ON A DIFFERENTIABILITY CONDITION FOR REFLEXIVITY OF A BANACH SPACE

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In studying the geometry of normed linear space it is useful to draw attention to the following mapping.

DEFINITION. A mapping $x \to f_x$ from a normed linear space X into its dual X^* is called a *support mapping* if, for each $x \in S \equiv \{X \in X : ||x|| = 1\}$ and real $\lambda \geqq 0$,

$$f_x \in D(x) \equiv \{f \in S^* : f(x) = ||f|| = 1\} \quad \text{and} \quad f_{\lambda x} = \lambda f_x.$$

(The Hahn-Banach theorem guarantees that $D(x)$ is non-empty for each $x \in S$ so that such a mapping exists for every normed linear space.)

In his paper [3] the author formulated a characterisation of strong (Fréchet) differentiability of the norm of a normed linear space in terms of support mappings:

LEMMA 1. *The norm of a normed linear space X is strongly differentiable at $x \in S$ if and only if there exists a support mapping $x \to f_x$ from X into X^* which is continuous on S at x.* [3, Theorem 1(ii)].

Such a characterisation is particularly valuable used in conjunction with the subreflexivity property of Banach spaces.

DEFINITION. A normed linear space X is said to be *subreflexive* if the set P of continuous linear functionals which attain their norm on S, is dense in X^*.

E. Bishop and R. R. Phelps [1] have proved the significant result that every Banach space is subreflexive.

From Lemma 1 using the subreflexivity property the following known result can be easily deduced.

THEOREM 1. *For a Banach space X, if the norm of X^* is strongly differentiable on S^* then $P = X^*$ and X is reflexive.* [3, Theorem 2].

It is the purpose of this note to deduce the following improvement of Theorem 1.

NOTATION. For a set A in a linear space X we denote by sp (A) the linear span of A.

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THEOREM 1*. *For a Banach space X, if the norm of X^* is strongly differentiable on* sp $(P) \cap S^*$ *then $P = X^*$ and X is reflexive.*

LEMMA 2. *Let X be a normed linear space and $x \to f_x$ be a support mapping from X into X^*. Consider the linear space X with metric*

$$d(x, y) = \tfrac{1}{2}\{||x-y||+||f_x-f_y||\}.$$

The topology of the metric d is compatible with the linear structure of X if and only if the support mapping is continuous on S.

PROOF. Suppose the support mapping is continuous on S, then from the homogeneity property it is clear that the mapping is continuous on X. When $d(x, x_0) \to 0$ and $d(y, y_0) \to 0$, then $||x-x_0|| \to 0$ and $||y-y_0|| \to 0$ and so $||(x+y)-(x_0+y_0)|| \to 0$. But the continuity of the support mapping implies that $||f_{x+y}-f_{x_0+y_0}|| \to 0$, and it follows that $d(x+y, x_0+y_0) \to 0$. For a continuous support mapping it is clear that $f_{\lambda x} = \bar{\lambda} f_x$ for all $x \in S$ and all complex λ, and so it can be directly verified that $d(\lambda x, \lambda_0 x_0) \to 0$ as $|\lambda-\lambda_0| \to 0$ and $d(x, x_0) \to 0$.

Conversely, suppose that the topology of d is compatible with the linear structure. Then $d(x+y, x_0+y_0) \to 0$ as $d(x, x_0) \to 0$ and $d(y, y_0) \to 0$. For $x, y \in S$ and λ real

$$d(x+\lambda y, x) = \tfrac{1}{2}\{|\lambda|+||f_{x+\lambda y}-f_x||\} \to 0 \quad \text{as } d(\lambda y, 0) = |\lambda| \to 0.$$

Therefore, $||f_{x+\lambda y}-f_x|| \to 0$, and uniformly for all $y \in S$, as $|\lambda| \to 0$. But this condition is equivalent to the support mapping being continuous at x. [3, Lemma 1(ii)].

LEMMA 3. *For a Banach space X where X^* is smooth on $P \cap S^*$, given a support mapping $f \to F_f$ from X^* into X^{**}, then P is complete in X^* with respect to the metric*

$$d(f_1, f_2) = \tfrac{1}{2}\{||f_1-f_2||+||F_{f_1}-F_{f_2}||\}.$$

PROOF. Consider a sequence $\{f_n\}$ which is Cauchy in $P \cap S^*$ with respect to the metric d. Then $\{f_n\}$ is Cauchy in $P \cap S^*$ with respect to the norm of X^* and convergentto $f \in S^*$ since X^* is complete. Also $\{\hat{x}_n\}$, where $\hat{x}_n = F_{f_n}$ for $n = 1, 2, \cdots$, is Cauchy in $\hat{S}$ and so convergent to $\hat{x} \in \hat{S}$ since X is complete. But

$$\begin{aligned}|1-f(x)| &\leqq |f_n(x_n)-f_n(x)|+|f_n(x)-f(x)| \\ &\leqq ||f_n||\,||x_n-x||+||f_n-f||\,||x||.\end{aligned}$$

So $f(x) = 1$ and $f \in P \cap S^*$.

These lemmas are used in establishing the result.

PROOF OF THEOREM 1*. Since the norm of X^* is strongly differentiable on sp $(P) \cap S^*$, from Lemma 1, the unique support mapping $f \to F_f$ from sp (P) into sp $(P)^*$ is continuous on sp $(P) \cap S^*$. Since X is complete it is subreflexive, so for $f \in (\operatorname{sp}(P) \setminus P) \cap S^*$ there exists a sequence $\{f_n\} \in P \cap S^*$ which converges to f. Then $\{\hat{x}_n\}$, where $\hat{x}_n = F_{f_n}$, is convergent to F_f. But $\{\hat{x}_n\}$ is Cauchy in $\hat{S}$ so $F_f \in \hat{S}$, i.e. $f \in P \cap S^*$. Therefore $\operatorname{sp}(P) = P$.

For a support mapping $f \to F_f$ from X^* into X^{**}, P is a linear space with metric

$$d(f_1, f_2) = \tfrac{1}{2}\{\|f_1 - f_2\| + \|F_{f_1} - F_{f_2}\|\},$$

and since the support mapping is continuous on $P \cap S^*$, we have from Lemma 2 that the topology of the metric d is compatible with the linear structure of P.

From the Metrisation theorem for linear topological spaces [4, p. 48] it follows that there exists an invariant metric on P which generates the same topology as the metric d. Since a support mapping is norm preserving the balls centred on 0 for the metric d and for the norm are equivalent. Therefore the invariant metric which generates the same topology as the metric d is that induced by the norm.

But further, from Lemma 3, P is complete with respect to the metric d. It then follows from a result of V. L. Klee [7, p. 84] that P is complete as a normed linear space, and so P is a closed subspace of X^*. However, P is dense in X^*. Therefore $P = X^*$.

The result then follows as in Theorem 1.

It should be noted that

1. a Banach space X whose norm is strongly differentiable on S, is not necessarily reflexive, and

2. a Banach space Y where the norm of Y^* is strongly differentiable on $P \cap S^*$, is not necessarily reflexive.

The following example constructed by R. R. Phelps [6, p. 447] illustrates both these points.

Let Y be the linear space l_1 of sequences $y = \{y_n\}$ where $\sum_n |y_n| < \infty$, with the norm

$$\|y\| = \left\{\Big(\sum_n |y_n|\Big)^2 + \sum_n \left(\frac{y_n}{2^n}\right)^2\right\}^{\frac{1}{2}},$$

and X be the linear space c_0 of sequences $x = \{x_n\}$ which converge to zero, with the norm

$$\|x\| = \sup\Big\{\sum_n x_n y_n : y = \{y_n\} \in Y \text{ and } \|y\| \leqq 1\Big\}.$$

Phelps has shown that X is a non-reflexive Banach space, $Y = X^*$ and the norm of Y is locally uniformly rotund on S. He deduced from a theorem of A. R. Lovaglia [5, p. 232] that the norm of X is strongly differentiable on S. But also, from results of D. F. Cudia [2, p. 308 and p. 296] we can deduce that the norm of Y^* is strongly differentiable on $P \cap S^*$.

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