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A PROOF OF AN IDENTITY FOR MULTIPLICATIVE FUNCTIONS

BY
K. KRISHNA

Introduction. An arithmetic function f is said to be multiplicative if $f(mn) = f(m)f(n)$, whenever $(m, n) = 1$ and $f(1) = 1$. The Dirichlet convolution of two arithmetic functions f and g, denoted by $f \cdot g$, is defined by $f \cdot g(n) = \sum_{d \mid n} f(d)g(n/d)$. Let $w(n)$ denote the product of the distinct prime factors of n, with $w(1) = 1$. R. Vaidyanathaswamy [3] proved the following identical equation for any multiplicative arithmetic function f:

$$f(mn) = \sum_{\substack{a \mid m \\ b \mid n}} f(m/a)f(n/b)f^{-1}(ab)C(a, b), \tag{1}$$

where m and n are arbitrary positive integers, f^{-1} is the Dirichlet inverse of f defined by

$$\sum_{d \mid n} f(d)f^{-1}(n/d) = E_o(n) = \begin{cases} 1 & \text{if} \quad n = 1, \\ 0 & \text{if} \quad n > 1, \end{cases}$$

and $C(a, b)$ is a multiplicative function of two variables defined by

$$C(a, b) = \begin{cases} (-1)^k & \text{if} \quad w(a) = w(b) = k, \\ 0 & \text{otherwise.} \end{cases}$$

The K-product of any two arithmetic functions f and g is the arithmetic function $f \times g$ defined by

$$f \times g(n) = \sum_{d \mid n} f(d)g(n/d)K((d, n/d)),$$

where $K(n)$ is a fixed arithmetic function satisfying $K(1) = 1$ and, for arbitrary positive integers a, b, c,

$$K((a, b))K((ab, c)) = K((a, bc))K((b, c)). \tag{2}$$

It has been shown [1] that (2) assures the associativity of the K-product and, together with the condition $K(1) = 1$, it implies that $K(n)$ is multiplicative.

M. V. Subba Rao and A. A. Gioia [2] gave a generalization of the identity

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(1), which holds in the case of the K-product. The generalized identity is

$$(3)\quad f(mn)=\sum_{\substack{a\mid m\\ b\mid n}} f(m/a)f(n/b)f^{-1}(ab)K((mn/ab, ab))K((m/a, n/b))C(a, b).$$

Their proof of (3) is based on the observation that the right side of (3) actually defines a multiplicative function of both the variables m and n so that one need only evaluate it when m and n are prime powers. The object of this note is to point out a new proof of (3) which is a straightforward generalization of Vaidyanathaswamy's proof of (1).

Lemma 1. *Let f be any multiplicative function and f^{-1} be its inverse with respect to the K-product operation. Then, for arbitrary positive integers m_1, m_2 and n, the sum*

$$\sum f(m_1d)f^{-1}(m_2n/d)K((m_1d, m_2n/d)),$$

extended over all the divisors d of n, vanishes unless every prime factor of n divides m_1m_2.

Proof. Let $n=n_1n_2$, where all the prime factors of n_1 divide m_1m_2, and n_2 is relatively prime to m_1m_2. Then it is clear that $(n_1, n_2)=1$, and therefore any factor d of n can be expressed uniquely in the form d_1d_2, where d_1 is a divisor of n_1 and d_2 is a divisor of n_2.

Hence we have

$$\begin{aligned}\sum f(m_1d)f^{-1}(m_2n/d)K((m_1d, m_2n/d))&\\ &=\sum f(m_1d_1d_2)f^{-1}(m_2n_1/d_1\cdot n_2/d_2)K((m_1d_1d_2, m_2n_1n_2/d_1d_2))\\ &=\left\{\sum f(m_1d_1)f^{-1}(m_2n_1/d_1)K((m_1d_1, m_2n_1/d_1))\right\}\\ &\quad\times\left\{\sum f(d_2)f^{-1}(n_2/d_2)K((d_2, n_2/d_2))\right\},\end{aligned}$$

where we have used the multiplicativity of f and f^{-1} together with the relation (see Lemma in section 3 of [2]):

$$(4)\quad K((ab, cd))=K((a, c))K((b, d))\quad\text{if}\quad (a, b)=1, (a, d)=1\quad\text{and}\quad (b, c)=1.$$

Now the summation in the second curly bracket above vanishes unless $n_2=1$, which proves the result. □

Corollary. *Calling a factor n_1 of n a block factor if $(n_1, n/n_1)=1$, we have*

$$\sum f(n/d)f^{-1}(d)K((n/d, d))=0,$$

where the summation extends over all the divisors d of a block factor n_1 $(\neq 1)$ of n. □

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LEMMA 2. *Let $w(n)=\nu$. Then*

$$\sum_{\substack{d\mid n\\ w(d)=w(n)}} f(n/d)f^{-1}(d)K((n/d,d))=(-1)^{\nu}f(n).$$

Proof. Let $n_{i1}, n_{i2}, \ldots, n_{ik}$ $(k=\binom{\nu}{i})$ denote the distinct block factors of n which contain exactly i of the prime factors. Consider the sum

$$\begin{aligned}A=\sum_{n} f(n/d)f^{-1}(d)K((n/d,d))-\sum_{k=1}^{\nu}\left\{\sum_{n_{\nu-1k}} f(n/d)f^{-1}(d)K((n/d,d))\right\}\\ +\sum_{k=1}^{\nu(\nu-1)/2}\left\{\sum_{n_{\nu-2k}} f(n/d)f^{-1}(d)K((n/d,d))\right\}-\cdots\\ +(-1)^{\nu-1}\sum_{k=1}^{\nu}\left\{\sum_{n_{1k}} f(n/d)f^{-1}(d)K((n/d,d))\right\},\end{aligned}$$

where the n_{ij} below $\sum$ indicates that the sum is extended over all the divisors d of n_{ij}. We evaluate the expression A in two ways. First, we observe that every partial sum in A, except the first, vanishes by the corollary to Lemma 1. Hence we have,

$$A=\sum_{n} f(n/d)f^{-1}(d)K((n/d,d))=0, \qquad (n>1).$$

On the other hand consider a particular divisor d of n, containing i distinct prime factors. The coefficient of $f(n/d)f^{-1}(d)K((n/d,d))$ in A is

$$1-\binom{\nu-i}{1}+\binom{\nu-i}{2}-\cdots=\begin{cases}0 & \text{if } 0<i<\nu,\\ 1 & \text{if } i=\nu.\end{cases}$$

If $d=1$, the coefficient of $f(n/1)f^{-1}(1)K((n/1,1))$ is

$$1-\binom{\nu}{1}+\binom{\nu}{2}-\cdots+(-1)^{\nu-1}\binom{\nu}{\nu-1}=(-1)^{\nu-1}.$$

Therefore we have

$$A=\sum_{\substack{d\mid n\\ w(d)=w(n)}} f(n/d)f^{-1}(d)K((n/d,d))+(-1)^{\nu-1}f(n).$$

But we have already observed that $A=0$. Hence we obtain the required identity. □

LEMMA 3. *Let $w(m)=w(n)=\nu$. Then*

$$\sum_{b\mid n} f(mn/b)f^{-1}(b)K((mn/b,b))=(-1)^{\nu}\sum_{\substack{a\mid m\\ w(a)=w(m)}} f(m/a)f^{-1}(na)K((m/a,na)). \tag{5}$$

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Proof. The proof is analogous to the proof of Theorem 3 of [3]. We shall just outline the proof here.

Let $m=m_{ik}m'_{ik}$ and $n=n_{ik}n'_{ik}$, where m_{ik} and n_{ik} $(k=1,2,\ldots,\binom{\nu}{i})$ are the block factors of m and n respectively, which contain the same i prime factors. Hence $(m_{ik}, m'_{ik})=1$, $(n_{ik}, n'_{ik})=1$, and m'_{ik} and n'_{ik} are the block factors of m and n respectively, containing the same $(\nu-i)$ prime factors.

Consider the expression

$$\begin{aligned}B=&\sum f(mn/b)f^{-1}(b)K((mn/b,b))\\&+\sum_{k=1}^{\nu}\left\{\sum\sum f(m_{1k}/a\cdot m'_{1k}n'_{1k}/b)f^{-1}(n_{1k}ab)K((m_{1k}/a\cdot m'_{1k}n'_{1k}/b,\, n_{1k}ab))\right\}\\&-\sum_{k=1}^{\nu(\nu-1)/2}\left\{\sum\sum f(m_{2k}/a\cdot m'_{2k}n'_{2k}/b)f^{-1}(n_{2k}ab)K((m_{2k}/a\cdot m'_{2k}n'_{2k}/b,\, n_{2k}ab))\right\}\\&+\cdots+(-1)^{\nu-1}\sum f(m/a)f^{-1}(nb)K((m/a,\, nb)).\end{aligned}$$

Here the first term of B is a summation over all divisors b of n. Every succeeding term contains three summations; the two inner summations relate respectively to all divisors b of $m'_{ik}n'_{ik}$ and to all such divisors a of m'_{ik} which contain all its distinct prime factors; the outer summation relates to all possible resolutions of m and n into corresponding block factors containing i and $(\nu-i)$ primes. The signs of the $(\nu+1)$ terms in B alternate from the second term onwards. In the last term $i=\nu$, and so the outer summation as well as the summation relating to b, has disappeared, leaving only the summation over all factors a of m containing all its ν prime factors.

The proof is now complete after the evaluation of the expression B in two ways, as we have done in the previous lemma. □

COROLLARY 1. *Let $w(m)=w(n)=\nu$ and $(m_1, m)=1$, and hence $(m_1, n)=1$. Put $m'=m_1m$. Then, multiplying both sides of* (5) *by $f(m_1)K((m_1, 1))$, we get, on using* (4) *and the multiplicativity of f*

$$\sum_{b\mid n} f(m'n/b)f^{-1}(b)K((m'n/b,b)) = (-1)^{\nu}\sum_{\substack{a\mid m'\\ w(a)=w(n)}} f(m'/a)f^{-1}(na)K((m'/a,\, na)). \quad \square$$

COROLLARY 2. *Let m and n be any two positive integers, with $w(n)=\nu$. Then*

$$\sum_{b\mid n} f(mn/b)f^{-1}(b)K((mn/b,b))=(-1)^{\nu}\sum_{\substack{a\mid m\\ w(a)=w(n)}} f(m/a)f^{-1}(na)K((m/a,\, na)).$$

Proof. If $w(n)\mid w(m)$, then this reduces to Corollary 1 above. If $w(n)\nmid w(m)$, the left side is zero by Lemma 1, while the right side is an empty sum. □

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We can now prove the generalized identical equation for K-products:

THEOREM. *If f is multiplicative, then for any two positive integers m and n,*

$$f(mn)=\sum_{\substack{a\mid m\\ b\mid n}} f(m/a)f(n/b)f^{-1}(ab)K((mn/ab,\, ab))K((m/a,\, n/b))C(a,b).$$

Proof. From Corollary 2, with n_1 in the place of n, we have

$$(6)\quad \sum_{b\mid n_1} f(mn_1/b)f^{-1}(b)K((mn_1/b,\, b)) = (-1)^{\nu} \sum_{\substack{a\mid m\\ w(a)=w(n_1)}} f(m/a)f^{-1}(n_1a)K((m/a,\, n_1a)),$$

where $\nu=w(n_1)$.

We multiply both sides of (6) by $f(n_2)K((n_2, mn_1))$, and sum over all values of n_1 and n_2 with $n_1n_2=n$. The summation is carried out in two stages; namely, we first keep n_1/b fixed, and sum over all values of n_2 and b such that $n_2b=nb/n_1$. On the left side, by using relation (2), we get

$$\begin{aligned}\sum_{n_1n_2=n}\sum_{b\mid n_1} f(mn_1/b)f^{-1}(b)K((mn_1/b,\, b))f(n_2)K((n_2,\, mn_1))\\ = \sum_{n_1n_2=n}\sum_{b\mid n_1} f(mn_1/b)f^{-1}(b)f(n_2)K((mn_1/b,\, n_2b))K((n_2,\, b)),\\ =\sum f(mn_1/b)K((mn_1/b,\, n_2b)) \sum_{n_2b=nb/n_1} f(n_2)f^{-1}(b)K((n_2,\, b)).\end{aligned}$$

The second summation here vanishes, by Lemma 1, unless $nb/n_1=1$ (equivalently $n_2b=1$), that is, unless $n_1=nb$, in which case it is 1. Therefore the left side of (6) reduces to $f(mn)K((mn, 1))=f(mn)$.

The right side of (6), after multiplying by $f(n_2)K((n_2, mn_1))$, is

$$\sum_{n_1n_2=n} \sum_{\substack{a\mid m\\ w(a)=w(n_1)}} (-1)^{\nu} f(m/a)f^{-1}(n_1a)K((m/a,\, n_1a))f(n_2)K((n_2,\, mn_1)),$$

which is equal to

$$\sum\sum(-1)^{\nu}f(m/a)f(n/b)f^{-1}(ab)K((m/a,\, ab))K((n/b,\, mb)),$$

where we sum over all the divisors b of n and all the divisors a of m with $w(a)=w(b)$.

This, by the definition of $C(a, b)$ and by the relation (2), is clearly equal to

$$\sum_{\substack{a\mid m\\ b\mid n}} f(m/a)f(n/b)f^{-1}(ab)K((mn/ab,\, ab))K((m/a,\, n/b))C(a,b),$$

and the proof of the theorem is complete. □

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Department of Mathematics
University of Pittsburgh
Pittsburgh, Penn, 15260