due to the Earth's figure. The coefficients of cosine and sine terms are given in arcsec. Values between parenthesis are reproduced from (Pešek, 1982) and are given for comparison

	Argument	cos	sin
p_1	ζ	0	−0.0729 (−0.0725)
	F	0	0.0121 (0.0108)
p_2	ζ	−0.0729 (−0.0726)	0
	F	0.0121 (0.0108)	0
τ	$\zeta - F$	0	−0.0099 (−0.0067)

i, j, k verify $\varepsilon_{ijk} = 1$ and D stands for $\xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3$. ξ_i are the components of the unit vector pointing towards the Earth's pole referred to the lunar principal axes of inertia (respectively here $-\sin\zeta \sin\varepsilon$, $-\cos\zeta \sin\varepsilon$, and $\cos\varepsilon$, ε being the mean obliquity of date). α_i stand respectively for α, $-\beta$, and γ.

Table II gives the resulting perturbations on the forced libration, and, for comparison, the results obtained by Pešek (1982). Coefficients of terms whose amplitudes are smaller than $0''.001$ in both solutions are not given. Our results are in good agreement with Pešek's ones for p_1 and p_2. The difference of $0''.003$ in τ comes from our second iteration in the resolution of Eq. (2).

2.4. NON PERIODIC INDIRECT PERTURBATIONS OF τ

By disregarding all terms of upper orders in Eq. (1), we obtain the following separate equation in τ:

$$\ddot{\tau} + 3\nu^2\gamma\tau = 3\nu^2\gamma(L - \overline{\lambda}) \tag{4}$$

where L is the lunar longitude, $\overline{\lambda}$ the mean mean longitude, and ν the sidereal mean motion.

For the main problem $\overline{\lambda}$ is a linear function of time t. Secular variations of the solar eccentricity and tidal perturbations introduce in L secular terms in t^2, t^3's Eq. (4) shows that term At^n in L induces the same term At^n in τ. It induces also terms at lower powers of t which may be disregarded because their coefficients are either zero or quantities much smaller than A for successive divisions by $3\nu^2\gamma$ (about 48 000 rad/cy). The existence of a t^2 term in τ has been mentioned yet by Bois et al. (1996). Nevertheless, since τ always appears through $\overline{\lambda} + \tau$, e.g. in the expression of matrix M transforming ecliptic coordinates x_i to lunar body fixed coordinates y_i (Chapront-Touzé, 1990), it is simpler to consider that secular terms At^n are involved in $\overline{\lambda}$, constituting the mean mean longitude W_1 of the orbital motion, and that τ contains only periodic and Poisson terms.

Eq. (4) shows that Poisson term $At \sin\varphi$ in L induces in τ the terms $A't \sin\varphi + B' \cos\varphi$ with :

$$A' = \frac{3\nu^2\gamma A}{3\nu^2\gamma - \dot{\varphi}^2}, \qquad B' = \frac{-6\nu^2\gamma A\dot{\varphi}}{(3\nu^2\gamma - \dot{\varphi}^2)^2}$$

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Hence, the Poisson terms

$$\Delta L = 0''.254\,25\, t \sin(18V - 16T - l + 114^\circ.565\,50) + 1''.676\,80\, t \sin l'$$

in the lunar longitude (Chapront-Touzé and Chapront, 1983) induce in τ Poisson terms which, following Eq. (4), are:

$$\Delta\tau = 0''.2543\, t \sin(18V - 16T - l + 114^\circ.5655) - 0''.2334\, t \sin l' \qquad (5)$$

t is the time in century reckoned from J2000.0, V and T are the mean mean longitudes of Venus and the Earth respectively, l' is the solar mean anomaly. Eq. (5), which is only an approximation, shows the interest of computing Poisson terms in the forced libration.

3. Comparisons with JPL Numerical Integrations

Several kinds of comparisons with JPL numerical integrations DE245 and DE403 have been performed. Two time spans Δt have been chosen to cover the periods of comparison which are of 300 and 600 years in the case of DE245 and DE403 respectively. We have used two different JPL integrations as reference models to insure the numerical consistency of our analysis and provide several sets of libration parameters depending on the model. The general scheme of our analysis is the following: Euler angles in JPL integrations are transformed into the libration variables p_1, p_2 and τ. The analytical solution (A) is computed using a set of parameters consistent with the JPL numerical integration (N). We compute the differences $\delta = (\mathrm{N}) - (\mathrm{A})$ and perform a frequency analysis of the "residuals" δ. Once we have determined the significant frequencies ω_i of the spectrum, a least square fit of the residuals is done in order to obtain an approximate "solution" for δ on the time interval Δt:

$$\delta = \sum_i A_i \sin(\omega_i t + \phi_i)$$

δ stands for any of the three residuals among the variables p_1, p_2 and τ ; t is the time reckoned from J2000 ; A_i and ϕ_i are the quantities provided by the least square fit. The quality of the frequency analysis strongly depends on the choice of the filtering. We have used a method proposed by (Laskar et al., 1993) which has been already successfully applied in the case of the construction of planetary ephemerides (Chapront, 1995).

Table III gives the values of the lunar physical parameters substituted in Moons' series for the comparisons, except for the values of the harmonic coefficients of degree 4 which are those of (Ferrari et al., 1980). The values of β, γ, C_{ij} and S_{ij} are the one used in the numerical integrations. The values of $C/m_L R_L^2$ are derived from the values of C_{22} used in the numerical integrations by means of the relation for a rigid body $C/m_L R_L^2 = 4C_{22}/\gamma$. β, γ, $C/m_L R_L^2$, and C_{22} are assumed to involve the constant tidal perturbation due to the lunar rotation.

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TABLE III
Physical parameters adopted for the comparison to DE245 and DE403 (from numerical integrations except for $C/m_L R_L^2$). Units of 10^{-4}

DE245		DE403	
$\beta = 6.31619133$		$\beta = 6.31610707$	
$\gamma = 2.27885980$		$\gamma = 2.27864190$	
$C_{30} = -0.086802$		$C_{30} = -0.086474$	
$C_{31} = 0.307083$	$S_{31} = 0.046115$	$C_{31} = 0.307083$	$S_{31} = 0.044875$
$C_{32} = 0.048737$	$S_{32} = 0.016975$	$C_{32} = 0.048727$	$S_{32} = 0.016962$
$C_{33} = 0.017161$	$S_{33} = -0.002844$	$C_{33} = 0.017655$	$S_{33} = -0.002744$
$C/m_L R_L^2 = 3948.72400$		$C/m_L R_L^2 = 3950.29692$	

A first type of analysis has been done to test the improvements due to the analytical complements described in Sect. 2 (solution SOL2) with respect to Moons' original solution (SOL1). The tidal perturbations introduced in SOL2 correspond to a constant time delay. Missing arguments in SOL1 were detected in the frequency analysis, and compared with those of Tables I and II. These comparisons show a good agreement between numerical A_i, ϕ_i and ω_i and the analytical ones for all the arguments of Table I, in particular for the constant terms of p_1 and p_2 which differ from those of Yoder (1979). This agreement verifies the validity of our spectral analysis. The error is estimated to less than $0''.005$ on amplitudes A_i and less than 10^{-6} radian per day on frequencies ω_i. The agreement is not so good for the terms with argument ζ in p_1 and p_2, and $\zeta - F$ in τ (Table II). The discrepancy amounts to $0''.08$ for p_1 and p_2 and $0''.03$ for τ, but it may be due to the indirect perturbations by the Earth's figure.

A second type of analysis has been done to estimate the accuracy of the free libration series in Moons' solution, and also to determine the numerical values of the free libration parameters $\sqrt{2P}$, $\sqrt{2Q}$ and $\sqrt{2R}$. The general procedure is the following: we compute the residuals δ with SOL2 but without free libration series. Three terms of importance appear in the spectrum whose related frequencies are close to the libration frequencies ω_q, ω_p and ω_{F+r} in Moons' solution. The dominant terms of the residuals are $A[p_2]\sin(\omega_q t+\phi_q)$ in p_2, and $B[\tau]\sin(\omega_p t+\phi_p)$ in τ. The argument $F+r$ appears in p_1 (and p_2) through $C[p_1]\sin(\omega_{F+r}t+\phi_{F+r})$ with smaller amplitude. Moons' solution provides an analytical form for the free libration series which allows to compute the free libration parameters, $\sqrt{2P}$, $\sqrt{2Q}$ and $\sqrt{2R}$ from the coefficients of the above arguments, respectively $B[\tau]$, $A[p_2]$ and $C[p_1]$. The phases ϕ_p, ϕ_q, ϕ_{F+r} give the values p_0, q_0, r_0 of the free libration fundamental arguments p, q, r in J2000.0. The "observed" frequencies ω_p and ω_q replace the computed ones. ω_r is poorly determined by frequency analysis through the combination $F+r$. The theoretical computed value is retained.

The free libration parameters obtained repectively from the two integrations, as well as the main terms mentioned above, are gathered in Table IV. A complete numerical evaluation of coefficients of Moons' free libration series is given

TABLE IV
Determination of free libration parameters. The three fundamental libration arguments are: $p = \omega_p t + p_0$, $q = \omega_q t + q_0$, $r = \omega_r t + r_0$. t is reckoned from J2000.0. The frequencies are in radian per day

	DE245		DE403	
	$B[\tau] = 1''.8235$	$\phi_p = 224^\circ.303$	$B[\tau] = 1''.8122$	$\phi_p = 224^\circ.310$
	$A[p_2] = 8''.1557$	$\phi_q = 251^\circ.651$	$A[p_2] = 8''.1825$	$\phi_q = 251^\circ.777$
	$C[p_1] = 0''.0208$	$\phi_r = 217^\circ.678$	$C[p_1] = 0''.0218$	$\phi_r = 202^\circ.965$
	$\sqrt{2P} = 0.2933$	$p_0 = 224^\circ.303$	$\sqrt{2P} = 0.2915$	$p_0 = 224^\circ.310$
	$\sqrt{2Q} = 5.1924$	$q_0 = 161^\circ.640$	$\sqrt{2Q} = 5.2095$	$q_0 = 161^\circ.766$
	$\sqrt{2R} = 0.0208$	$r_0 = 124^\circ.394$	$\sqrt{2R} = 0.0218$	$r_0 = 109^\circ.681$
	Computed	"Observed"	Computed	"Observed"
ω_p	0.0060467320	0.0059492451	0.0060466648	0.0059492451
ω_q	0.0002281306	0.0002304932	0.0002281236	0.0002304970
ω_{F+r}	0.2301836354	0.2301811833	0.2301836363	0.2301820813

in (Chapront and Chapront-Touzé, 1997) with free libration parameters fitted to DE245. We mention only here that, after substitution of the values of $\sqrt{2P}$, $\sqrt{2Q}$, $\sqrt{2R}$, ϕ_p, ϕ_q, ϕ_r, ω_p, and ω_q quoted in Table IV in Moons' free libration series, all the terms are in good agreement with the terms of same frequencies (within the estimated error) in the development of δ provided by the frequency analysis.

In the following comparisons, the free libration parameters of Table IV and lunar physical constants of Table III in agreement with (N) have been introduced in SOL1 and SOL2, to render the residuals δ independent of the model as much as possible.

To illustrate the improvements due to our analytical complements, we show on Fig. 1-a a comparison of SOL1 to DE245 for the variable τ (Δt = 300 years): δ = (DE245) − (SOL1). On the contemporary period the main difference is a constant ($0''.4$) due to tidal effects and oscillations whose maximum amplitudes are about $0''.2$. In the past the Poisson terms due to planetary perturbations dominate, and the total differences reach $1.''6$. On Fig 1-b, we show the differences evaluated with the improved analytical solution SOL2: δ = (DE245) − (SOL2). It remains now oscillations whose total amplitudes are less than $0''.15$ on the whole time span. Fig. 2-a and 2-b illustrate the same comparisons for variable p_1. The gain of precision between SOL1 and SOL2 is not so good in the past because Poisson terms have not yet been introduced in SOL2 for p_1 and p_2, and the beating effect in Fig. 2-b shows clearly this lack.

4. Numerical Complements to the Analytical Libration Series

The solution SOL2 contains now free libration series and again we analyze the residuals δ. The spectrum of δ is very clean. This means that, at the level of accuracy of $0.''005$, few terms are lacking in the analytical series and that the residuals can be represented by complementary series with a small number of sensible components. In Table V we have listed the complementary series as

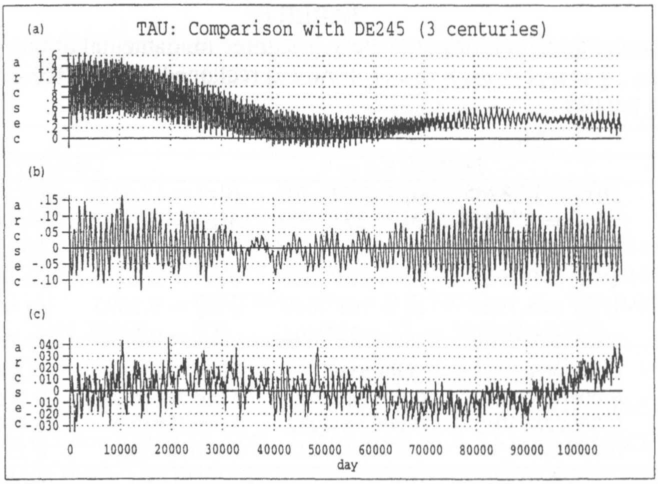

Fig. 1. Residuals on the variable τ from 1750 to 2050. 1-a : (DE245) – (SOL1). 1-b : (DE245) – (SOL2). 1-c : (DE245) – (SOL3)

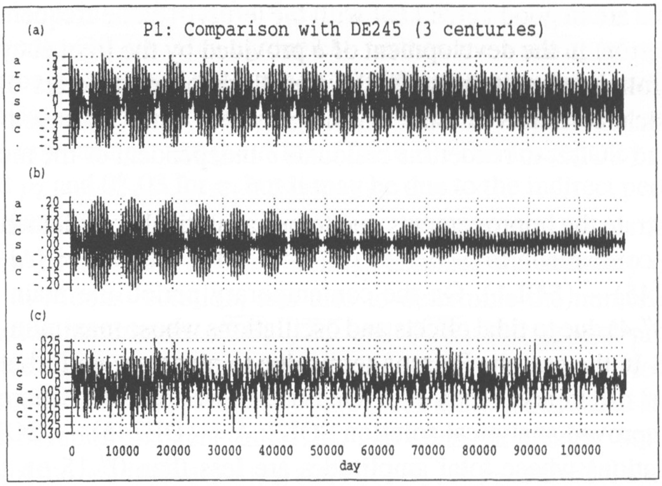

Fig. 2. Residuals on the variable p_1 from 1750 to 2050. 2-a : (DE245) – (SOL1). 2-b : (DE245) – (SOL2). 2-c : (DE245) – (SOL3)

they come from the frequency analysis on the residuals δ = (DE245) – (SOL2). Fig. 1-c and 2-c illustrate the final results of solution SOL3 (SOL2 + numerical complements of Table V) compared with the source DE245. Residuals are below $0''.03$ on the whole time span. The results of the comparison is the same when SOL3 is compared to DE403 over the same time span of three centuries, using constants of DE403 and the free libration series with values listed in Table IV for DE403. Note that numerical complements of Table V are valid only over the time span 1750 – 2050. Outside this time span, residuals slowly diverge because of

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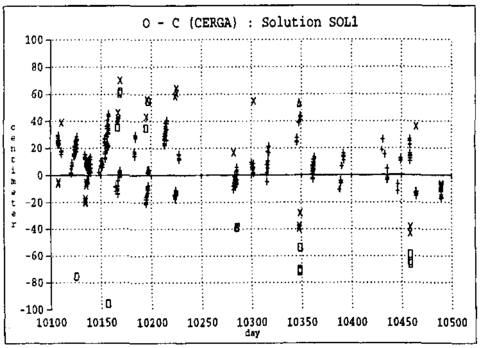

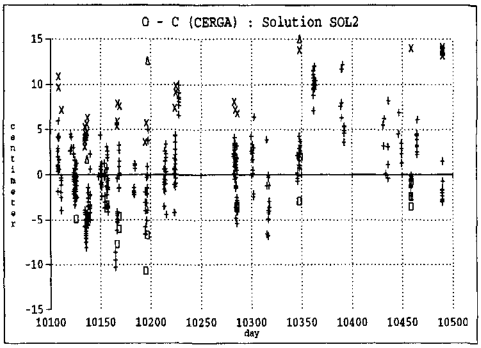

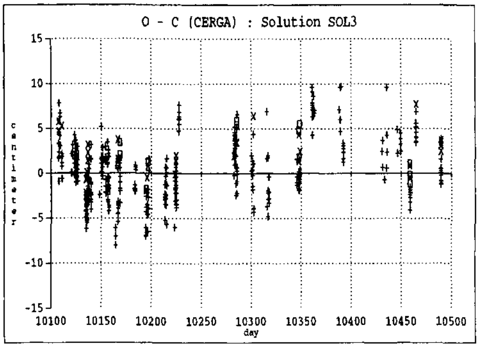

Fig. 3. O-C for the distance Observer-Reflector (Feb. 1997 – March 1998). Reflectors are: Apollo 11 (×), Apollo 14 (□), Apollo 15 (+), and Lunakhod 2 (△)

missing secular and Poisson terms.

As a final test, the solutions SOL1, SOL2 and SOL3 have been compared directly to lunar-laser observations themselves. In the three cases a large set of parameters, including reflector coordinates, has been fitted to the observations as described in (Chapront et al., 1998). We see on Fig. 3, over a time span of 400 days, the residuals O-C (Observation minus Computation) on the one-way range observer-reflector in centimeter, for CERGA observations. The most frequently observed reflector is Apollo 15, that is represented with a sign (+) on the graph; this reflector contributes mainly to the fit of parameters. We observe that the introduction of the analytical

TABLE V

Numerical complements. Series of the differences δ = (DE245) − (SOL2). Units: 10^{-4} arcsec for A, degree for ϕ, and rad/day for ω

Variable p_1			Variable p_2			Variable τ		
A	ϕ	ω	A	ϕ	ω	A	ϕ	ω
73	106.98	0.22987004	52	274.41	0.00567265	137	155.61	0.00008575
295	7.48	0.22990957	54	331.96	0.00580103	46	223.41	0.00089398
935	252.19	0.22994149	46	144.16	0.22969520	349	70.79	0.00092222
821	75.97	0.22996514	150	313.56	0.22986707	61	336.73	0.00094330
183	226.15	0.23001653	385	218.42	0.22991443	66	262.58	0.00432017
688	268.26	0.23089745	760	66.54	0.22994551	48	88.57	0.00436703
164	64.93	0.23092684	508	290.12	0.23001560	134	73.84	0.00572887
52	252.04	0.24813066	252	172.79	0.23004927	89	176.59	0.00585279
			691	5.12	0.23089738	218	271.48	0.00589170
			123	198.97	0.23094800	637	287.48	0.00596264
			48	32.81	0.24807360	146	241.37	0.00601362
			49	226.21	0.24809876	51	195.68	0.01720365

complements of this paper in Moons' solution (SOL2 instead of SOL1) produces a significant decrease of the O-C. The introduction of SOL3 instead of SOL2 makes the dispersion of the O-C with the reflectors smaller.

5. Conclusion

This study shows the good quality of Moons' libration theory. Nevertheless it needs to be completed by few missing perturbations. Two kinds of complements are given in this paper. Some futher complements should be achieved in the case of direct planetary perturbations and Poisson term contributions.

References

Bois, E., Boudin, F., Journet, A.: 1996, *Astron. Astrophys.*, **314**, 989.
Chapront, J.: 1995, *Astron. Astrophys. Suppl. Ser.*, **109**, 181.
Chapront, J., Chapront-Touzé, M.: 1997, *Celest. Mech.*, **66**, 31.
Chapront, J., Chapront-Touzé, M., Francou, G.: 1998, submitted to *Astron. Astrophys.*
Chapront-Touzé, M.: 1990, *Astron. Astrophys.*, **235**, 447.
Chapront-Touzé, M., Chapront, J.: 1983, *Astron. Astrophys.*, **124**, 50.
Danjon, A.: 1959, *Astronomie générale*, Sennac, Paris, p. 343.
Eckhardt, D.H.: 1981, *The Moon and the Planets*, **25**, 3.
Ferrari, A.J., Sinclair, W.S., Sjogren, W.L., Williams, J.G., Yoder, C.F.: 1980, *J.G.R.*, **85**, 3939.
Hilton, J.L.: 1992, in *Explanatory Supplement to the Astronomical Almanac*, P.K. Seidelmann (ed.), University Science Books, California, p. 398.
Laskar, J., Froeschlé, C, Celleti, A.: 1993, *Phys. D*, **67**, 257.
Lambeck, K.: 1980, *The Earth's Variable Rotation*, Cambridge University Press.
Moons, M.: 1981, *Libration Physique de la Lune*, Thesis, Facultés Universitaires de Namur.
Moons, M.: 1982, *The Moon and the Planets*, **27**, 257.
Moons, M.: 1984, *Celest. Mech.*, **34**, 263.
Pešek, I.: 1982, *Bull. Astron. Inst. Czechosl.*, **33**, 176.
Schutz, B.E.: 1981, *Celest. Mech.*, **24**, 173.
Standish, E.M.: 1981, *Astron. Astrophys.*, **101**, L17.
Yoder, C.F.: 1979, in *Natural and Artificial Satellite Motion*, P. Nacozy and S. Ferraz-Mello (eds), Texas University Press, p. 211.