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ROOT CLOSURE IN INTEGRAL DOMAINS, III

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ABSTRACT. If A is a subring of a commutative ring B and if n is a positive integer, a number of sufficient conditions are given for "$A[[X]]$ is n-root closed in $B[[X]]$" to be equivalent to "A is n-root closed in B." In addition, it is shown that if S is a multiplicative submonoid of the positive integers $\mathbb{P}$ which is generated by primes, then there exists a one-dimensional quasilocal integral domain A (resp., a von Neumann regular ring A) such that $S = \{n \in \mathbb{P} \mid A \text{ is } n\text{-root closed}\}$ (resp., $S = \{n \in \mathbb{P} \mid A[[X]] \text{ is } n\text{-root closed}\}$).

1. **Introduction.** All rings considered below are commutative with identity. As usual, if A is a subring of B (with the same 1) and $n \geq 1$ is an integer, we say that A is n-root closed in B if $b^n \in A$ with $b \in B$ implies $b \in A$; in case B is the total quotient ring of A and the above condition holds, we say that A is n-root closed. Much of the literature on n-root closedness has focused on domains and connections with seminormality (*cf*. [6], [7], [1], [3], [8], [5]); the emphasis on von Neumann regularity in [14] is a notable exception.

It is known [8, Theorem 1] that if A is a subring of B, then the polynomial ring $A[X]$ is n-root closed in $B[X]$ if and only if A is n-root closed in B. Hence if A is a domain with quotient field K, then $A[X]$ is n-root closed if and only if $A[X]$ is n-root closed in $K[X]$, since $K[X]$ is integrally closed and contained in the quotient field $K(X)$ of $A[X]$. Thus if A is a domain, then $A[X]$ is n-root closed if and only if A is n-root closed. However, the analogous assertions for rings of formal power series do not hold. Indeed, [14, Example 1] shows that $\mathbb{Z}[[\mathrm{X}]]$ is not n-root closed in $\mathbb{Q}[[X]]$ for any $n \geq 2$, even though $\mathbb{Z}$ is root closed (that is, n-root closed for each n).

In [5], necessary and sufficient conditions were given on an extension $A \subset B$ of domains for $A[[\mathbf{X}]]$ to be n-root closed in $B[[\mathbf{X}]]$. Moreover, it was shown that $A[[X]]$ is n-root closed for a single indeterminate X if and only if $A[[\mathbf{X}]]$ is n-root closed for any nonempty family of indeterminates $\mathbf{X}$ [5, Corollary 2.7]. (Here, $A[[\mathbf{X}]]$ is defined as the union of the rings $A[[\mathbf{Y}]]$, where $\mathbf{Y}$ ranges over the finite subsets of $\mathbf{X}$: *cf.* [10, (1.1)].)

In the second section, we give several cases where "$A[[X]]$ is n-root closed in $B[[X]]$" is equivalent to "A is n-root closed in B." We also show that if $A[[X]]$ is n-root closed in $K[[X]]$ for a subring A of a field K, then A is a field. Some of the material in the second section can be deduced from [5] (we specify the relevant connections at the appropriate points below), but it is included here for two reasons. First, the approach and methods of

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proof here are often more direct than those of [5] because the papers have different goals. Second, much of the material in the second section is preparatory for our main result in the third section and, thus, helps to make the paper self-contained.

The final section is devoted to the multiplicative monoid $C(A)$, in the sense of [1] and [2], namely $C(A) = \{n \in \mathbb{P} \mid A$ is n-root closed$\}$, where $\mathbb{P}$ is the set of positive integers. Any such monoid is generated by primes since for integers $m, n \geq 2$, A is mn-root closed (in B) if and only if A is both m-root closed (in B) and n-root-closed (in B). Conversely, it was shown in [1, Theorem 2.7] that any multiplicative submonoid of $\mathbb{P}$ generated by primes can be realized as $C(A)$ for a suitable domain A; if the monoid does not contain 2, then it was shown in [2, Theorem 6] that A can be arranged to be a one-dimensional Noetherian local domain. In Theorem 3.3, we remove the restriction on the prime 2, at the expense of replacing "Noetherian local" with "quasilocal," while preserving the one-dimensionality of the exhibited domain A. We use some results from Section 2 and [5] to show that the domain A in Theorem 3.3 can also be chosen so that $C(A[[\mathbf{X}]]) = C(A)$. This equality fails for an arbitrary ring A, although $C(A[\mathbf{X}]) = C(A)$ for all A and any $\mathbf{X}$. Our final result, Theorem 3.5, states that any multiplicative submonoid of $\mathbb{P}$ generated by primes can be realized as $C(A[[X]])$ for a suitable von Neumann regular ring A. In particular, if a ring A is root closed (that is, if $C(A) = \mathbb{P}$), then $C(A[[X]])$ can be essentially arbitrary.

2. **Root closure.** As usual, a domain A is said to be *completely integrally closed* (in its quotient field K) if $cu^i \in A$ for all integers $i \geq 1$, $0 \neq c \in A$, $u \in K$ implies $u \in A$. Recall that any completely integrally closed domain is integrally closed [10, Theorem 13.1(2)], and hence root closed. Moreover, $A[[X]]$ is completely integrally closed if and only if A is completely integrally closed; also, if and only if $A[X]$ is completely integrally closed [10, Theorem 13.9]. Similarly, we recalled in the introduction that A is n-root closed if and only if $A[X]$ is n-root closed. However, as the next example shows, "n-root closed" does not behave so simply for power series rings.

EXAMPLE 2.1. (a) In [14, Example 1], it is shown that although $\mathbb{Z}$ is root closed, $\mathbb{Z}[[X]]$ is not n-root closed in $\mathbb{Q}[[X]]$ for any $n \geq 2$. Note that $\mathbb{Z}[[X]]$ is however root closed, in fact, completely integrally closed since $\mathbb{Z}$ is completely integrally closed.

(b) $A = \mathbb{Z} + T\mathbb{Q}[[T]]$ is root closed in $B = \mathbb{Q}[[T]]$ by [4, Theorem 1.7] since $\mathbb{Z}$ is root closed in $\mathbb{Q}$ and A and B have common ideal $T\mathbb{Q}[[T]]$. In fact, A is a two-dimensional Bézout domain [10, p.202, Exercise 11(4), and p. 286, Exercise 13(3)], and hence integrally closed (*cf*. [10, Theorem 23.4(1)]). However, $A[[X]]$ is not n-root closed in $B[[X]]$ for any $n \geq 2$ since $\mathbb{Z}[[X]]$ is not n-root closed in $\mathbb{Q}[[X]]$. ∎

Clearly, A is n-root closed (in B) if $A[[X]]$ is n-root closed (in $B[[X]]$). The converse is false; indeed, Example 2.1(b) shows that $A[[X]]$ need not be root closed even when A is integrally closed. Such results should be contrasted with the behavior of "seminormality." (As in [8], a domain A with quotient field K is said to be seminormal if $u \in K, u^2 \in A, u^3 \in A$ implies $u \in A$. If A is n-root closed for some $n \geq 2$, then A is seminormal;

but the converse is false.) On the positive side, $A[[X]]$ is seminormal if and only if A is seminormal [9].

We often restrict to the case where A is p-root closed in B for p a prime. When A has positive characteristic, our next lemma often allows us to restrict to studying n-root closure when $(n, \operatorname{char} A) = 1$, that is, when n is invertible in A. Lemma 2.2 (a) was stated without proof in [5, Remark 1.20] as it also follows from a much deeper result ([5, Theorem 1.18], fortified with [5, Remark 1.5(v)]).

LEMMA 2.2. *(a) Let A be a subring of a ring B with* $\operatorname{char} A = p$ *prime. Then* $A[[X]]$ *is p-root closed in* $B[[X]]$ *if and only A is p-root closed in B.*

(b) The following conditions are equivalent for a domain A with quotient field K and $\operatorname{char} A = p$ *prime:*

(i) A is p-root closed;

(ii) $A[[X]]$ *is p-root closed;*

(iii) $A[[X]]$ *is p-root closed in* $K[[X]]$.

PROOF. These assertions all follow easily from the fact that $(\sum a_i X^i)^p = \sum a_i^p X^{ip}$ for $\sum a_i X^i \in B[[X]]$ when $\operatorname{char} A = p$ is prime. ∎

Our next result isolates a reason that the domains in Example 2.1 fail to be n-root closed.

PROPOSITION 2.3. *Let A be a subring of a domain B with A n-root closed in B and n nonzero in A. Let $f = \sum b_i X^i \in B[[X]]$ with b_0 nonzero and $f^n \in A[[X]]$. Then $b_0 \in A$ and $f \in A[1/(nb_0)][[X]]$. If, in addition, b_0 and n are each invertible in A, then $f \in A[[X]]$.*

PROOF. Let $f = \sum b_i X^i \in B[[X]]$ with $f^n \in A[[X]]$. Then $b_0 \in A$ since A is n-root closed in B. Suppose now by induction that $b_0, \ldots, b_{m-1} \in A[1/nb_0)]$. Then, by comparing coefficients of X^m, we have that $nb_0^{n-1} b_m + b \in A \subset A[1/nb_0)]$, where b is a sum of products of $b_0, \ldots, b_{m-1}$. By the induction hypothesis, $b \in A[1/(nb_0)]$, and thus also $b_m \in A[1/(nb_0)]$. Hence $f \in A[1/nb_0)][[X]]$. The final assertion follows immediately. ∎

It may be helpful to record the fact that Proposition 2.3 remains valid if the hypothesis "A is n-root closed in B" is replaced with "$b_0 \in A$."

The special case of the next corollary when B is also a field is actually a special case of [14, Theorem 1], since any field is von Neumann regular. Corollary 2.4 is also a special case of [5, Theorem 1.27], since any field is a p-injective ring. However, the proof given below is much easier than the proofs of the results just cited from [14] and [5].

COROLLARY 2.4. *Let k be a subfield of a ring B. Then* $k[[X]]$ *is n-root closed in* $B[[X]]$ *if and only k is n-root closed in B.*

PROOF. Clearly, k is n-root closed in B if $k[[X]]$ is n-root closed in $B[[X]]$. For the converse, we may assume by Lemma 2.2(a) that n is nonzero in k. Let $f = \sum b_i X^i \in B[[X]]$ with $f^n \in k[[X]]$. By factoring out a suitable power of X from f, we may assume that b_0 is nonzero. Then $k[1/(nb_0)] = k$, and so $f \in k[[X]]$ by Proposition 2.3. ∎

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Our next theorem, which concentrates on the case when A is a subring of a field K, shows that Example 2.1 should not really be too surprising. The proof is a reworking of the proof of [13, Theorem, p. 171] and [12, footnote (2), p. 321]. The special case of Theorem 2.5 in which n is assumed invertible in A also follows from [5, Proposition 1.28].

THEOREM 2.5. *Let A be a subring of a field K and $n \geq 2$ such that n is nonzero in A. If $A[[X]]$ is n-root closed in $K[[X]]$, then A is a field. Thus, if A has quotient field K, then $A[[X]]$ is n-root closed in $K[[X]]$ if and only if $A = K$.*

PROOF. Let $0 \neq a \in A$. Recursively, one can verify that $1 + (n/a^2)X \in K[[X]]$ has an n-th root $f = 1 + (X/a^2) + c_2(X/a^2)^2 + \cdots \in K[[X]]$ with each c_i contained in the prime subfield of K. Thus $(af)^n = a^n + na^{n-2}X \in A[[X]]$. If $A[[X]]$ is n-root closed in $K[[X]]$, then $af \in A[[X]]$, and hence $1/a \in A$. Thus each nonzero $a \in A$ is a unit; hence A is a field. The final assertion is now clear. ∎

Our next two results, which will be used in Section 3, give some more cases in which "$A[[X]]$ is n-root closed in $B[[X]]$" is equivalent to "A is n-root closed in B." Theorem 2.6 may also be proved by applying [5, Proposition 1.30(2)].

THEOREM 2.6. *Let A and B be quasilocal rings, with A a subring of B and common maximal ideal M. Then the following conditions are equivalent:*

(1) A/M is n-root closed in B/M;

(2) A is n-root closed in B;

(3) $A[[X]]$ is n-root closed in $B[[X]]$.

PROOF. By Corollary 2.4, we may assume that M is nonzero. The equivalence of (1) and (2) then follows from [4, Theorem 1.7]. Since A and B have common maximal ideal M, $A[[X]]$ and $B[[X]]$ have $M[[X]]$ as a common ideal. Since $(A/M)[[X]]$ and $A[[X]]/M[[X]]$ are naturally isomorphic, (1) and (3) are equivalent by Corollary 2.4 and [4, Theorem 1.7] again. ∎

COROLLARY 2.7. *Let A and B be quasilocal domains, with A a subring of B and common nonzero maximal ideal M. Suppose that $B[[X]]$ is n-root closed. Then $A[[\mathbf{X}]]$ is n-root closed for any nonempty family of indeterminates $\mathbf{X}$ if and only if A/M is n-root closed in B/M.*

PROOF. By Theorem 2.6, $A[[X]]$ is n-root closed in $B[[X]]$ if and only A/M is n-root closed in B/M. Moreover, since $B[[X]]$ is n-root closed, $A[[X]]$ is n-root closed if $A[[X]]$ is n-root closed in $B[[X]]$. Since $A[[X]]$ and $B[[X]]$ have common nonzero ideal $M[[X]]$, they also have the same quotient field. Thus $A[[X]]$ is n-root closed in $B[[X]]$ if and only if $A[[X]]$ is n-root closed. By [5, Corollary 2.7], $A[[\mathbf{X}]]$ is n-root closed for any nonempty family of indeterminates $\mathbf{X}$ if and if $A[[X]]$ is n-root closed. Hence $A[[\mathbf{X}]]$ is n-root closed if and only if A/M is n-root closed in B/M. ∎

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3. **Realizing $C(A)$.** As in [1] and [2], for any ring A, we define $C(A) = \{n \in \mathbb{P} \mid A$ is n-root closed$\}$. Then $C(A)$ is a multiplicative submonoid of $\mathbb{P}$ generated by primes. In [1, Theorem 2.7], a monoid domain construction was used to show that any multiplicative submonoid S of $\mathbb{P}$ generated by primes can be realized as $C(A)$ for some domain A. In that construction, the domain A was typically quite large. In [2, Theorem 6], it was shown that A can be chosen to be a one-dimensional Noetherian local domain as long as $2 \notin S$. In Theorem 3.3, we show that A can be chosen to be a one-dimensional quasilocal domain. In fact, A will be of classical "$D + M$" type. Example 3.1 identifies $C(A)$ for such A. The particular domain A exhibited in Theorem 3.3 will be shown to satisfy $C(A[[\mathbf{X}]]) = C(A)$. Our final result, Theorem 3.5, states that any multiplicative submonoid of $\mathbb{P}$ generated by primes can also be realized as $C(A[[X]])$ for a suitable von Neumann regular ring A. For any such ring A, in contrast with the domains in Theorem 3.3, $C(A[[X]])$ differs by as much as possible from $C(A) = \mathbb{P}$.

EXAMPLE 3.1. Let B be a one-dimensional valuation domain of the form $K + M$, where K is a field contained in B and M is the maximal ideal of B. For k a proper subfield of K, the subring $A = k + M$ is a one-dimensional quasilocal domain which is not completely integrally closed. (A is integrally closed if and only if k is algebraically closed in K [10, p. 202, Exercise 12(2)]). Note that $B[[X]]$ is root closed, in fact completely integrally closed, since a one-dimensional valuation domain is completely integrally closed [10, Theorem 17.5]. Hence, by Corollary 2.7, $A[[\mathbf{X}]]$ is n-root closed if and only if k is n-root closed in K. Thus $C(A[[\mathbf{X}]]) = C(A) = \{n \in \mathbb{P} \mid k$ is n-root closed in $K\}$ for any nonempty family of indeterminates $\mathbf{X}$. Note that $A[[\mathbf{X}]]$ is always seminormal [9], but for suitable choices of k and K, $C(A[[\mathbf{X}]])$ may be generated by any prescribed set of primes (see Lemma 3.2 and Theorem 3.3). ∎

Let S be a multiplicative submonoid of $\mathbb{P}$ generated by some set $\mathcal{P}$ of positive primes. The key step in the following work is to identify a suitable associated field (denoted K_S below). We define inductively an increasing sequence of subfields of $\mathbb{R}$ by $K_0 = \mathbb{Q}$ and $K_{n+1} = K_n(\{x \in \mathbb{R} \mid x^p \in K_n$ for some $p \in \mathcal{P}\})$. Let $K_S = \cup K_n$. In particular, $K_S = \mathbb{Q}$ if $\mathcal{P}$ is the empty set.

LEMMA 3.2. *K_S is n-root closed in $\mathbb{R}$ if and only if $n \in S$.*

PROOF. We may assume that $n = p$ is prime. First , suppose that $p \in S$ and $x^p \in K_S$ for some $x \in \mathbb{R}$. Then $x^p \in K_n$ for some n, so $x \in K_{n+1} \subset K_S$. Hence K_S is n-root closed in $\mathbb{R}$ if $n \in S$. Conversely, we next show that $2^{1/p} \notin K_S$ for p any prime such that $p \notin \mathcal{P}$. Suppose that $2^{1/p} \in K_S$. Then $2^{1/p} \in K_{n+1} - K_n$ for some $n \geq 0$. Hence $2^{1/p} \in K_n(\alpha_1, \ldots, \alpha_r)$, where each $\alpha_i^{p_i} \in K_n$ for some $p_i \in \mathcal{P}$. Thus we may assume that $2^{1/p} \in k(\alpha^{1/q}) - k$ for some subfield $k \subset \mathbb{R}$, $\alpha \in k$, and prime $q \in \mathcal{P}$. Since $X^p - 2$ and $X^q - \alpha$ are each irreducible over k [11, Theorem 9.1, p. 297], we have that $[k(2^{1/p}) : k] = p$ divides $[k(\alpha^{1/q}) : k] = q$, a contradiction. Hence K_S is not n-root closed if $n \notin S$. ∎

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THEOREM 3.3. *Let S be a multiplicative submonoid of $\mathbb{P}$ generated by primes. Then there is a one-dimensional quasilocal domain A with $\mathcal{C}(A) = \mathcal{C}(A[[\mathbf{X}]]) = S$ for any family of indeterminates* $\mathbf{X}$.

PROOF. Let $A = K_S + T\mathbb{R}[[T]]$, where K_S is the subfield of $\mathbb{R}$ constructed above. By Example 3.1 and Lemma 3.2, $\mathcal{C}(A[[\mathbf{X}]]) = \mathcal{C}(A) = \{n \in \mathbb{P} \mid K_S \text{ is } n\text{-root closed in } \mathbb{R}\} = S$. ∎

REMARK 3.4. (a) Note that the domain A constructed in Theorem 3.3 is never Noetherian, since K_S is countable and hence $[\mathbb{R} : K_S]$ is infinite [10, p. 271, Exercise 8(3)]. However, if the set J of primes not in P is finite, then we may modify the construction to make A Noetherian. In this case, define $L = K_S(\{2^{1/q} \mid q \in J\})$. Since $[L : K_S] < \infty$, $A = K_S + TL[[T]]$ is Noetherian; also, $\mathcal{C}(A[[\mathbf{X}]]) = \mathcal{C}(A) = S$ as above.

(b) Note that K_S is not algebraically closed in $\mathbb{R}$, and hence the domain A in Theorem 3.3 is not integrally closed. Thus when $S = \mathbb{P}$, A and $A[[\mathbf{X}]]$ are each root closed, but not integrally closed. (A similar example of such a domain A is given in [10, p. 184, Exercise 6].) ∎

An open question [5, Question 2.6] asks whether $A[[X]]$ must be integrally closed, given that A is an integrally closed domain such that $A[[X]]$ is root closed. By an example of Watkins [14, Example 4], the answer is negative if "domain" is changed to "ring." In Watkin's example, the ring A is Boolean, hence von Neumann regular. The ring A in Theorem 3.5 is also von Neumann regular; its analysis depends on using Lemma 3.2 to modify another example of Watkins [14, Example 6].

THEOREM 3.5. *Let S be a multiplicative submonoid of $\mathbb{P}$ generated by primes. Then there is a ring A which is root closed (that is, $\mathcal{C}(A) = \mathbb{P}$) and satisfies $\mathcal{C}(A[[X]]) = S$. It can be arranged that A is von Neumann regular.*

PROOF. Let $A = (\prod_{n\geq 0} K_S) + (\bigoplus_{n\geq 0} \mathbb{R})$, viewed as a subring of $B = \prod_{n\geq 0} \mathbb{R}$, a direct product of denumerably many copies of $\mathbb{R}$. Thus, A is the ring of all the real-valued sequences which after some point (depending on the sequence) take all subsequent values in K_S. Notice that A is von Neumann regular, and hence equals its own total quotient ring. In particular, A is root closed. Thus, it remains only to prove that if $n \geq 1$, then $A[[X]]$ is n-root closed if and only if $n \in S$. Note first that $B[[X]]$ is torsion-free over $A[[X]]$, by an application of [14, Proposition 1]. (This application is straightforward after one interprets the support $\sup_B$ as the pointwise supremum—the verification then reduces to the triviality that 0 and 1 are the only idempotents of $\mathbb{R}$). Hence, according to [14, Theorem 2(b), (i) and (ii)], it suffices to show that if $n \geq 1$, then $n \in S$ if and only if $\{b \in B \mid b^n \in A, b = \sup_B\{a_k\}$ for some countable subset $\{a_k\}$ of A such that $a_i a_j = 0$ if $i \neq j\} \subset A$. (The results quoted from [14] would require us to replace "$\subset$" with equality, but it is evident that "$\supset$" holds in general.)

We may prove the (contrapositive of the) "if" assertion by modifying the proof of [14, Example 6]. Consider $n \in \mathbb{P} - S$. Then, by Lemma 3.2, K_S is not n-root closed in $\mathbb{R}$, and so we may pick $\alpha \in \mathbb{R} - K_S$ such that $\alpha^n \in K_S$. If $k \geq 0$, let $a_k = (0, \ldots, 0, \alpha, 0, \ldots)$,

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with α in the k-th place and zeros elsewhere. Each $a_k \in A$ and $a_i a_j = 0$ if $i \neq j$. Put $b := \sup_B\{a_k\}$. Then $b = (\alpha, \alpha, \dots) \in B$, whence $b^n = (\alpha^n, \alpha^n, \dots) \in \prod_{n \geq 0} K_S \subset A$, although $b \notin A$ since $\alpha \notin K_S$. Therefore, the above "$\subset$" fails, and the "if" assertion has been proved.

Finally, to prove the "only if" assertion, let $n \in S$, and consider $b = (b_0, b_1, b_2, \dots) \in B$ such that $b^n \in A$. (It will not be necessary to specify that $b = \sup_B\{a_k\}$ for some countable subset $\{a_k\}$ of A such that $a_i a_j = 0$ if $i \neq j$.) We need only show that $b \in A$; that is, that there exists $\mu \in \mathbb{P}$ such that $b_m \in K_S$ for all $m \geq \mu$. Since $(b_0^n, b_1^n, b_2^n, \dots) = b^n \in A$, there exists $\mu \in \mathbb{P}$ such that $b_m^n \in K_S$ for all $m \geq \mu$. The proof then concludes, since Lemma 3.2 ensures that K_S is n-root closed in $\mathbb{R}$. ∎

NOTE ADDED IN PROOF (JULY 2, 1997). For S as in Theorem 3.3, M. Roitman has independently given another construction of a domain A such that $C(A) = S$: see Theorem 2.11(2) of *On root closure in Noetherian domains*, pp. 417–428, in Lecture Notes Pure Appl. Math., Vol. 189, Dekker, New York, 1997.

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