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MODERATE DEVIATIONS FOR STABLE RANDOM WALKS IN RANDOM SCENERY

YUQIANG LI,* *East China Normal University*

Abstract

In this paper, a moderate deviation theorem for one-dimensional stable random walks in random scenery is proved. The proof relies on the analysis of maximum local times of stable random walks, and the comparison of moments between random walks in random scenery and self-intersection local times of the underlying random walks.

Keywords: Random walk in random scenery; stable random walk; moderate deviation

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1. Introduction

Let $\xi_1, \xi_2, \dots$ be a sequence of independent, identically distributed (i.i.d.) $\mathbb{Z}^d$-valued random vectors. A random walk $\{S_n,\ n \geq 0\}$ in $\mathbb{Z}^d$ with $S_0 = 0$ is defined as $S_n = \sum_{k=1}^{n} \xi_k$ for each $n \geq 1$. Let $\{\zeta_i, i \in \mathbb{Z}\}$ be a sequence of i.i.d. nondegenerate random variables taking values in $\mathbb{R}$. We refer to $\{\zeta_i,\ i \geq 0\}$ as the random scenery. Then the process $\{X_n,\ n \geq 1\}$ defined by

$$X_n = \sum_{k=0}^{n-1} \zeta_{S_k}, \qquad n \geq 1,$$

is called a random walk in random scenery (with underlying random walk $\{S_n\}$). The random walk in random scenery is often rewritten as

$$X_n = \sum_{x \in \mathbb{Z}^d} \zeta_x L_n(x), \qquad n \geq 1,$$

where $L_n(x) = \sum_{j=0}^{n-1} \mathbf{1}_{\{x\}}(S_j)$ is the local time of $\{S_n\}$ at x before time n. Random walks in random scenery have a heuristic interpretation. If a random walker has to pay Y_z units every time he/she visits the site z, then X_n is the total amount he/she pays before time n.

A random walk in random scenery in the $d = 1$ case was formally introduced and analyzed by Kesten and Spitzer [13]. They proved that if $\{\xi_k,\ k \geq 0\}$ and $\{\zeta_x,\ x \in \mathbb{Z}\}$ belong to domains of attraction of different stable laws with indices $1 < \alpha \leq 2$ and $0 < \beta \leq 2$, respectively, then $n^{-\delta} X_n$ converges in distribution as $n \to \infty$ to a nondegenerate variable, where $\delta = 1 - 1/\alpha + 1/\alpha\beta$. Since then there has been much work on random walks in scenery. For example, when S_n is a simple random walk in $\mathbb{Z}^d$, Csáki *et al.* [8] studied the strong invariance principle for X_n in the $d = 2$ case, Asselah and Castell [2] estimated the probability that X_n is large in the $d \geq 5$ case, and Asselah [1] investigated the moderate deviation for X_n in the $d = 3$ case. More generally, if the underlying random walks have finite variance, Gantert

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* Postal address: School of Finance and Statistics, East China Normal University, Shanghai 200241, P. R. China.
Email address: yqli@stat.ecnu.edu.cn

et al. [11] analyzed the deviations $\mathrm{P}(X_n/n > b_n)$ for various choices of sequences $\{b_n\}_{n\in\mathbb{N}}$ in $[1, \infty)$ with $b_n \to \infty$ as $n \to \infty$ in the case of arbitrary sceneries unbounded to $+\infty$ and Fleischmann *et al.* [10] proved moderate deviation principles for X_n in the $d \geq 2$ case with the random sceneries satisfying Cramér's condition. In addition, when the underlying random walk has infinite variance, Lewis showed in [15] that a law of the iterated logarithm for X_n can be obtained provided random normalizers are employed, and provided in [16] some sufficient conditions to obtain the deterministic normalizers.

We note that Khoshnevian and Lewis [14] proved a large deviation theorem for stable processes in Brownian scenery which, in essence, is the limit processes of random walks in random scenery under certain conditions. Their results (see [14, Theorems 1.1 and 5.1]) read as follows.

Theorem 1.1. ([14].) *Suppose that $X = \{X_t : t \geq 0\}$ is a strictly stable Lévy processes with index $\alpha \in (1, 2]$ and $Y = \{Y(t),\ t \in \mathbb{R}\}$ is a two-sided Brownian motion. Let $L = \{L_t^x : t \geq 0, x \in \mathbb{R}\}$ denote the processes of local times of X. Define $G(t) := \int_{\mathbb{R}} L_t^x \,\mathrm{d}Y(x)$. Then there exist a positive real number $\gamma_1 = \gamma(\alpha)$ and some constant $\gamma_2 > 0$ such that*

$$\lim_{\lambda\to\infty} \lambda^{-2\alpha/(1+\alpha)} \ln \mathrm{P}(G_1 \geq \lambda) = -\gamma_1 \tag{1.1}$$

and

$$\limsup_{t\to\infty} \left(\frac{\ln\ln t}{t}\right)^{1-1/2\alpha} \frac{G(t)}{(\ln\ln t)^{3/2}} = \gamma_2 \quad \textit{almost surely.}$$

Although the constants γ_1 and γ_2 are not specific, these results are still very interesting. It is natural to ask whether similar results hold for the random walks in random scenery when $\xi_{\cdot}$ belongs to the domain of attraction of stable distributions and $\zeta_{\cdot}$, as in, e.g. [8] and [11], is generally non-Gaussian. This question does not seem to have been addressed directly by Khoshnevian and Lewis' approach, since their proofs are heavily dependent on the properties of Gaussian processes and/or Brownian motions.

The purpose of this short paper is to partly solve the problem for a one-dimensional random walk in random scenery. We note that Chen [4], by means of the large deviation results of intersection local times of random walks in [6], investigated limit laws for the energy of a charged polymer and got interesting moderate deviation principles in different dimensional cases. Moreover, Chen and Khoshnevisan [5], under the assumption that the underlying random walk has finite second moments, pointed out that the model of polymers is close to random walks in random scenery. This result motivates us to study the problem by modifying the methods used in Chen [4], if we have large deviation principals for the self-intersection local times of the corresponding stable random walks.

The rest of this paper is organized as follows. In Section 2 we specify the assumptions and introduce the main result of this paper, and then prove a comparison lemma on moments between the random walk in random scenery and the self-intersection local times of underlying random walks. In Section 3 we prove in detail a limit theorem for the logarithmic moment generating function of the maximum local times of stable random walks. The proof of the main result is given in Section 4.

Throughout this paper, we use E^{ω} to denote the expectation with respect to the scenery variables only, and E and P to respectively denote the expectation and probability with respect to both the random walk and scenery.

We use the notation C and C_k, $k \geq 1$, to denote positive, finite constants, whose values can change at every occurrence, and which never depend on random quantities.

2. Main result

Let $\xi_1, \xi_2, \dots$ be nondegenerate, symmetric i.i.d. random variables taking values in $\mathbb{Z}$. Let $S_0 = 0$ and $S_n = \sum_{k=1}^{n} \xi_k$ for each $n \geq 1$. In the remainder of this paper, we suppose that $\{S_n\}$ is strongly aperiodic with support $\mathbb{Z}$ and assume that there exists $g(x)$, a function of regular variation with index $1/\alpha$, such that $S_n/g(n) \to X$ in law as $n \to \infty$, where $\mathrm{E}(\mathrm{e}^{\mathrm{i}\lambda X}) = \mathrm{e}^{-|\lambda|^\alpha}$ and $\alpha \in (1, 2]$. Then, the ξ_k, $k \geq 1$, belong to the domain of attraction of the symmetric α-stable distribution, and, hence, in this paper we call $S = \{S_n\}$ the symmetric stable random walk (with α) for convenience.

Let $L_n(x)$ be the local time of $\{S_n\}$ at x before time n, i.e. $L_n(x) = \sum_{j=0}^{n-1} \mathbf{1}_{\{x\}}(S_j)$. From Theorem 4 of [7] we know that, for any positive sequence $\{b_n, n \geq 0\}$ with $b_n \to \infty$ and $b_n/n \to 0$,

$$\lim_{n\to\infty} \frac{1}{b_n} \ln \mathrm{P}\biggl(\sum_{x\in\mathbb{Z}} L_n^2(x) \geq \lambda \frac{n^2}{g(n/b_n)}\biggr) = -\lambda^\alpha \frac{1}{2\alpha}\biggl(\frac{2\alpha-1}{2\alpha M_{\alpha,2}}\biggr)^{2\alpha-1}, \tag{2.1}$$

where

$$M_{\alpha,2} = \sup_{f\in\mathcal{F}_\alpha}\biggl\{\|f\|_4^2 - \int_{-\infty}^{\infty} |\lambda|^\alpha |\hat{f}(\lambda)|^2 \,\mathrm{d}\lambda\biggr\} < \infty,$$

$\hat{f}$ is the Fourier transform of f, and

$$\mathcal{F}_\alpha = \biggl\{f \in L^2(\mathbb{R})\colon \|f\|_2 = 1 \text{ and } \int_{-\infty}^{\infty} |\lambda|^\alpha |\hat{f}(\lambda)|^2 \,\mathrm{d}\lambda < \infty\biggr\}.$$

For simplicity, we always assume in the sequel that

$$\sigma := \lim_{x\to\infty} x^{-1/\alpha} g(x) > 0.$$

Therefore, from (2.1), we obtain

$$\lim_{n\to\infty} \frac{1}{b_n} \ln \mathrm{P}\biggl(\sum_{x\in\mathbb{Z}} L_n^2(x) \geq \lambda \frac{n^2}{(n/b_n)^{1/\alpha}}\biggr) = -\lambda^\alpha \frac{\sigma^\alpha}{2\alpha}\biggl(\frac{2\alpha-1}{2\alpha M_{\alpha,2}}\biggr)^{2\alpha-1}. \tag{2.2}$$

For convenience, in the rest of this paper, let

$$C_\alpha := \sigma^\alpha \biggl(\frac{2\alpha-1}{2\alpha M_{\alpha,2}}\biggr)^{2\alpha-1}.$$

Consider the stable random walk in random scenery

$$X_n = \sum_{k=0}^{n-1} \zeta_{S_k} = \sum_{x\in\mathbb{Z}} \zeta_x L_n(x),$$

where the random scenery $\zeta = \{\zeta_x, x \in \mathbb{Z}\}$, independent of $\{S_n, n \geq 0\}$, is a family of nondegenerate, symmetric, i.i.d. real-valued random variables satisfying Equation (1.2) of [4], i.e.

$$\mathrm{E}(\zeta_1^2) = 1 \quad \text{and} \quad \mathrm{E}(\mathrm{e}^{\lambda_0 \zeta_1^2}) < \infty \quad \text{for some } \lambda_0 > 0. \tag{2.3}$$

Our main result is a moderate deviation theorem for X_n.

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Theorem 2.1. *Under the above assumptions,*

$$\lim_{n\to\infty}\frac{1}{b_n}\ln \mathrm{P}(\pm X_n \geq \lambda n^{1-1/2\alpha} b_n^{1/2+1/2\alpha}) = -\frac{\alpha+1}{\alpha}\lambda^{2\alpha/(\alpha+1)}\bigg(\frac{C_\alpha}{2\sqrt{8^\alpha}}\bigg)^{1/(\alpha+1)}$$

for any positive sequence $\{b_n\}$ *satisfying*

$$b_n \to \infty \quad \text{and} \quad b_n = o(n^{1/(2\alpha+1)}).$$

Remark 2.1. Theorem 2.1 generalizes the corresponding result in [14] (see (1.1)) to random walks in general random sceneries. Furthermore, we specify that

$$\gamma_1 = \frac{\alpha+1}{\alpha}\bigg(\frac{C_\alpha}{2\sqrt{8^\alpha}}\bigg)^{1/(\alpha+1)}.$$

The proof of Theorem 2.1 is similar to that of Theorem 1.2 of [4] with some necessary modifications to handle the technical complexities caused by the stable random walks. The basic idea is to compare the moments between the random walks in random scenery and the self-intersection local times of underlying random walks by using localization. See [4] for more details and [6] for some related tricks.

In the rest of this section we develop an analogue of Proposition 2.1 of [4] to compare the moments of the localized random walks in random scenery with those of the corresponding self-intersection local times. Recall that the self-intersection local times of $S = \{S_n\}$ are

$$H_n = \sum_{x\in\mathbb{Z}} L_n^2(x).$$

For positive constants $K_n = M_n n^{1-1/\alpha} b_n^{1/\alpha} > 0$, where

$$M_n \to \infty \quad \text{and} \quad \frac{M_n^{3\alpha} b_n^{2\alpha+1}}{n} \to 0 \tag{2.4}$$

as $n \to \infty$, define

$$\tilde{X}_n = X_n \mathbf{1}_{\{\sup_{x\in\mathbb{Z}} L_n(x) \leq K_n\}},$$
$$\tilde{H}_n = H_n \mathbf{1}_{\{\sup_{x\in\mathbb{Z}} L_n(x) \leq K_n\}},$$

and

$$A_m(n) = \sum_{(y_1,y_2,\ldots,y_m)\in B_m} \mathrm{E}\bigg(\mathbf{1}_{\{\sup_{x\in\mathbb{Z}} L_n(x)\leq K_n\}} \prod_{k=1}^m L_n^2(y_k)\bigg),$$

where $m, n = 1, 2, \ldots$ and $B_m = \{(y_1, \ldots, y_m) \in \mathbb{Z}^m;\ y_1, \ldots, y_m$ are distinct$\}$. By applying similar arguments used in the proof of Proposition 2.1 of [4] we obtain the following result.

Lemma 2.1. *There exists a constant* C *independent of* n, m, *and the choice of* K_n *such that*

$$\mathrm{E}(\tilde{X}_n^m) \leq m! \sum_{l=1}^{[m/2]} \frac{1}{l!} 2^{-l/2} K_n^{m-2l} C^{(m-2l)/2} \binom{m-l-1}{m-2l} \mathrm{E}(\tilde{H}_n^l). \tag{2.5}$$

On the other hand, for any integers $m, n \geq 1$,

$$\mathrm{E}(\tilde{X}_n^{2m}) \geq \frac{(2m!)}{2^m m!} A_m(n) \tag{2.6}$$

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and

$$\mathrm{E}(\tilde{H}_n^m) \le \sum_{l=1}^{m} \binom{m}{l} l^{m-l} K_n^{2(m-l)} A_l(n). \tag{2.7}$$

Proof. As the proof follows the same lines as that of Proposition 2.1 of [4] with some modifications, to save space, we only show the modification necessary for the proof of (2.5).

By replacing $\Lambda_n(x)$ in Equation (2.10) of [4, p. 644] with $2\zeta_x L_n(x)$ and applying the same arguments that lead to Equation (2.14) of [4, p. 646], we obtain

$$\begin{aligned}\mathrm{E}(\tilde{X}_n^m) = &\sum_{l=1}^{[2^{-1}m]} \frac{1}{l!} \sum_{\substack{i_1+\cdots+i_l=m\\ i_1,\dots,i_l\ge 2}} \frac{m!}{(i_1)!\cdots(i_l)!}\\ &\times \sum_{(y_1,\dots,y_m)\in B_l} \mathrm{E}\biggl(\mathbf{1}_{\{\sup_{x\in\mathbb{Z}} L_n(x)\le K_n\}} \prod_{k=1}^{l} L_n(y_k)^{i_k}\, \mathrm{E}^{\omega}(\zeta_{y_k}^{i_k})\biggr).\end{aligned} \tag{2.8}$$

Condition (2.3) implies that there exists a constant $C_1 > 1$ such that

$$\mathrm{E}(\zeta_x^i) \le (\mathrm{E}(\zeta_x^{2i}))^{1/2} \le (i!\, C_1^i)^{1/2}$$

for all $i \ge 3$. Therefore, from (2.8) we obtain

$$\begin{aligned}\mathrm{E}(\tilde{X}_n^m) \le m! &\sum_{l=1}^{[2^{-1}m]} \frac{1}{l!} \sum_{\substack{i_1+\cdots+i_l=m\\ i_1,\dots,i_l\ge 2}} \frac{C_{i_1}^{i_1/2}\cdots C_{i_l}^{i_l/2}}{\sqrt{(i_1)!\cdots(i_l)!}}\\ &\times \sum_{(y_1,\dots,y_m)\in B_l} \mathrm{E}\biggl(\mathbf{1}_{\{\sup_{x\in\mathbb{Z}} L_n(x)\le K_n\}} \prod_{k=1}^{l} L_n(y_k)^{i_k}\biggr),\end{aligned}$$

where $C_2 = 1$ and $C_k = C_1$ for $k \ge 3$. Since

$$\mathbf{1}_{\{\sup_{x\in\mathbb{Z}} L_n(x)\le K_n\}} \prod_{k=1}^{l} L_n(y_k)^{i_k} \le K_n^{m-2l} \mathbf{1}_{\{\sup_{x\in\mathbb{Z}} L_n(x)\le K_n\}} \prod_{k=1}^{l} L_n(y_k)^2,$$

when $i_1 + \cdots + i_l = m$ and $i_1, \dots, i_l \ge 2$, we further have

$$\begin{aligned}\mathrm{E}(\tilde{X}_n^m) &\le m! \sum_{l=1}^{[2^{-1}m]} \frac{1}{l!} \sum_{\substack{i_1+\cdots+i_l=m\\ i_1,\dots,i_l\ge 2}} \frac{C_{i_1}^{i_1/2}\cdots C_{i_l}^{i_l/2}}{\sqrt{(i_1)!\cdots(i_l)!}} K_n^{m-2l}\, \mathrm{E}(\tilde{H}^l)\\ &\le m! \sum_{l=1}^{[2^{-1}m]} \frac{2^{-l/2}}{l!} K_n^{m-2l}\, \mathrm{E}(\tilde{H}^l) \sum_{\substack{i_1+\cdots+i_l=m\\ i_1,\dots,i_l\ge 2}} C_{i_1}^{i_1/2}\cdots C_{i_l}^{i_l/2}.\end{aligned} \tag{2.9}$$

Let k be the number of elements in the set $\{0 \le j \le l, i_j = 2\}$. Then $m \ge 2k + 3(l-k)$,

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which leads to $l - k \le m - 2l$. Therefore,

$$
\begin{aligned}
\sum_{\substack{i_1+\cdots+i_l=m\\ i_1,\dots,i_l\ge 2}} C_{i_1}^{i_1/2}\cdots C_{i_l}^{i_l/2} &\le \sum_{\substack{i_1+\cdots+i_l=m\\ i_1,\dots,i_l\ge 2}} C_1^{(m-2k)/2}\\
&= \sum_{\substack{i_1+\cdots+i_l=m\\ i_1,\dots,i_l\ge 2}} C_1^{(m-2l+2l-2k)/2}\\
&\le \sum_{\substack{i_1+\cdots+i_l=m\\ i_1,\dots,i_l\ge 2}} C_1^{3(m-2l)/2}\\
&= C^{(m-2l)/2}\binom{m-l-1}{m-2l},
\end{aligned}
\tag{2.10}
$$

where $C = C_1^3$. Substituting (2.10) into (2.9) yields (2.5).

Remark 2.2. It is readily seen that $\tilde{X}_n$ is symmetric, and, hence, by (2.5), $\mathrm{E}(\tilde{X}_n^{2m+1}) = 0$ for each integer $m \ge 0$.

3. Deviations of the maximum local time of stable random walks

In this section we prove a limit theorem for the logarithmic moment generating function of the maximum local times of the symmetric stable random walk S_n. This is not only an important step in the proof of the main result, but is also of independent interest. The main result of this section is as follows.

Theorem 3.1. *There exists a constant $C > 0$ such that, for any $\lambda > 0$ and positive sequence $\{b_n\}$ with $b_n \to \infty$ and $b_n = o(n)$ as $n \to \infty$,*

$$
\limsup_{n\to\infty} \frac{1}{b_n} \ln \mathrm{E}\biggl(\exp\biggl\{\lambda\Bigl(\frac{b_n}{n}\Bigr)^{1-1/\alpha} \sup_x L_n(x)\biggr\}\biggr) < \frac{1}{2}(2C\lambda \vee 1)^{\alpha/(\alpha-1)} < +\infty. \tag{3.1}
$$

In the case where the random walks have finite second moment, the finiteness of the lim sup in (3.1) seems to be well known; it was used by Chen [4] without specific proof. Our result deals with the case of stable random walks and provides an explicit constant as the upper bound. The proof of the theorem is based on the refinements of some results in [12] and Chen's ideas on local times (see [3]). To avoid unnecessary repetition with the material included in [12], identical parts are omitted.

For convenience, let $c_n := (n/b_n)^{1/\alpha}$ in this section.

The following lemma generalizes Lemma 9 of [12].

Lemma 3.1. *There is a constant $C > 0$ such that if $|x - y| \le \eta c_n$ then*

$$
\mathrm{P}\biggl(\frac{c_n}{n}|L_n(x) - L_n(y)| \ge C\eta^{(\alpha-1)/4}\biggr) \le \eta^{b_n(\alpha-1)/2}
$$

for all n, $0 < \eta < 1$.

Proof. By the same arguments as those used in the proof of the first part of Lemma 9 of [12] we obtain, for some constant $C > 0$,

$$
\mathrm{E}[(L_n(x) - L_n(y))^{2r}] \le (2r)!\, C^r (m a_m^{-1})^r (|x - y| Q(|x - y|))^{-r},
$$

where $m = [n/r] + 1$, $Q(x) = \mathrm{E}[(x^{-1}|\xi_1| \wedge 1)^2]$ for $x \geq 0$, and a_x satisfies $Q(a_x) = 1/x$ for $x > 1/Q(1)$ and $a_x = 1$ for $x \in [0, 1/Q(1)]$. Note that ξ_1 is in the domain of attraction of an α-stable law with $\alpha \in (1, 2]$. There exists a constant $C_1 > 0$ such that $\lim_{x\to\infty} x^\alpha Q(x) = C_1$. Therefore, for any $1 \leq x \leq y$, $x^\alpha Q(x) \geq C_2 y^\alpha Q(y)$ for some constant C_2 and $a_n = O(n^{1/\alpha})$. Now, since $|x - y| \leq \eta c_n$, we obtain

$$|x - y| Q(|x - y|) \geq C_2 \eta^{-(\alpha-1)} c_n Q(c_n),$$

and, hence,

$$\begin{aligned}
\mathrm{E}\bigg[\bigg(\frac{c_n}{n}(L_n(x) - L_n(y))\bigg)^{2r}\bigg] &\leq \bigg(\frac{C}{C_2}\bigg)^r (2r)^{2r} \bigg(\frac{c_n}{n}\bigg)^{2r} \bigg(\frac{m}{a_m}\bigg)^r \bigg(\frac{1}{c_n Q(c_n)}\bigg)^r \eta^{r(\alpha-1)} \\
&\leq C_3^r r^{2r} \bigg(\frac{c_n}{n}\bigg)^{2r} \bigg(\frac{n}{r}\bigg)^{(1-1/\alpha)r} c_n^{(\alpha-1)r} \eta^{r(\alpha-1)} \\
&= C_3^r r^{r+r/\alpha} c_n^{r+\alpha r} \frac{1}{n^{r+r/\alpha}} \eta^{r(\alpha-1)} \\
&= C_3^r \bigg(\frac{r}{b_n}\bigg)^{r+r/\alpha} \eta^{r(\alpha-1)}.
\end{aligned}$$

Let $r = b_n$. Then

$$\mathrm{E}\bigg[\bigg(\frac{c_n}{n}(L_n(x) - L_n(y))\bigg)^{2b_n}\bigg] \leq C_3^{b_n} \eta^{b_n(\alpha-1)}.$$

Using Markov's inequality, we obtain

$$\begin{aligned}
\mathrm{P}\bigg(\frac{c_n}{n}|L_n(x) - L_n(y)| \geq \sqrt{C_3}\eta^{(\alpha-1)/4}\bigg) &\leq \frac{1}{C_3^{b_n} \eta^{b_n(\alpha-1)/2}} \mathrm{E}\bigg[\bigg(\frac{c_n}{n}(L_n(x) - L_n(y))\bigg)^{2r}\bigg] \\
&\leq \eta^{b_n(\alpha-1)/2},
\end{aligned}$$

which is the desired conclusion if we take $C = \sqrt{C_3}$.

Lemma 3.2. *For any $\varepsilon > 0$, there exist a constant $0 < \delta < 1$ independent of $\{c_n\}$ and a constant $n_0 > 0$ such that*

$$\mathrm{P}\bigg(\sup_{|x-y|\leq\delta c_n} \frac{c_n}{n}|L_n(x) - L_n(y)| \geq \varepsilon\bigg) \leq \mathrm{e}^{-b_n} \quad \textit{for all } n > n_0.$$

Proof. Since the random walk is α-stable with $\alpha \in (1, 2]$, by Lemma 3 of [12], there exist a constant $C > 0$ and $\lambda > 1$ such that $\mathrm{P}(\max_{1\leq k\leq n} |S_k| \geq M a_n) \leq C M^{-\lambda}$ for all n and $M > 1$. Therefore,

$$\mathrm{P}(L_n(x) \neq 0 \text{ for some } |x| \geq \mathrm{e}^{2b_n} a_n) \leq C\mathrm{e}^{-2\lambda b_n} < C\mathrm{e}^{-2b_n}.$$

Let $\gamma = 2^{-(\alpha-1)/4}$. Choose $\delta < \mathrm{e}^{-16/(\alpha-1)}$ such that $C\delta^{(\alpha-1)/4} \leq \varepsilon(1 - \gamma)$, where C is the constant in Lemma 3.1, and let $j = \min\{k, \delta c_n \leq 2^k\}$. For $0 < x < 2^j \leq 2\delta c_n < c_n$, we may write $x = \sum_{i=0}^{j} \chi_i 2^i$, where each $\chi_i = 0$ or 1. Let $x_m = \sum_{i=m}^{j} \chi_i 2^i$. By the same arguments used in the proof of Lemma 11 of [12] (see the last three lines on page 79), from Lemma 3.1

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we obtain

$$\begin{aligned}
&\mathrm{P}\bigg(\max_{0<x\leq\delta c_n}\frac{c_n}{n}|L_n(0)-L_n(x)|\geq\varepsilon\bigg)\\
&\quad\leq \mathrm{P}\bigg(\max_{0<x\leq 2^j}\frac{c_n}{n}|L_n(0)-L_n(x)|\geq\varepsilon\bigg)\\
&\quad\leq \sum_{m=0}^{j}2^{j-m}\,\mathrm{P}\bigg(\frac{c_n}{n}|L_k(x_{m+1})-L_k(x_m)|\geq\varepsilon\gamma^{j-m}(1-\gamma)\bigg)\\
&\quad\leq \sum_{m=0}^{j}2^{j-m}\,\mathrm{P}\bigg(\frac{c_n}{n}|L_k(x_{m+1})-L_k(x_m)|\geq C(2^{m-j}\delta)^{(\alpha-1)/4}\bigg)\\
&\quad\leq \sum_{m=0}^{j}2^{j-m}(2^{m-j}\delta)^{b_n(\alpha-1)/2}.
\end{aligned}$$

Note that $b_n\to\infty$ implies that

$$\sum_{m=0}^{j}2^{(m-j)(b_n(\alpha-1)/2-1)}\leq 2$$

for sufficiently large n. Therefore,

$$\mathrm{P}\bigg(\max_{0<x\leq\delta c_n}\frac{c_n}{n}|L_k(0)-L_k(x)|\geq\varepsilon\bigg)\leq 2\delta^{b_n(\alpha-1)/2}$$

for sufficiently large n. Since there are at most $2(\mathrm{e}^{2b_n}a_n+c_n)/\delta c_n+2$ disjoint short intervals of length δc_n in $[-(\mathrm{e}^{2b_n}a_n+c_n),\mathrm{e}^{2b_n}a_n+c_n]$, by the same arguments used in the proof of Lemma 11 of [12] (see pages 79 and 80), there exist constants $C_1, C_2>0$ which are independent of c_n and δ such that

$$\begin{aligned}
&\mathrm{P}\bigg(\sup_{|x-y|\leq\delta c_n}\frac{c_n}{n}|L_n(x)-L_n(y)|\geq\varepsilon\bigg)\\
&\quad\leq C\mathrm{e}^{-2b_n}+C_1\delta^{b_n(\alpha-1)/2}\frac{\mathrm{e}^{2b_n}a_n+c_n}{\delta c_n}\\
&\quad\leq C_2(\mathrm{e}^{-2b_n}+\delta^{b_n(\alpha-1)/2-1}\mathrm{e}^{2b_n}b_n^{1/\alpha})\\
&\quad= C_2\bigg(\mathrm{e}^{-2b_n}+\exp\bigg\{b_n\bigg(\frac{\ln b_n}{\alpha b_n}+\bigg(\frac{\alpha-1}{2}-\frac{1}{b_n}\bigg)\ln\delta+2\bigg)\bigg\}\bigg).
\end{aligned}$$

Since $b_n\to\infty$, we can choose suitable n_0 such that, for $n>n_0$, $2C_2\mathrm{e}^{-b_n}<1$ and

$$\frac{\ln b_n}{\alpha b_n}+\bigg(\frac{\alpha-1}{2}-\frac{1}{b_n}\bigg)\ln\delta+2\leq\frac{\ln b_n}{\alpha b_n}+\bigg(\frac{\alpha-1}{2}-\frac{1}{b_n}\bigg)\frac{-16}{\alpha-1}+2\leq -2.$$

Consequently, for all $n\geq n_0$,

$$\mathrm{P}\bigg(\sup_{|x-y|\leq\delta c_n}\frac{c_n}{n}|L_n(x)-L_n(y)|\geq\varepsilon\bigg)\leq 2C_2\mathrm{e}^{-2b_n}\leq\mathrm{e}^{-b_n},$$

which completes the proof.

Lemma 3.3. *For sufficiently large n,*

$$\mathrm{P}\left(\frac{c_n}{n}\sup_x L_n(x) \ge \frac{1}{\delta} + \varepsilon\right) \le \mathrm{e}^{-b_n}, \tag{3.2}$$

where δ and ε are the constants in Lemma 3.2.

Proof. Suppose that, for some $n > n_0$ with $c_n > 2/\delta$,

$$\mathrm{P}\left(\frac{c_n}{n}\sup_x L_n(x) \ge \frac{1}{\delta} + \varepsilon\right) > \mathrm{e}^{-b_n}.$$

Then Lemma 3.2 implies that

$$\begin{aligned}
&\mathrm{P}\left(\frac{c_n}{n}\sup_x L_n(x) \ge \frac{1}{\delta} + \varepsilon,\ \sup_{|x-y|\le\delta c_n}\frac{c_n}{n}|L_n(x) - L_n(y)| < \varepsilon\right)\\
&\quad \ge \mathrm{P}\left(\frac{c_n}{n}\sup_x L_n(x) \ge \frac{1}{\delta} + \varepsilon\right) - \mathrm{P}\left(\sup_{|x-y|\le\delta c_n}\frac{c_n}{n}|L_n(x) - L_n(y)| \ge \varepsilon\right)\\
&\quad \ge \mathrm{P}\left(\frac{c_n}{n}\sup_x L_n(x) \ge \frac{1}{\delta} + \varepsilon\right) - \mathrm{e}^{-b_n}\\
&\quad > 0.
\end{aligned}$$

Note that

$$\begin{aligned}
&\left\{\frac{c_n}{n}\sup_x L_n(x) \ge \frac{1}{\delta} + \varepsilon,\ \sup_{|x-y|\le\delta c_n}\frac{c_n}{n}|L_n(x) - L_n(y)| < \varepsilon\right\}\\
&\qquad \subset \left\{\sum_{x\in\mathbb{Z}}\frac{c_n}{n}L_n(x) \ge \frac{2}{\delta}(\delta c_n - 1)\right\}\\
&\qquad \subset \left\{\sum_{x\in\mathbb{Z}}\frac{c_n}{n}L_n(x) > c_n\right\}.
\end{aligned}$$

It follows that

$$\mathrm{P}\left(\sum_{x\in\mathbb{Z}}\frac{c_n}{n}L_n(x) > c_n\right) > 0,$$

which contradicts the fact that

$$\sum_{x\in\mathbb{Z}}\frac{c_n}{n}L_n(x) = c_n.$$

Therefore, for any $n > \max\{n_0, \min\{n\colon c_n > 2/\delta\}\}$, (3.2) holds.

Lemma 3.4. *For any nonnegative integer r, there exists a positive constant C independent of $\{c_n\}$ such that*

$$\mathrm{E}\left[\left(\frac{c_n}{n}\sup_x L_n(x)\right)^r\right] \le C^r\left(1 + \frac{r!}{b_n^r}\right).$$

Proof. By Lemma 1 of [3] we obtain, for any $a, b > 0$,

$$\mathrm{P}\left(\frac{c_n}{n}\sup_x L_n(x) > a + b\right) \le \mathrm{P}\left(\frac{c_n}{n}\sup_x L_n(x) > a\right)\mathrm{P}\left(\frac{c_n}{n}\sup_x L_n(x) > b\right).$$

Therefore, we can infer from Lemma 3.3 that

$$F(u) := \mathrm{P}\bigg(\frac{c_n}{n}\sup_x L_n(x) > u\bigg) \le (\mathrm{e}^{-b_n})^{[u/C_1]} \le \mathrm{e}^{-b_n(u/C_1-1)} \tag{3.3}$$

for all $u > C_1 := 1/\delta + \varepsilon$. Hence,

$$\begin{aligned}
\mathrm{E}\bigg[\bigg(\frac{c_n}{n}\sup_x L_n(x)\bigg)^r\bigg] &= \int_0^\infty r x^{r-1} F(x)\,\mathrm{d}x \\
&\le (2C_1)^r + \int_{2C_1}^\infty r x^{r-1} \mathrm{e}^{-b_n(x/C_1-1)}\,\mathrm{d}x \\
&\le (2C_1)^r + \int_0^\infty r x^{r-1} \mathrm{e}^{-b_n x/2C_1}\,\mathrm{d}x \\
&= (2C_1)^r + \frac{r!}{(b_n/2C_1)^r} \\
&\le (2C_1)^r\bigg(1 + \frac{r!}{b_n^r}\bigg).
\end{aligned}$$

Letting $C = 2C_1$ completes the proof.

Proof of Theorem 3.1. If $\lambda \le 1/2C$, where C is the constant in Lemma 3.4, then, by Lemma 3.4 and Taylor's expansion, for any $\{b_n\}$ satisfying the conditions in Theorem 3.1 and sufficiently large n, we have

$$\begin{aligned}
\ln \mathrm{E}\bigg(\exp\bigg\{\lambda\bigg(\frac{b_n}{n}\bigg)^{1-1/\alpha}\sup_x L_n(x)\bigg\}\bigg) &= \ln \mathrm{E}\bigg(\exp\bigg\{\lambda b_n \frac{c_n}{n}\sup_x L_n(x)\bigg\}\bigg) \\
&= \ln \sum_{k=0}^\infty \frac{(\lambda b_n)^k}{k!}\mathrm{E}\bigg[\bigg(\frac{c_n}{n}\sup_x L_n(x)\bigg)^k\bigg] \\
&\le \ln \sum_{k=0}^\infty \frac{(\lambda b_n)^k}{k!}\bigg(C^k + C^k\frac{k!}{b_n^k}\bigg) \\
&= \ln\bigg(\mathrm{e}^{\lambda C b_n} + \frac{1}{1-\lambda C}\bigg).
\end{aligned} \tag{3.4}$$

If $\lambda > 1/2C$ then

$$\begin{aligned}
&\ln \mathrm{E}\bigg(\exp\bigg\{\lambda\bigg(\frac{b_n}{n}\bigg)^{1-1/\alpha}\sup_x L_n(x)\bigg\}\bigg) \\
&\quad = \ln \mathrm{E}\bigg(\exp\bigg\{\frac{1}{2C}\bigg(\frac{b_n(2C\lambda)^{\alpha/(\alpha-1)}}{n}\bigg)^{1-1/\alpha}\sup_x L_n(x)\bigg\}\bigg).
\end{aligned}$$

Letting $b_n' = b_n(2C\lambda)^{\alpha/(\alpha-1)}$, from (3.4) we obtain

$$\ln \mathrm{E}\bigg(\exp\bigg\{\lambda\bigg(\frac{b_n}{n}\bigg)^{1-1/\alpha}\sup_x L_n(x)\bigg\}\bigg) \le \ln(\mathrm{e}^{(2C\lambda)^{\alpha/(\alpha-1)}b_n/2} + 2) \tag{3.5}$$

for sufficiently large n. From (3.4) and (3.5), it is easy to obtain the desired conclusion. This completes the proof.

4. Proof of the main result

In this section we complete the proof of Theorem 2.1. Although this part is in essence the same as the proof in [4, pp. 651–655], we make some modifications owing to the stable random walks and introduce some simplifications by using the symmetry of the sceneries.

Owing to the fact that $H_n \le n \sup_{x\in\mathbb{Z}} L_n(x)$ for any $\lambda > 0$,

$$\mathrm{E}\biggl(\exp\biggl\{\lambda \frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}} H_n\biggr\}\biggr) \le \mathrm{E}\biggl(\exp\biggl\{\lambda\Bigl(\frac{b_n}{n}\Bigr)^{1-1/\alpha} \sup_{x\in\mathbb{Z}} L_n(x)\biggr\}\biggr),$$

which, together with Theorem 3.1, implies that

$$\lim_{n\to\infty} \frac{1}{b_n} \ln \mathrm{E}\biggl(\exp\biggl\{\lambda \frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}} H_n\biggr\}\biggr) < +\infty. \tag{4.1}$$

Therefore, by Varadhan's integral lemma (see Theorem 4.3.1 of [9, p. 137]), it follows from (2.2) and (4.1) that

$$\lim_{n\to\infty} \frac{1}{b_n} \ln \mathrm{E}\biggl(\exp\biggl\{\lambda \frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}} H_n\biggr\}\biggr) = \sup_{y>0}\biggl\{y\lambda - \frac{C_\alpha}{2\alpha} y^\alpha\biggr\} = \frac{\alpha-1}{2\alpha} C_\alpha \Bigl(\frac{2\lambda}{C_\alpha}\Bigr)^{\alpha/(\alpha-1)}. \tag{4.2}$$

The symmetry of $\tilde{X}_n$ yields

$$\mathrm{E}\biggl(\exp\biggl\{-\theta \frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}} \tilde{X}_n\biggr\}\biggr) = \mathrm{E}\biggl(\exp\biggl\{\theta \frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}} \tilde{X}_n\biggr\}\biggr), \tag{4.3}$$

and, by Taylor's expansion and (2.5),

$$\begin{aligned}
&\mathrm{E}\biggl(\exp\biggl\{\theta \frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}} \tilde{X}_n\biggr\}\biggr) \\
&\quad = 1 + \sum_{k=2}^{\infty} \frac{\theta^k}{k!}\biggl(\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\biggr)^k \mathrm{E}(\tilde{X}_n^k) \\
&\quad \le 1 + \sum_{k=2}^{\infty} \frac{\theta^k}{k!}\biggl(\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\biggr)^k k! \sum_{l=1}^{[k/2]} \frac{1}{l!} 2^{-l/2} K_n^{k-2l} C^{(k-2l)/2} \binom{k-l-1}{k-2l} \mathrm{E}(\tilde{H}_n^l).
\end{aligned}$$

Furthermore, for any $\theta > 0$ and sufficiently large n,

$$\begin{aligned}
&\mathrm{E}\biggl(\exp\biggl\{\theta \frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}} \tilde{X}_n\biggr\}\biggr) \\
&\quad \le 1 + \sum_{l=1}^{\infty} \frac{\theta^{2l}}{l!}\biggl(\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\biggr)^{2l} 2^{-l/2} \mathrm{E}(\tilde{H}_n^l) \\
&\qquad\qquad \times \sum_{k=2l}^{\infty} \theta^{k-2l}\biggl(\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\biggr)^{k-2l} K_n^{k-2l} C^{(k-2l)/2} \binom{k-l-1}{k-2l} \\
&\quad = 1 + \sum_{l=1}^{\infty} \frac{\theta^{2l}}{l!}\biggl(\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\biggr)^{2l} 2^{-l/2}\biggl(1 - \frac{\sqrt{C}\theta K_n b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\biggr)^{-(l-1)} \mathrm{E}(\tilde{H}_n^l) \\
&\quad \le \mathrm{E}\biggl(\exp\biggl\{\frac{\theta^2}{\sqrt{2}} \frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}}\biggl(1 - \frac{\sqrt{C}\theta K_n b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\biggr)^{-1} \tilde{H}_n\biggr\}\biggr),
\end{aligned} \tag{4.4}$$

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where we have used the facts that

$$(1-x)^{-(l-1)} = \sum_{k=0}^{\infty} \binom{l-1+k}{k} x^k$$

for $|x| < 1$, and, as $n \to \infty$,

$$\frac{\sqrt{C}\theta K_n b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}} = \frac{\sqrt{C}\theta M_n b_n^{1/2+1/2\alpha}}{n^{1/2\alpha}} \to 0$$

by (2.4). Combining (4.4) with (4.3) and (4.2) yields

$$\limsup_{n\to\infty} \frac{1}{b_n} \mathrm{E}\biggl(\exp\biggl\{\pm\theta \frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}} \tilde{X}_n\biggr\}\biggr) \le \frac{\alpha-1}{2\alpha} C_\alpha \biggl(\frac{\sqrt{2}\theta^2}{C_\alpha}\biggr)^{\alpha/(\alpha-1)}. \tag{4.5}$$

On the other hand, from Remark 2.2 and Taylor's expansion, it follows that

$$\mathrm{E}\biggl(\exp\biggl\{\pm\theta \frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}} \tilde{X}_n\biggr\}\biggr) = 1 + \sum_{k=1}^{\infty} \frac{\theta^{2k}}{(2k)!} \biggl(\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\biggr)^{2k} \mathrm{E}(\tilde{X}_n^{2k}). \tag{4.6}$$

By (2.6),

$$\sum_{k=1}^{\infty} \frac{\theta^{2k}}{(2k)!} \biggl(\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\biggr)^{2k} \mathrm{E}(\tilde{X}_n^{2k}) \ge \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\theta^{2k}}{2^k} \biggl(\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\biggr)^{2k} A_k(n). \tag{4.7}$$

Letting

$$\bar{\theta} = \theta \exp\biggl\{-\theta^2 \frac{K_n^2 b_n^{1-1/\alpha}}{2n^{2-1/\alpha}}\biggr\} \le \theta,$$

by (2.7) we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{\bar{\theta}^{2k}}{2^k k!} \biggl(\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\biggr)^{2k} \mathrm{E}(\tilde{H}_n^k) \\
&\quad \le \sum_{k=1}^{\infty} \frac{\bar{\theta}^{2k}}{2^k k!} \biggl(\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\biggr)^{2k} \sum_{l=1}^{k} \binom{k}{l} l^{k-l} K_n^{2(k-l)} A_l(n) \\
&\quad = \sum_{l=1}^{\infty} \frac{\bar{\theta}^{2l}}{2^l l!} \biggl(\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\biggr)^{2l} A_l(n) \sum_{k=l}^{\infty} \frac{l^{k-l}}{(k-l)!} K_n^{2(k-l)} \biggl(\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\biggr)^{2(k-l)} \biggl(\frac{\bar{\theta}^2}{2}\biggr)^{k-l} \\
&\quad = \sum_{l=1}^{\infty} \frac{\bar{\theta}^{2l}}{2^l l!} \biggl(\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\biggr)^{2l} A_l(n) \exp\biggl\{\bar{\theta}^2 \frac{l K_n^2 b_n^{1-1/\alpha}}{2n^{2-1/\alpha}}\biggr\} \\
&\quad \le \sum_{l=1}^{\infty} \frac{1}{2^l l!} \biggl(\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\biggr)^{2l} A_l(n) \biggl(\bar{\theta}^2 \exp\biggl\{\theta^2 \frac{K_n^2 b_n^{1-1/\alpha}}{2n^{2-1/\alpha}}\biggr\}\biggr)^l.
\end{aligned}$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{\bar{\theta}^{2k}}{2^k k!} \biggl(\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\biggr)^{2k} \mathrm{E}(\tilde{H}_n^k) \le \sum_{l=1}^{\infty} \frac{\theta^{2l}}{2^l l!} \biggl(\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\biggr)^{2l} A_l(n),$$

which, together with (4.7), yields

$$1+\sum_{k=1}^{\infty}\frac{\theta^{2k}}{(2k)!}\bigg(\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\bigg)^{2k}\mathrm{E}(\tilde{X}_n^{2k})\geq 1+\sum_{k=1}^{\infty}\frac{\bar{\theta}^{2k}}{2^k k!}\bigg(\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\bigg)^{2k}\mathrm{E}(\tilde{H}_n^{k})$$
$$=\mathrm{E}\bigg(\exp\bigg\{\frac{\bar{\theta}^2}{2}\frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}}\tilde{H}_n\bigg\}\bigg). \tag{4.8}$$

Note that $\tilde{H}_n \leq nK_n$ and that

$$\frac{K_n^3 b_n^{1-1/\alpha}}{n^{2-1/\alpha}}\frac{b_n^{1-1/\alpha}}{n^{1-1/\alpha}}=\frac{M_n^3 b_n^{2+1/\alpha}}{n^{1/\alpha}}\to 0$$

by (2.4). We have

$$\begin{aligned}\frac{\bar{\theta}^2}{2}\frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}}\tilde{H}_n&=\frac{\theta^2}{2}\frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}}\tilde{H}_n-\frac{\theta^2}{2}\bigg[1-\exp\bigg\{-\theta^2\frac{K_n^2 b_n^{1-1/\alpha}}{n^{2-1/\alpha}}\bigg\}\bigg]\frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}}\tilde{H}_n\\&\geq\frac{\theta^2}{2}\frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}}\tilde{H}_n-\frac{\theta^4}{2}\frac{K_n^3 b_n^{1-1/\alpha}}{n^{2-1/\alpha}}\frac{b_n^{1-1/\alpha}}{n^{1-1/\alpha}}\\&=\frac{\theta^2}{2}\frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}}\tilde{H}_n-o(1).\end{aligned} \tag{4.9}$$

Combining (4.6) with (4.7)–(4.9), we further obtain

$$\mathrm{E}\bigg(\exp\bigg\{\pm\theta\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\tilde{X}_n\bigg\}\bigg)\geq\mathrm{E}\bigg(\exp\bigg\{\frac{\theta^2}{2}\frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}}\tilde{H}_n\bigg\}\bigg)-o(1). \tag{4.10}$$

Observe that

$$\begin{aligned}&\mathrm{E}\bigg(\exp\bigg\{\frac{\theta^2}{2}\frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}}\tilde{H}_n\bigg\}\bigg)\\&\quad\geq\mathrm{E}\bigg(\exp\bigg\{\frac{\theta^2}{2}\frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}}H_n\bigg\}\mathbf{1}_{\{\sup_{x\in\mathbb{Z}}L_n(x)\leq K_n\}}\bigg)\\&\quad=\mathrm{E}\bigg(\exp\bigg\{\frac{\theta^2}{2}\frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}}H_n\bigg\}\bigg)-\mathrm{E}\bigg(\exp\bigg\{\frac{\theta^2}{2}\frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}}H_n\bigg\}\mathbf{1}_{\{\sup_{x\in\mathbb{Z}}L_n(x)>K_n\}}\bigg).\end{aligned} \tag{4.11}$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned}&\frac{1}{b_n}\ln\mathrm{E}\bigg(\exp\bigg\{\frac{\theta^2}{2}\frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}}H_n\bigg\}\mathbf{1}_{\{\sup_{x\in\mathbb{Z}}L_n(x)>K_n\}}\bigg)\\&\quad\leq\frac{1}{b_n}\ln\bigg[\mathrm{E}\bigg(\exp\bigg\{\theta^2\frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}}H_n\bigg\}\bigg)^{1/2}\mathrm{P}\Big(\sup_{x\in\mathbb{Z}}L_n(x)>K_n\Big)^{1/2}\bigg]\\&\quad=\frac{1}{2b_n}\ln\mathrm{E}\bigg(\exp\bigg\{\theta^2\frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}}H_n\bigg\}\bigg)+\frac{1}{2b_n}\ln\mathrm{P}\Big(\sup_{x\in\mathbb{Z}}L_n(x)>K_n\Big).\end{aligned} \tag{4.12}$$

In addition, since $K_n = M_n n^{1-1/\alpha} b_n^{1/\alpha}$ and $M_n\to\infty$, by (3.3) we obtain

$$\frac{1}{2b_n}\ln\mathrm{P}\Big(\sup_{x\in\mathbb{Z}}L_n(x)>K_n\Big)\leq\frac{1}{2b_n}\ln\mathrm{e}^{-[M_n/C_1]b_n}\to-\infty, \tag{4.13}$$

which, together with (4.1) and (4.12), implies that

$$\frac{1}{b_n}\ln \mathrm{E}\bigg(\exp\bigg\{\frac{\theta^2}{2}\frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}}H_n\bigg\}\mathbf{1}_{\{\sup_{x\in\mathbb{Z}}L_n(x)>K_n\}}\bigg) \to -\infty. \tag{4.14}$$

Consequently, it follows from (4.11) and (4.14) that

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{b_n}\ln \mathrm{E}\bigg(\exp\bigg\{\frac{\theta^2}{2}\frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}}\tilde{H}_n\bigg\}\bigg) &\geq \liminf_{n\to\infty}\frac{1}{b_n}\ln \mathrm{E}\bigg(\exp\bigg\{\frac{\theta^2}{2}\frac{b_n^{1-1/\alpha}}{n^{2-1/\alpha}}H_n\bigg\}\bigg)\\ &= \frac{\alpha-1}{2\alpha}C_\alpha\bigg(\frac{\sqrt{2}\theta^2}{C_\alpha}\bigg)^{\alpha/(\alpha-1)}.\end{aligned} \tag{4.15}$$

Combining (4.5), (4.10), and (4.15), we obtain

$$\lim_{n\to\infty}\mathrm{E}\bigg(\exp\bigg\{\pm\theta\frac{b_n^{1/2-1/2\alpha}}{n^{1-1/2\alpha}}\tilde{X}_n\bigg\}\bigg) = \frac{\alpha-1}{2\alpha}C_\alpha\bigg(\frac{\sqrt{2}\theta^2}{C_\alpha}\bigg)^{\alpha/(\alpha-1)}. \tag{4.16}$$

According to the Gärtner–Ellis theorem (see [9, Theorem 2.3.6, p. 44]), (4.16) implies that $\tilde{X}_n$ satisfies the moderate deviation given in Theorem 2.1. By [9, Theorem 4.2.13, p. 130], the moderate deviation passes from $\tilde{X}_n$ to X_n through the exponential equivalence given by

$$\limsup_{n\to\infty}\frac{1}{b_n}\ln \mathrm{P}(\tilde{X}_n\neq X_n) = \lim_{n\to\infty}\frac{1}{b_n}\ln \mathrm{P}\Big(\sup_{x\in\mathbb{Z}}L_n(x)>K_n\Big) = -\infty,$$

which follows from (4.13). The proof is complete.

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