PRIME AND MAXIMAL IDEALS IN POLYNOMIAL RINGS

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In this paper we study prime and maximal ideals in a polynomial ring $R[X]$, where R is a ring with identity element. It is well-known that to study many questions we may assume R is prime and consider just R-disjoint ideals. We give a characterizaton for an R-disjoint ideal to be prime. We study conditions under which there exists an R-disjoint ideal which is a maximal ideal and when this is the case how to determine all such maximal ideals. Finally, we prove a theorem giving several equivalent conditions for a maximal ideal to be generated by polynomials of minimal degree.

0. Introduction. Let R be a ring with identity element and $R[X]$ the polynomial ring over R in an indeterminate X. If P is a prime ideal of $R[X]$, by factoring out the ideals $P \cap R$ and $(P \cap R)[X]$ from R and $R[X]$, respectively, we may assume that R is prime and $P \cap R = 0$. That is why we will assume here that R is a prime ring. Then $R[X]$ is also prime. A non-zero ideal (resp. prime ideal) P of $R[X]$ with $P \cap R = 0$ will be called an R-disjoint ideal (resp. prime ideal).

In [**3**] we studied R-disjoint prime ideals of $R[X]$. In particular, we proved that an R-disjoint ideal P of $R[X]$ is prime if and only if $P = Q[X]f_0 \cap R[X]$, where Q is a ring of right quotients of R and $f_0 \in C[X]$ is an irreducible polynomial, C being the extended centroid of R.

The main purpose of this paper is to study maximal ideals of $R[X]$. There are two interesting questions concerning maximal ideals that we want to consider here. First, determine all the prime ideals L of R such that there exists a maximal ideal M of $R[X]$ with $M \cap R = L$. As we said above, by factoring out L and $L[X]$ from R and $R[X]$, respectively, we may assume that $L = 0$. Second, assume that there exists a maximal ideal M of $R[X]$ which is R-disjoint. Then determine all these ideals. In Section 2 of this paper we study these questions.

It is well-known that an R-disjoint prime ideal of $R[X]$ is not necessarily generated by its polynomials of minimal degree, even if R is a commutative integral domain (see Example 4.1). In Section 3 we prove a theorem (Theorem 3.1) giving several equivalent conditions for an R-disjoint maximal ideal of $R[X]$ to be generated by polynomials of minimal degree. It follows that in this case the maximal ideal is a principal ideal generated by just one central polynomial of minimal degree.

Independent of the above in Section 1 we recall and complete the results on R-disjoint prime ideals. Our purpose here is twofold. First, we give an intrinsic characterization for R-disjoint ideals to be prime. This characterization was obtained for skew polynomial rings in [**5**, Corollary 1.8] and [**2**, Theorem 2.8]. However, as far as we

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know the characterization is not explicitly written in any place directly for polynomial rings. Secondly, the results of Section 1 are also required in the rest of the paper.

Finally, in Section 4 we give some additional remarks and examples.

Throughout this paper R is a prime ring with identity element. If I is an R-disjoint ideal of $R[X]$, then by $\rho(I)$ (resp. $\tau(I)$) we denote the ideal of R consisting of 0 and all the leading coefficients of all the polynomials (resp. polynomials of minimal degree) in I. If $f \in R[X]$, ∂f denotes the degree of f and $\mathrm{lc}(f)$ denotes the leading coefficient of f. The minimality of I is defined by $\mathrm{Min}(I) = \min\{\partial f : 0 \neq f \in I\}$. The notations $\supset$ and $\subset$ indicate strict inclusions and we denote by $Z = Z(R)$ the center of R.

We point out that *ideal* means always two sided ideal and *R-disjoint maximal ideal* means maximal ideal which is R-disjoint.

1. Prime ideals. First we recall the following well-known lemma. A proof of it has appeared in several places (e.g. [**9**, Corollary 2.13]).

Lemma 1.1. *Let P be an R-disjoint ideal of $R[X]$. The following are equivalent*:
(i) *P is a prime ideal of $R[X]$.*
(ii) *P is maximal in the set of R-disjoint ideals of $R[X]$.*

As in [**3**] we let

$$\Gamma = \{f \in R[X] : \partial f \geq 1 \text{ and } arf = fra, \text{ for every } r \in R, \text{ where } a = \mathrm{lc}(f)\}.$$

For $f \in \Gamma$ with $a = \mathrm{lc}(f)$ we put

$$[f] = \{g \in R[X] : \text{there exists } 0 \neq H \lhd R \text{ such that } gHa \subseteq R[X]f\}.$$

Then $[f]$ is an R-disjoint ideal of $R[X]$. An ideal of this type is said to be a (principal) closed ideal of $R[X]$. It follows from [**3**, Theorem 1.5] that $[f]$ is the unique closed ideal of $R[X]$ containing f and satisfying $\mathrm{Min}([f]) = \partial f$.

Let Q be the maximal (or the Martindale) right quotient ring of R and C the extended centroid of R, i.e., the center of Q. An R-disjoint ideal P of $R[X]$ is prime if and only if $P = Q[X]f_0 \cap R[X]$ for some irreducible polynomial $f_0 \in C[X]$ ([**3**, Corollary 2.7]). Now we give an intrinsic characterization for P to be prime (see also [**5**, Corollary 1.8; **2**, Theorem 2.8]).

Definition 1.2. We say that a polynomial $f \in \Gamma$ is *completely irreducible* in Γ (or Γ-completely irreducible) if the following condition is satisfied.
If there exist $b \in R$, $g \in \Gamma$ and $h \in R[X]$ such that $0 \neq fb = hg$, then $\partial g = \partial f$.

This definition is symmetrical. In fact, we have the following result.

Lemma 1.3. *A polynomial $f \in \Gamma$ is Γ-completely irreducible if and only if for every $b \in R$, $h \in \Gamma$ and $g \in R[X]$ such that $0 \neq fb = hg$ we necessarily have $\partial h = \partial f$.*

Proof. If $f \in \Gamma$ and $fb \neq 0$, $b \in R$, we have that $fb \in \Gamma$ and $[fb] = [f]$. In fact, assume that $ab = 0$, where $a = \mathrm{lc}(f)$. Then $arfb = frab = 0$, for every $r \in R$. Hence $fb = 0$ since R is prime, a contradiction. Thus $\partial(fb) = \partial f$ and $[fb] = [f]$ follows from [**3**, Theorem 1.5].

Assume that there exist $b \in R$, $h \in \Gamma$ and $g \in R[X]$ such that $0 \neq fb = hg$ with $\partial h < \partial f$. Then $[f] = [fb] \subset [h]$. Therefore there exists $0 \neq H \lhd R$ such that $fHc \subseteq R[X]h$, where $c = \mathrm{lc}(h)$. Take $d \in H$ and $p \in R[X]$ with $0 \neq fdc = ph$. This shows that f is not Γ-completely irreducible. So the result holds in one direction. The converse can be proved in a similar way using an obvious symmetric version of the definition of $[f]$ (see [**4**, Remarks 1.5 and 1.8]). $\square$

Now we are in position to prove the main result of this section.

THEOREM 1.4. *Let P be an R-disjoint ideal of $R[X]$. Then the following conditions are equivalent*:

(i) *P is prime.*

(ii) *P is closed and every $f \in P$ with $\partial f = \mathrm{Min}(P)$ is completely irreducible in Γ.*

(iii) *P is closed and there exists $f \in P$ with $\partial f = \mathrm{Min}(P)$ which is completely irreducible in Γ.*

Proof. (i)$\Rightarrow$(ii). If P is prime, then P is closed by [**3**, Corollary 1.9, (i)]. Assume $f \in P$, $\partial f = \mathrm{Min}(P)$ and $0 \neq fb = hg$, for $b \in R$ and $g \in \Gamma$. Hence $P = [f] = [fb] \subseteq [g]$. Thus $P = [g]$ by Lemma 1.1. Consequently $\partial g = \mathrm{Min}(P) = \partial f$.

(iii)$\Rightarrow$(i). Assume that $P = [f]$, where f is a Γ-completely irreducible polynomial. If P is not prime there exists a closed ideal $[g]$ of $R[X]$, $g \in \Gamma$, such that $P \subset [g]$. It follows that $\partial g < \partial f$ and there exists $0 \neq H \lhd R$ such that $fHc \subseteq R[X]g$, where $c = \mathrm{lc}(g)$. Now we get a contradiction as in the proof of Lemma 1.3. $\square$

It can easily be checked that if R is a commutative domain and F is the field of fractions of R, then a polynomial $f \in R[X]$ with $\partial f \geq 1$ is completely irreducible in Γ if and only if f is irreducible in $F[X]$.

More generally, note that if $f \in Z[X]$, Z the center of R, and $\partial f \geq 1$, then $f \in \Gamma$. We have

COROLLARY 1.5. *Assume that $f \in Z[X]$. Then the following are equivalent*:

(i) *f is completely irreducible in Γ.*

(ii) *f is irreducible in $C[X]$.*

Proof. (i)$\Rightarrow$(ii). Assume that $f = gh$, for g, h in $C[X]$, $\partial g < \partial f$ and $\partial h < \partial f$. Then $[f] \subset Q[X]g \cap R[X]$. This is a contradiction by Lemma 1.1 because $[f]$ is prime.

(ii)$\Rightarrow$(i). Since $f \in \Gamma$ there exists a monic polynomial $f_0 \in C[X]$ such that $[f] = Q[X]f_0 \cap R[X]$. Then $f = f_0 c$, for $c = \mathrm{lc}(f) \in Z \subseteq C$. Thus f_0 is irreducible in $C[X]$ by the assumption and so $[f]$ is prime [**3**, Corollary 2.7]. Consequently f is Γ-completely irreducible by Theorem 1.4. $\square$

REMARK 1.6. Assume that R is a unique factorization commutative domain and $f \in R[X]$. Then f is completely irreducible in Γ if and only if f is irreducible in $R[X]$.

To prove this fact we first recall that a polynomial $g \in R[X]$ is said to be primitive if the greatest common divisor of the coefficients of g is 1. Also, a primitive polynomial is irreducible if and only if it is irreducible in $F[X]$, where F is the field of fractions of R.

Now, if f is completely irreducible in Γ, then f is clearly irreducible in $R[X]$. Conversely, if f is irreducible in $R[X]$ and f is primitive, then f is irreducible in $F[X]$ and we apply Corollary 1.5. In general, there exist $d \in R$ and a primitive polynomial $g \in R[X]$ such that $f = dg$. Since f is irreducible so is g. Hence g is completely irreducible in Γ and consequently so is f. $\square$

In Section 4 we give an example of a polynomial of $R[X]$ which is irreducible in $R[X]$, but is not completely irreducible, where R is a commutative domain (Example 4.1).

2. Maximal ideals. As we said in the introduction, there are two interesting questions concerning maximal ideals that we want to consider here. First, determine all the prime ideals L of R such that there exists a maximal ideal M of $R[X]$ with $M \cap R = L$. By factoring out convenient ideals we may assume that $L = 0$. Second, determine all the R-disjoint maximal ideals of $R[X]$ in case the set of all these ideals is not empty.

We extend the terminology used for commutative rings [**7**]. If R is a prime ring, then the intersection of all the non-zero prime ideals of R is called the pseudo-radical of R and is denoted by $\mathrm{ps}(R)$.

We begin this section with the following extension of [**6**, Lemma 3].

LEMMA 2.1. *Assume that P is an R-disjoint prime ideal of $R[X]$ and L is a non-zero prime ideal of R. If $\rho(P) \nsubseteq L$, then $(P + L[X]) \cap R = L$.*

Proof. If $X \in P$, then $P = XR[X]$ and for $r \in (P + L[X]) \cap R$ we easily obtain $r \in L$. So we may assume $X \notin P$.

Suppose that there exists $r \in R \backslash L$ such that $r = h_1 + h_2$, for some $h_1 \in P$ and $h_2 \in L[X]$. It follows that there exists $g = X^m a_m + \ldots + a_0 \in P$ with $a_0 \notin L$ and $a_i \in L$ for $i \geq 1$. Take such a g of minimal degree with respect to these conditions and choose a polynomial $f = X^n b_n + \ldots + b_0 \in P$ of minimal degree with respect to $b_n \notin L$.

By the assumption there exists $c \in R$ such that $a_0 c b_n \notin L$. If $m \geq n$ we have $g c b_n - a_m c X^{m-n} f \in P$, which contradicts the minimality of ∂g. In case $m < n$ put $h = a_0 c f - g c b_0 \in P$. Then $h = (X^{n-1} c_{n-1} + \ldots + c_0) X$, where $c_i \in R$ and $c_{n-1} = a_0 c b_n \notin L$. Since P is prime and $X \notin P$ we have $X^{n-1} c_{n-1} + \ldots + c_0 \in P$, contradicting the minimality of ∂f. □

COROLLARY 2.2. *Let M be a maximal ideal of $R[X]$. with $M \cap R = 0$. Then $0 \neq \rho(M) \subseteq \mathrm{ps}(R)$.*

Proof. Since $M \neq 0$ we have $\rho(M) \neq 0$. If L is a non-zero prime ideal of R we have $M + L[X] = R[X]$. So $\rho(M) \subseteq L$ by Lemma 2.1. □

Corollary 2.2 shows that if there exists an R-disjoint maximal ideal of $R[X]$, then $\mathrm{ps}(R) \neq 0$. The converse is not true, in general (see Example 4.2). However, it is true under some additional assumption which is satisfied, for example, if R is a *PI* ring [**11**, Theorem 1.6.27].

PROPOSITION 2.3. *Let R be a prime ring such that every non-zero ideal of R contains a central element. Then there exists an R-disjoint maximal ideal of $R[X]$ if and only if $\mathrm{ps}(R) \neq 0$.*

Proof. Assume that $\mathrm{ps}(R) \neq 0$ and take $0 \neq c \in Z \cap \mathrm{ps}(R)$. Put $f = Xc + 1$. Then $f \in \Gamma$ and $[f]$ is an R-disjoint prime ideal of $R[X]$. If I is a maximal ideal with $I \supset [f]$ we have $I \cap R \neq 0$. Hence $c \in \mathrm{ps}(R) \subseteq I \cap R$ and $1 \in I$ follows, a contradiction. Thus $[f]$ is a maximal ideal. The proof is complete by Corollary 2.2. □

Now we give a more general criterion. Denote by $1 + X\,\mathrm{ps}(R)[X]$ the set of all the polynomials of the type $f = X^n a_n + \ldots + X a_1 + 1 \in R[X]$, where $a_i \in \mathrm{ps}(R)$ for $1 \leq i \leq n$.

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Proposition 2.4. *Assume that* M *is an* R*-disjoint maximal ideal of* $R[X]$. *Then one of the following possibilities occurs*:

(i) $X \in M$, *R is simple and* $M = XR[X]$.

(ii) $X \notin M$ *and* $M \cap (1 + X\,\mathrm{ps}(R)[X]) \neq \varnothing$.

Proof. If $X \in M$ (i) follows. Assume $X \notin M$. Then $X\,\mathrm{ps}(R)[X] \nsubseteq M$ and so $M + X\,\mathrm{ps}(R)[X] = R[X]$. Thus there exist $f \in M$ and $g \in \mathrm{ps}(R)[X]$ such that $f + Xg = 1$. Consequently $f \in 1 + X\,\mathrm{ps}(R)[X]$ and we are done. □

Corollary 2.5. *Let* R *be a prime ring which is not simple and let* M *be a prime ideal of* $R[X]$ *with* $M \cap R = 0$. *Then* M *is a maximal ideal if and only if* $M \cap (1 + X\,\mathrm{ps}(R)[X]) \neq \varnothing$.

Proof. Assume that $f = X^n a_n + \ldots + Xa_1 + 1 \in M$, where $a_i \in \mathrm{ps}(R)$ for $1 \leq i \leq n$. If I is a maximal ideal of $R[X]$ such that $I \supset M$ we obtain $1 \in I$ as in Proposition 2.3. Then M is a maximal ideal. The rest is clear. □

The intersection of a finite family of closed ideals is closed [**3**, Corollaries 3.4 and 3.5]. So for any $f \in R[X]$ with $\partial f \geq 1$ there exists the smallest closed ideal of $R[X]$ containing f. We denote this ideal again by $[f]$. If there is no closed ideal which contains f we put $[f] = R[X]$.

Corollary 2.6. *There exists an* R*-disjoint maximal ideal of* $R[X]$ *if and only if either* R *is simple or there exists* $f \in 1 + X\,\mathrm{ps}(R)[X]$ *such that* $[f] \neq R[X]$.

Proof. If there exists a maximal ideal M of $R[X]$ with $M \cap R = 0$ we have that either $X \in M$ and R is simple or there exists $f \in M \cap (1 + X\,\mathrm{ps}(R)[X])$. Thus $[f] \subseteq M \neq R[X]$.

Conversely, if R is simple, then $XR[X]$ is a maximal ideal. If R is not simple we choose a polynomial $f \in 1 + X\,\mathrm{ps}(R)[X]$ such that $[f] \neq R[X]$. Then there exists an R-disjoint ideal M which is maximal with respect to $[f] \subseteq M$. Hence M is a maximal ideal by Corollary 2.5. □

Remark 2.7. The set of all the R-disjoint maximal ideals $\mathcal{M}$ of $R[X]$ can now be determined as follows. If there is no $f \in 1 + X\,\mathrm{ps}(R)[X]$ such that $[f] \neq R[X]$, then $\mathcal{M} = \varnothing$. Assume there exists $f \in 1 + X\,\mathrm{ps}(R)[X]$ with $[f] \neq R[X]$. Then for such a polynomial f there exist a uniquely determined finite family of R-disjoint prime ideals P_{1f}, $P_{2f}, \ldots, P_{n_f f}$ such that $[f] = \bigcap_i [P_{if}^{e_i}]$, where $e_i \geq 1$ [**3**, Theorem 3.1]. Then $\mathcal{M} = \{P_{if}\}$, where $f \in 1 + X\,\mathrm{ps}(R)[X]$, $[f] \neq R[X]$ and $1 \leq i \leq n_f$.

An equivalent formulation can be given. First, we say that two polynomials g and h of Γ are essentially different if $gr\,\mathrm{lc}(h) - \mathrm{lc}(g)rh \neq 0$ for some $r \in R$. By [**3**, Corollary 1.3] g and h are essentially different if and only if $[g] \neq [h]$.

Given $f \in R[X]$ we say that a polynomial $g \in \Gamma$ essentially divides f if $f \in [g]$, i.e., there exists $0 \neq H \lhd R$ with $fH\,\mathrm{lc}(g) \subseteq R[X]g$.

Take any polynomial $f \in 1 + X\,\mathrm{ps}(R)[X]$ with $[f] \neq R[X]$. Then there exist a finite family of essentially different Γ-completely irreducible polynomials $g_{if} \in \Gamma$ which essentially divide f, $1 \leq i \leq n_f$. Then $\mathcal{M} = \{[g_{if}]\}$, where f is as above and $1 \leq i \leq n_f$.

Remark 2.8. It is well-known that the Brown-McCoy radical $G(R[X])$ of $R[X]$ is equal to $I[X]$ for the ideal $I = G(R[X]) \cap R$ of R. Corollary 2.6 shows that the ideal I

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equals the intersection of all the ideals L of R such that either R/L is simple or there exists $f \in 1 + X\,\mathrm{ps}(R/L)[X]$ such that $[f] \neq (R/L)[X]$.

3. Maximal ideals generated by poynomials of minimal degree. As we have seen in Section 1, if P is an R-disjoint prime ideal of $R[X]$, then P is determined by just one polynomial of minimal degree in P. However, it is well-known that P is not necessarily generated by its polynomials of minimal degree (see Example 4.1).

The purpose of this section is to study when a maximal ideal M of $R[X]$ with $M \cap R = 0$ is generated by the polynomials of minimal degree in M.

Suppose that f, g are in $R[X]$ and $f = gX^i$. In this case we will denote g by fX^{-i}.

In this section we will use frequently polynomials $f \in 1 + XR[X]$. A polynomial of this type will be called *comonic*.

Let S be a ring. Recall that an element $a \in S$ is said to be normal if $Sa = aS$. In this section we will consider ideals which are right principal generated by normal elements. An ideal of this type is, of course, also a left principal ideal and will be called simply a principal ideal. Also, *ideal generated* by some elements means *generated as right ideal* by those elements. We will see that in our case this is the same as saying generated *as left ideal*, instead of *right ideal.*

For an R-disjoint ideal M of $R[X]$ we will consider the following conditions:

(M_1) M is generated by polynomials of minimal degree.

(M_2) M is a principal ideal generated by a central polynomial.

(M_3) M is a principal ideal generated by a normal polynomial of minimal degree.

(M_4) $\rho(M) = \tau(M)$.

(M_5) $\rho(M) = \tau(M)$ is a principal ideal generated by a normal element of R.

(M_6) M contains a polynomial of minimal degree which is comonic.

(M_7) The right ideal of R generated by all the coefficients of polynomials of minimal degree of M is R.

The main purpose of this section is to prove the following theorem.

THEOREM 3.1. *Let M be an R-disjoint maximal ideal of $R[X]$ with $X \notin M$. Then conditions* (M_1) – (M_7) *are equivalent.*

Note that for an R-disjoint maximal ideal M with $X \in M$ conditions (M_1) – (M_7), with the exception of (M_6), are all satisfied. Thus from Theorem 3.1 the following is clear.

COROLLARY 3.2. *Let M be an R-disjoint maximal ideal of $R[X]$. Then M is generated by polynomials of minimal degree if and only if either $X \in M$ or M contains a polynomial of minimal degree f which is comonic. In the first case $M = XR[X]$ and in the second case $M = fR[X]$. In particular, in this case conditions* (M_2) – (M_5) *and* (M_7) *are satisfied.*

REMARK 3.3. We consider above ideals which are generated in some way as right ideals. Since there are several conditions (M_i) which are symmetrical, the same result holds if we change right by left.

We prove the theorem in several steps.

LEMMA 3.4. *Let M be an R-disjoint ideal of $R[X]$. Then the following implications hold*: (M_2) ⇒ (M_3) ⇒ (M_5) ⇒ (M_4) ⇒ (M_1) *and* (M_6) ⇒ (M_7).

Proof. $(M_2) \Rightarrow (M_3)$ If $M = fR[X]$, where $f \in Z[X]$, we have that f is a polynomial of minimal degree in M because $\operatorname{lc}(f)$ is not a zero divisor in R. Then (M_3) holds.

$(M_3) \Rightarrow (M_5)$. Assume that $M = fR[X] = R[X]f$, where $\partial f = \operatorname{Min}(M)$, and put $a = \operatorname{lc}(f)$. If $ba = 0$ we have $bfra = barf = 0$ for every $r \in R$, since $f \in \Gamma$. Thus $bf = 0$. Using this fact we easily see that $\rho(M) = \tau(M) = aR$. Similarly we obtain that $\tau(M) = Ra$ and (M_5) holds.

$(M_4) \Rightarrow (M_1)$. Let L denote the right ideal of $R[X]$ generated by all the polynomials of minimal degree in M. If $f \in M$ and $\partial f = \operatorname{Min}(M)$ we clearly have $f \in L$. Assume that every $g \in M$ with $\operatorname{Min}(M) \leq \partial g \leq m-1$ is in L and take $h \in M$ of degree m with $\operatorname{lc}(h) = b$. Then $b \in \rho(M) = \tau(M)$ and so there exists $f \in M$ such that $\partial f = \operatorname{Min}(M)$ and $\operatorname{lc}(f) = b$. Thus $h - fX^{m-n} \in M$ and $\partial(h - fX^{m-n}) < m$. Therefore $h - fX^{m-n} \in L$ and since $f \in L$ we obtain $h \in L$. Consequently $L = M$.

The proof is complete since the other implications are evident. □

LEMMA 3.5. *Let M be an R-disjoint prime ideal of $R[X]$ with $X \notin M$. Then* $(M_6) \Rightarrow (M_2)$.

Proof. Put $\operatorname{Min}(M) = n$ and take a comonic polynomial $f = X^n a_n + \ldots + Xa_1 + 1 \in M$, where $a_i \in R$. Since $(rf - fr)X^{-1} \in M$, for every $r \in R$, and $\partial((rf - fr)X^{-1}) < n$, we have that $f \in Z[X]$. Suppose $g = X^n b_n + \ldots + b_0 \in M$. Then $(g - fb_0)X^{-1} \in M$ and so $g = fb_0$. Assume, by induction, that every polynomial of M of degree smaller than m is in $fR[X]$ and take $h = X^m c_m + \ldots + c_0 \in M$, $c_m \neq 0$. Then $(h - fc_0)X^{-1} \in M$ and $\partial((h - fc_0)X^{-1}) < m$. Thus $(h - fc_0)X^{-1} \in fR[X]$ and it follows that $h \in fR[X]$. Consequently $M = fR[X]$. □

LEMMA 3.6. *Let M be an R-disjoint maximal ideal of $R[X]$ with $X \notin M$. Then conditions* $(M_1) - (M_6)$ *are equivalent.*

Proof. It is enough to prove $(M_1) \Rightarrow (M_6)$, by Lemmas 3.4 and 3.5. Take a polynomial $f = X^m a_m + \ldots + Xa_1 + 1 \in M$ (Proposition 2.4, (ii)). Then there exist polynomials of minimal degree $g_1, \ldots, g_t$ in M and $h_1, \ldots, h_t$ in $R[X]$ such that $f = \sum_{i=1}^{t} g_i h_i$. Hence $\sum_{i=1}^{t} g_{i0} h_{i0} = 1$, where g_{i0} and h_{i0} are the constant terms of g_i and h_i, respectively. Consequently $\sum_{i=1}^{t} g_i h_{i0}$ is a polynomial of minimal degree in M which is comonic. □

To complete the proof of Theorem 3.1 it remains to prove that $(M_7) \Rightarrow (M_6)$. This is the most difficult part of the proof and requires some preparation.

Let I be an R-disjoint ideal of $R[X]$ and set $t \geq \operatorname{Min}(I)$. We put

$$\mu_t(I) = \{b \in R;\ \text{there exists } h \in I \text{ with } \partial h = t \text{ and } \operatorname{lc}(h) = b\} \cup \{0\}.$$

Then $\mu_t(I)$ is an ideal of R and for $n = \operatorname{Min}(I) \leq t \leq s$ we have $\tau(I) \subseteq \mu_t(I) \subseteq \mu_s(I) \subseteq \rho(I)$ and $\rho(I) = \bigcup_{t \geq n} \mu_t(I)$.

LEMMA 3.7. *Assume that P is an R-disjoint prime ideal of $R[X]$ which contains a comonic polynomial f of degree $m \geq \operatorname{Min}(P) = n$, where m is minimal. We have*:

(i) *If $m = n$, then $f \in Z[X]$ and $P = fR[X]$.*

(ii) *If $m > n$, then $\rho(P) = \mu_m(P) = \mu_{m-1}(P) + aR$, where $a = \operatorname{lc}(f) \notin \mu_{m-1}(P)$ and P is generated by its polynomials of degree $\leq m$.*

Proof. By Lemma 3.5 it remains to prove (ii). Assume that

$$f = X^m a + X^{m-1}a_{m-1} + \ldots + Xa_1 + 1 \in P,$$

where $a \neq 0$ and $m > n$. Put $\mu_m(P) = \mu(P)$ and $\mu_{m-1}(P) = \gamma(P)$.

Take $h = X^t c + \ldots + c_0 \in P$, where $c \neq 0$. If $t > m$ we have that $(h - fc_0)X^{-1} \in P$, $\delta((h - fc_0)X^{-1}) < t$ and $\text{lc}((h - fc_0)X^{-1}) = c$. Using an induction argument we easily obtain $c \in \mu(P)$ and $h \in fR[X] + I$, where I is the right ideal of $R[X]$ generated by all the polynomials of P of degree $\leq m - 1$. Consequently $\rho(P) = \mu(P)$ and $P = fR[X] + I$.

It is clear that $\gamma(P) + aR \subseteq \mu(P)$. A similar computation as above shows that if $t = m$, then $c - ac_0 \in \gamma(P)$, where $h = X^m c + \ldots + c_0 \in P$. Thus $\mu(P) = \gamma(P) + aR$.

Finally, if $a \in \gamma(P)$ there exists $g \in P$ with $\partial g = m - 1$ and $\text{lc}(g) = a$. Then $f - gX \in P$ is a comonic polynomial of degree smaller than m, a contradiction. □

Proposition 2.4 and Lemma 3.7 give the following corollary which is of independent interest.

COROLLARY 3.8. *Let M be an R-disjoint maximal ideal of $R[X]$. Then there exists an integer $m \geq \text{Min}(M)$ such that M is generated by its polynomials of degree at most m.*

Now we are ready to complete the proof of Theorem 3.1.

LEMMA 3.9. *Let M be an R-disjoint maximal ideal of $R[X]$ with $X \notin M$. Then* $(M_7) \Rightarrow (M_6)$.

Proof. By way of contradiction we assume that $m > n = \text{Min}(M)$ is the smallest integer such that M contains a comonic polynomial $f = X^m a + X^{m-1}a_{m-1} + \ldots + Xa_1 + 1$ of degree m. Take any $g = X^n b_n + \ldots + b_0 \in M$, a polynomial of degree n. We show by induction that $b_j Ra \subseteq \gamma(M)$, for $0 \leq j \leq n$, where $\gamma(M) = \mu_{m-1}(M)$. Then we can use (M_7) to obtain $a \in \gamma(M)$. This is a contradiction by Lemma 3.7, (ii).

Denote by I_j the right ideal of R generated by $b_0, b_1, \ldots, b_j$, $0 \leq j \leq n$. For any $r_1 \in R$ put $f_{r_1} = (b_0 r_1 f - g r_1)X^{-1} \in M$. Considering the leading coefficient of f_{r_1} we obtain $b_0 r_1 a \in \gamma(M)$. Thus $b_0 Ra \subseteq \gamma(M)$. Also the coefficient of X^j in f_{r_1}, for $0 \leq j \leq n - 1$, is of the type $c_{j,r_1} = -b_{j+1} r_1 + \alpha$, where $\alpha \in I_0$. We repeat the argument starting with f and f_{r_1}. For any $r_2 \in R$ we put $f_{r_1,r_2} = (c_{0,r_1} r_2 f - f_{r_1} r_2)X^{-1} \in M$. Then $c_{0,r_1} r_2 a \in \gamma(M)$ and it follows that $b_1 Ra \subseteq \gamma(M)$. Also, the coefficient of X^j in f_{r_1,r_2}, for $0 \leq j \leq n - 2$, is of the type $c_{j,r_1,r_2} = b_{j+2} r_1 r_2 + \beta$, where $\beta \in I_1$. The proof can easily be completed using an induction argument. □

Condition (M_7) of Theorem 3.1 has some interesting applications. First, assume that R is a local ring with the maximal ideal $\mathbf{M}$ and let M be a maximal ideal of $R[X]$ with $M \cap R = 0$. Denote by M_0 the set of all the polynomials of minimal degree of M. We have

COROLLARY 3.10. *Let R be a local ring with the maximal ideal $\mathbf{M}$ and let M be an R-disjoint maximal ideal of $R[X]$. Then M is generated by polynomials of minimal degree if and only if $M_0 \nsubseteq \mathbf{M}[X]$.*

Proof. Note that the right ideal I of R generated by all the coefficients of polynomials in M_0 is actually a two-sided ideal. So $I = R$ if and only if $I \nsubseteq \mathbf{M}$. □

The second application concerns localizations. Assume that M is a maximal ideal of

$R[X]$ which is R-disjoint. For any prime ideal $\mathfrak{p}$ of Z, the localization $R_{\mathfrak{p}}$ is a prime ring. It is easy to see that the ideal $M_{\mathfrak{p}}$ of $R_{\mathfrak{p}}[X]$ is an $R_{\mathfrak{p}}$-disjoint maximal ideal and $\mathrm{Min}(M_{\mathfrak{p}}) = \mathrm{Min}(M)$.

Let I be the right ideal of R generated by all the coefficients of all the polynomials of minimal degree in M. Then $I_{\mathfrak{p}}$ is the similar ideal for $M_{\mathfrak{p}}$. Since $I = R$ if and only if $I_{\mathfrak{p}} = R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of Z we have the following result.

COROLLARY 3.11. *Let M be a R-disjoint maximal ideal of $R[X]$. Then M is generated by polynomials of minimal degree if and only if $M_{\mathfrak{p}}$ is generated by polynomials of minimal degree, for every prime ideal $\mathfrak{p}$ of Z.*

REMARK 3.12. The above corollary can also be stated as follows: M is a principal ideal generated by a central polynomial if and only if $M_{\mathfrak{p}}$ is a principal ideal generated by a central polynomial, for every prime ideal $\mathfrak{p}$ of Z.

4. Examples and additional results. We begin this section with the following example showing that there exist irreducible polynomials which are not completely irreducible as well as R-disjoint prime ideals which are not generated by polynomials of minimal degree.

EXAMPLE 4.1. (c.f. [**3**, Example 3.6]). Let $\mathbb{Q}$ be the field of rational numbers and let R be the integral domain of all the power series of $\mathbb{Q}[Y]$ having the coefficient of Y equal to zero. The field of fractions of R is $F = \mathbb{Q}[Y, Y^{-1}]$. Then the polynomial $X^2 - Y^2 \in R[X]$ is irreducible in $R[X]$, but is not completely irreducible since $X^2 - Y^2 = (X + Y)(X - Y)$ in $F[X]$.

The ideal $P = (X - Y)F[X] \cap R[X]$ is a prime ideal of $R[X]$ which is not generated by polynomials of minimal degree. Indeed, we can easily see that if $f \in P$ and $\partial f = 1$, then

$$f = (X - Y)\sum_{i=2}^{\infty} q_i Y^i, \qquad q_i \in \mathbb{Q}.$$

If $X^2 - Y^2$ is in the ideal generated by the polynomials of minimal degree of P we have

$$X^2 - Y^2 = \sum_{i=1}^{t} (a_i X + b_i)(c_i X + d_i),$$

where a_i, b_i, c_i, d_i are in R and $c_i X + d_i \in P$. So $\sum_{i=1}^{t} a_i c_i = 1$; hence $h = \sum_{i=1}^{t} a_i(c_i X + d_i) \in P$, where h is a monic polynomial of degree one, a contradiction.

In Section 2 we proved that if there exists an R-disjoint maximal ideal of $R[X]$, then $\mathrm{ps}(R) \neq 0$. The next example shows that in general the converse is not true.

EXAMPLE 4.2. Let R be a subdirectly irreducible ring with identity and idempotent heart H (for example, take R as the ring of all infinite matrices over a field having only finitely many non-zero entries and adjoin an identity). Then R is a prime ring, $\mathrm{ps}(R) \neq 0$ and $R[X]$ contains no R-disjoint maximal ideal.

Indeed, if M is a maximal ideal of $R[X]$ with $M \cap R = 0$, then $M + H[X] = R[X]$. Hence $H[X]$ can be homomorphically mapped onto the ring with identity $R[X]/M$. This is a contradiction as easily follows from [**8**, Theorem 10].

Another example for the same question is the following.

EXAMPLE 4.3. Let A be a finitely generated nil algebra over a field F which is not nilpotent such that $A[X]$ is nil [**1**, Lemma 59]. Take an ideal I which is maximal with respect to the property $A^n \nsubseteq I$, for every integer n (apply Zorn's Lemma). Put $B = A/I$, a prime algebra with the same properties as A, and let R denote the prime algebra obtained from B by adjoining an identity (take the subring of the Martindale ring of quotients of B generated B and 1). Then $\mathrm{ps}(R) = B \neq 0$. If M is an R-disjoint maximal ideal of $R[X]$ with $M \cap R = 0$, then $B[X] + M = R[X]$. So $B[X]$ can be mapped onto a ring with 1. This is impossible since $B[X]$ is nil.

The question of whether there exists an R-disjoint maximal ideal of $R[X]$ is related to Question 13 in [**10**] of whether $T[X]$ is Brown-McCoy radical if T is a nil ring. If this last question has a negative answer, there exists a nil ring T such that $T[X]$ can be mapped onto a ring with 1. Then $T[X]$ can also be mapped onto a simple ring with identity S. Thus we have an epimorphism $\varphi: T[X] \to S$ and by factoring out the ideals $T \cap \mathrm{Ker}\, \varphi$ and $(T \cap \mathrm{Ker}\, \varphi)[X]$ from T and $T[X]$, respectively, we may assume that T is prime. We have

PROPOSITION 4.4. *Let T be a prime nil ring. Then $T[X]$ can be mapped onto a ring with* 1 *if and only if T is an ideal of a prime ring with identity R such that $R[X]$ has an R-disjoint maximal ideal.*

Proof. Assume that there exist a simple ring with identity S and an epimorphism $\varphi: T[X] \to S$. Let R be the extension of T to a prime ring with 1. Then the epimorphism φ can be extended to an epimorphism $\psi: R[X] \to S$ in a natural way. Thus $\mathrm{Ker}\, \psi$ is an R-disjoint maximal ideal of $R[X]$.

Conversely, if T is an ideal of a prime ring with identity R and M is a maximal ideal of $R[X]$ with $M \cap R = 0$, then $T[X] \to R[X] \to R[X]/M$ is an epimorphism of $T[X]$ onto a ring with 1. □

The question of finding conditions under which there exists an R-disjoint maximal ideal of $R[X]$ which is generated by polynomials of minimal degree has the following precise answer.

PROPOSITION 4.5. *Let R be a prime ring with* 1. *Then there exists an R-disjoint maximal ideal of $R[X]$ which is generated by polynomials of minimal degree if and only if* $\mathrm{ps}(R) \cap Z \neq 0$.

Proof. If $\mathrm{ps}(R) \cap Z \neq 0$ there exists $0 \neq c \in \mathrm{ps}(R) \cap Z$. Then $M = (Xc + 1)R[X]$ is a maximal ideal of the required type.

Conversely, let M be a maximal ideal which satisfies the above conditions. If $X \in M$, then R is simple and $\mathrm{ps}(R) = R$. We are done in this case. So we may assume $X \notin M$.

By Theorem 3.1 there exists a comonic polynomial $f = X^n a_n + \dots + Xa_1 + 1 \in M \cap Z[X]$ such that $M = fR[X]$, where $a_n \neq 0$ and $n = \mathrm{Min}(M)$. Also, by Proposition 2.4, there exists $g = X^m b_m + \dots + Xb_1 + 1 \in M$ with $b_i \in \mathrm{ps}(R)$ for $1 \le i \le m$. Thus $g = fh$, for some $h = X^t c_t + \dots + Xc_1 + c_0 \in R[X]$, where $t = m - n$. Clearly $c_0 = 1$. If $t = 0$, then $f = g$, so $a_n \in \mathrm{ps}(R) \cap Z$ and we are done. So we may assume $t \ge 1$.

Let L be a non-zero prime ideal of R. Then $a_n R c_t = R a_n c_t = R b_m \subseteq L$. It follows that either $a_n \in L$ or $c_t \in L$. Suppose that $a_n, \dots, a_{i+1} \in L$, $a_i \notin L$, $c_t, \dots, c_{j+1} \in L$ and $c_j \notin L$,

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for some i, $j \geq 1$. From $g = fh$ we easily obtain that $a_i R c_j \subseteq L$, a contradiction. Consequently $a_i \in L$ for every $1 \leq i \leq m$. In particular, $\mathrm{ps}(R) \cap Z \neq 0$. □

The proof of the above proposition shows that if M is an R-disjoint maximal ideal of $R[X]$ which is generated by polynomials of minimal degree and $X \notin M$, then the polynomial of minimal degree of the type $f = X^n a_n + \ldots + X a_1 + 1 \in M \cap Z[X]$ satisfies $a_i \in \mathrm{ps}(R)$ for $1 \leq i \leq n$. Also, f is Γ-completely irreducible by Theorem 1.3.

COROLLARY 4.6. *Assume that* $\mathrm{ps}(R) \cap Z \neq 0$. *Then there is a one-to-one correspondence between the following.*

(i) *The set of all the R-disjoint maximal ideals M of $R[X]$ with $X \notin M$ and which are generated by polynomials of minimal degree.*

(ii) *The set of all the polynomials of the type $f = X^n a_n + \ldots + X a_1 + 1$ which are completely irreducible in Γ, where $a_i \in \mathrm{ps}(R) \cap Z$ for $1 \leq i \leq n$.*

Moreover, this correspondence associates the maximal ideal M with the polynomial f if $M = fR[X]$.

Proof. If f is a polynomial of the type given in (ii), then $f \in \Gamma$ and put $M = [f]$. Then M is a maximal ideal which is R-disjoint, $M = fR[X]$ and f is Γ-completely irreducible (apply Theorem 1.3 and the results of Sections 2 and 3). The rest is clear. □

REMARK 4.7. If R is a commutative domain and M is an R-disjoint maximal ideal of $R[X]$ which is a principal ideal with generator f, then $\partial f = \mathrm{Min}(M)$ since the leading coefficient of f is not a zero divisor. The same result holds if R is a completely prime ring. However, it seems to be an open problem whether there exists a maximal R-disjoint ideal M of $R[X]$ which is a principal ideal but is not generated by polynomials of minimal degree.

We end the paper with the following example.

EXAMPLE 4.8. Let R be a non-simple prime algebra over a non-denumerable field F such that card $F > \dim_F R$. Then $R[X]$ contains no R-disjoint maximal ideal.

Indeed, if M is an R-disjoint maximal ideal of $R[X]$, then $\mathrm{ps}(R) \neq 0$. However, $\mathrm{ps}(R) \subseteq G(R)$, the Brown-McCoy radical of R ($\mathrm{ps}(R) \neq R$ because R is not simple). Hence $G(R) \neq 0$. Also, by [**8**, Theorem 9] we have $G(R)[X] = G(R[X])$. Thus $G(R) \subseteq M \cap R = 0$, a contradiction.

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