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A RIEMANN TYPE THEOREM FOR SERIES OF OPERATORS ON BANACH SPACES

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We study Kalton's theorem on the unconditional convergence of series of compact operators and we use some matrix techniques to obtain sufficient conditions, weaker than previous ones, on the convergence and unconditional convergence of series of compact operators. Finally, we characterise weak unconditionally Cauchy series in $C\mathcal{L}(X,Y)$ in terms of certain spaces of vector sequences.

1. INTRODUCTION

Let X, Y be two Banach spaces. We denote by $C\mathcal{L}(X,Y)$ (respectively $K(X,Y)$) the Banach space of bounded (respectively, compact) linear maps from X to Y. Kalton [8] proved that if $\sum_i T_i$ is subseries convergent in $K(X,Y)$, with respect to the weak operator topology, and X^* contains no subspace isomorphic to l_∞, then $\sum_i T_i$ is subseries convergent with respect to the norm topology of $K(X,Y)$. Kaftal and Weiss [7] proved that if H is a Hilbert space and $(T_n)_n$ is a sequence in $K(H)$ such that

(i) $\sum_n T_n$ is unconditionally convergent for the stronger operator topology, and

(ii) for every $F \subset \mathbb{N}$ the operator $\sum_{n\in F} T_n$ is compact,

then $\sum_n T_n$ is unconditionally convergent in $K(H)$.

Qingying Bu and Congxin Wu [5] proved the following result, which is equivalent to the previous result of Kalton. Every series $\sum_i K_i$ of operators in $C\mathcal{L}(X,Y)$ such that:

(i) $\sum_i K_i$ converges unconditionally in the weak operator topology (in the proof the series $\sum_i K_i$ should be subseries convergent with respect to the weak operator topology), and

(ii) $\sum_{i\in F} K_i$ is a compact operator for every index set $F \subset \mathbb{N}$,

is unconditionally convergent in the norm if and only if X^* contains no copy of c_0.

In this paper, we use some matrix techniques ([**2, 3, 4**]) to improve the above mentioned results.

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We shall say that $\mathcal{F}$ is a natural family if $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ and $\mathcal{F}$ contains the family $\phi_0(\mathbb{N})$ of the finite subsets of $\mathbb{N}$. We also say that $\mathcal{F}$ has the property S_1 if for any pair $[(A_i),(B_i)]$ of mutually disjoint sequences of disjoint subsets of $\phi_0(\mathbb{N})$ there exist $B \in \mathcal{F}$ and an infinite set $M \subset \mathbb{N}$ such that $A_i \subset B$ and $B_i \subset B^c$. We shall say that the family $\mathcal{F}$ is subsequentially complete if for any sequence $(A_i)_i$ of disjoint sets in $\mathcal{F}$ there exist an infinite set $M \subset \mathbb{N}$ and $B \in \mathcal{F}$ such that $B = \bigcup_{i \in M} A_i$.

We must recall that any atomic subalgebra $\mathcal{F}$ of $\mathcal{P}(\mathbb{N})$ is natural.

Let X be a vector space and let T be a vector space topology on X. Let $\sum_i x_i$ be a series in X and let $\mathcal{F}$ be a natural family. We shall say that $\sum x_i$ is $\mathcal{F}$-T convergent if for every $A \in \mathcal{F}$ the series $\sum_{j \in A} x_j$ converges in (X, T). It can be proved ([**2**]) that if $\mathcal{F}$ is a natural family with the property S_1 and $\sum x_i$ is a $\mathcal{F}$-weak convergent series in a Banach X then $\sum_i x_i$ is unconditionally convergent. The following result can also be proved ([**2**]) (we shall denote it as (1) throughout this paper). Let $(x_{ij})_{i,j}$ be a matrix in a normed space X such that $(x_{ij})_i$ is a Cauchy sequence for $j \in \mathbb{N}$, and for $i \in \mathbb{N}$ the series $\sum_j x_{ij}$ is subseries convergent. Then the following conditions are equivalent:

(1) there exists a natural family $\mathcal{F}$ with the property S_1 such that $\left(\sum_{j \in B} x_{ij}\right)_i$ is Cauchy, for $B \in \mathcal{F}$;

(2) the sequence $\left(\sum_{j \in A_n} x_{ij}\right)_i$ is Cauchy uniformly in $n \in \mathbb{N}$, for every disjoint sequence $(A_n)_n$ of $\phi_0(\mathbb{N})$.

In this paper we obtain several results on the convergence and unconditional convergence of series of compact operators. The results that appear in ([**7, 8, 9**]) can be obtained as an immediate consequence. Our results are based on the $\mathcal{F}$-convergence of series, where $\mathcal{F}$ denotes a natural family or a natural Boolean algebra with an appropriate property. We also obtain an interesting characterisation of weak unconditionally Cauchy series in $C\mathcal{L}(X, Y)$.

2. Convergence of series of compact operators

Theorem 2.1. *Let X, Y be two Banach spaces and let $\mathcal{F}$ be a natural family with the property S_1. Let $(T_i)_i$ be a sequence of compact operators from X to Y. The series $\sum_i T_i$ is unconditionally convergent in $K(X, Y)$ if and only if $\sum_i T_i^*$ is $\mathcal{F}$-weak operator topology convergent, and for any $B \in \mathcal{F}$ the operator defined by $T_B^*(g) = w \sum_{j \in B} T_j^* g$ is compact.*

Proof: If $x \in X$ then the series $\sum_j T_j x$ is $\mathcal{F}$-weak convergent and, therefore, is unconditionally convergent. We also have that, for $g \in Y^*$, $\sum_j T_j^* g$ is unconditionally convergent in X^*.

Since the rank of a compact operator is separable, we can suppose that Y is separable. Let us suppose that $\sum_j T_j$ is not unconditionally convergent in $C\mathcal{L}(X, Y)$. In this case,

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there exist $\varepsilon > 0$ and a sequence $(F_k)_k$ in $\phi_0(\mathbb{N})$ such that $\left\|\sum_{j\in F_k} T_j\right\| > \varepsilon$ and $\sup F_k < \inf F_{k+1}$ for $k \in \mathbb{N}$. Since $\left\|\sum_{j\in F_k} T_j^*\right\| > \varepsilon$, for every $k \in \mathbb{N}$ there exists $g_k \in B_{Y^*}$ such that $\left\|\sum_{j\in F_k} g_k T_j\right\| > \varepsilon$. There exists a subsequence of $(g_k)_k$ that is *weak** convergent to some $g_0 \in B_{Y^*}$. Without loss of generality we shall suppose that $w^* \lim_k g_k = g_0$.

Let us consider the matrix $(g_i T_j)_{i,j}$. We have that:

(1) Since, for $j \in \mathbb{N}$, T_j is compact, the sequence $(g_i T_j)_i$ is convergent.

(2) The sequence $\left(\sum_{j\in B} g_i T_j\right)_i$ is convergent in X^*; therefore, the matrix $(g_i, T_j)_{i,j}$ satisfies (1) (see introduction). This contradicts that $\left\|\sum_{j\in F_k} g_i T_j\right\| > \varepsilon$ for $i \in \mathbb{N}$. ∎

REMARK. Let us consider in $K(X,Y)$ the topology $T_{w'}$ defined as follows: a net $(T_\alpha)_{\alpha\in I}$ has $w' \lim_\alpha T_\alpha = T_0$ if and only if $\lim_\alpha T_\alpha^* = T_0^*$ for weak operator topology. It can be proved ([**7**]) that if $(T_n)_n$ is a sequence in $K(X,Y)$ such that $w' \lim_n T_n = T_0$ and $T_0 \in K(X,Y)$ then $\lim T_n = T_0$ for the weak topology in $K(X,Y)$. From this result, we obtain that, under the hypothesis of Theorem 2.1, $\sum_i T_i$ is $\mathcal{F}$-weak convergent, and therefore $\sum_i T_i$ is unconditionally convergent.

COROLLARY 2.2. *Let X, Y be two Banach spaces and let $\mathcal{F}$ be a natural family with the property S_1. Let $(T_i)_i$ be a sequence of compact operators from X to Y and let us suppose that X^* does not have a copy of l_∞. Then, the series $\sum_i T_i$ is unconditionally convergent in $K(X,Y)$ if and only if $\sum_j T_j$ is $\mathcal{F}$-weak operator topology convergent and, for $B \in \mathcal{F}$, the operator defined by $T_B(x) = \sum_{j\in B} T_j x$ is compact.*

PROOF: For $x \in X$, the series $\sum_j T_j x$ is unconditionally convergent. Hence, for $g \in Y^*$, we have that $\sum_j g(T_j x)$ is unconditionally convergent. This proves that $\sum_j gT_j$ is weak unconditionally of Cauchy. Since X^* does not have a copy of c_0, $\sum_j gT_j$ is unconditionally convergent in X^*. Now, it is sufficient to proceed as in the proof of Theorem 2.1. ∎

The Vitali-Hahn-Saks property for a Boolean algebra is related to classical Measure Theory ([**10**]). In the hypothesis of our next corollary, we consider a natural Boolean algebra $\mathcal{F}$ with the Vitali-Hahn-Saks property instead of a natural family $\mathcal{F}$ with the property S_1.

COROLLARY 2.3. *Let X, Y be two Banach spaces and let $\mathcal{F}$ be a natural Boolean algebra with the Vitali-Hahn-Saks property. Let $(T_i)_i$ be a sequence of compact operators from X to Y. Then,*

(a) *$\sum_j T_j$ is unconditionally convergent if and only if $\sum_{j\in B} T_j^*$ is $\mathcal{F}$-weak operator topology convergent and, for any $B \in \mathcal{F}$, the operator defined by $T_B^*(g) = \sum_{j\in B} T_j^* g$ is compact.*

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(b) *If X^* does not have a copy of l_∞, $\sum_j T_j$ is unconditionally convergent if and only if $\sum_j T_j$ is $\mathcal{F}$-weak operator topology convergent and, for any $B \in \mathcal{F}$, the operator defined by $T_B(x) = \sum_{j\in B} T_j x$ is compact.*

PROOF: Let us check that, for $x \in X$, the series $\sum_j T_j x$ is unconditionally convergent in Y. If $\sum_j T_j x$ is not unconditionally convergent, then there exist $\varepsilon > 0$ and a sequence $(F_k)_k$ in $\phi_0(\mathbb{N})$ such that $\sup F_k < \inf F_{k+1}$ and $\left\| \sum_{j\in F_k} T_j x \right\| > \varepsilon$, for $k \in \mathbb{N}$. We can assume that Y is separable and we have that, for $k \in \mathbb{N}$, there exists $f_k \in S_{Y^*}$ such that $f_k\left(\sum_{j\in F_k} T_j x\right) > \varepsilon$. We can also assume that there exists $f_0 \in B_{Y^*}$ such that $w^* \lim_k f_k = f_0$. For $j \in \mathbb{N}$, let us define the finitely additive measure $\mu_j : \mathcal{F} \to \mathbb{R}$ by $\mu_j(B) = f_j\left(w \sum_{j\in B} T_j x\right)$. We have that, for $B \in \mathcal{F}$, $\lim_j \mu_j(B) = f_0\left(w \sum_{j\in B} T_j x\right)$. Since $\mathcal{F}$ has the Vitali-Hahn-Saks property, $(\mu_j)_j$ is strongly uniformly additive. Therefore if $(B_i)_i$ is a disjoint sequence in $\mathcal{F}$ then $\lim_i \mu_j(B_i) = 0$ uniformly in $j \in \mathbb{N}$. This contradicts the fact that $\mu_k(F_k) = f_k\left(\sum_{j\in F_k} T_j x\right) > \varepsilon$, for $k \in \mathbb{N}$.

It can also be proved, as before, that the series $\sum_j gT_j$ is unconditionally convergent in X^* for every $g \in Y^*$.

If $\sum_j T_j$ is not unconditionally convergent then, a proof similar to Theorem 2.1 shows that the matrix $(g_i T_j)_{i,j}$ in X^* satisfies the following.

(i) $\sum_j g_i T_j$ is unconditionally convergent, for $i \in \mathbb{N}$;

(ii) the sequence $(g_i T_j)_i$ is convergent, for $j \in \mathbb{N}$;

(iii) the sequence $\left(\sum_{j\in B} g_i T_j\right)_i$ is convergent, for $B \in \mathcal{F}$;

(iv) there exist $\varepsilon > 0$ and a disjoint sequence $(F_k)_k$ in $\phi_0(\mathbb{N})$ such that $\left\| \sum_{j\in F_k} g_k T_j \right\| > \varepsilon$, for $k \in \mathbb{N}$.

For $i \in \mathbb{N}$ let us consider the vector measure $\mu_i : \mathcal{F} \to X^*$ defined by $\mu_i(B) = \sum_{j\in B} g_i T_j$, for $B \in \mathcal{F}$. It is clear that every μ_i is σ-additive and that $(\mu_i)_i$ is pointwise convergent in $\mathcal{F}$. Therefore, $(\mu_i)_i$ is uniformly strongly additive in $\mathcal{F}$. This contradicts that $\left\|\mu_i(F_i)\right\| > \varepsilon$ for $i \in \mathbb{N}$. ∎

REMARK. As a consequence of Corollary 2.2, by considering $\mathcal{F} = \mathcal{P}(\mathbb{N})$, we can prove Kalton's theorem (see the introduction). If $\mathcal{F}$ is a natural family that is subsequentially complete, we could use the proof of Kalton to obtain the corresponding results. However, we now prove that there exist natural families with the property S_1 that are not subsequentially complete.

Let B_1 be the family of the subsets $A \subset \mathbb{N}$ such that A and A^c have infinite even numbers and infinite odd numbers. Let $\mathcal{F} = B_1 \cup \phi_0(\mathbb{N})$. We first prove that $\mathcal{F}$ is not subsequentially complete. Let us consider the sequence $\left(\{2n\}\right)_{n\in\mathbb{N}}$. It is easy to check that the union of the terms of any subsequence is not in $\mathcal{F}$. Hence, $\mathcal{F}$ is not subsequentially

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complete. It is easy to check that $\mathcal{F}$ has property S_1.

REMARK. On the conditional convergence of series in $K(X,Y)$. Ronglu and Kang [**9**] considered several conditions for the conditional convergence of a series and for the convergence of double series and iterated series.

We say that a natural family $\mathcal{F}$ has property P_1 ([**3**]) if there exists a map $f:\mathbb{N}\to\mathbb{N}$ such that for any pair $(F_r)_r$, $(m_r)_r$ of sequences in $I_0(\mathbb{N})$ (the family of finite intervals of $\mathbb{N}$) and $\mathbb{N}$, respectively, with $m_r \leqslant \inf F_r \leqslant \sup F_r < m_{r+1}$ for $r\in\mathbb{N}$, there exist $B\in\mathcal{F}$ and an infinite set $M\subseteq\mathbb{N}$ that:

(a) $B\cap[m_r,m_{r+1}) = F_r$ for $r\in M$, and

(b) $B\cap(m_r,m_{r+1})$ is either empty, or for $r\in\mathbb{N}\setminus M$ and $r>1$, it can be written as the union of at most $f(r-1)$ intervals.

It has been proved, in [**3**], that if $\mathcal{F}$ is a natural family with the property P_1 and $(x_{ij})_{i,j}$ is a matrix in a Banach space X such that the sequence $\left(\sum_{j\in B} x_{ij}\right)_i$ is convergent, for $B\in\mathcal{F}$, then:

(i) $\sum_j x_{ij}$ is convergent, for $i\in\mathbb{N}$;

(ii) $\lim_i(\sum_j x_{ij}) = \sum_j x_j$, where $x_j = \lim_i x_{ij}$ for $j\in\mathbb{N}$; and

(iii) $\lim_i \sum_{j\in F} x_{ij} = \sum_{j\in F} x_j$ uniformly in the sets F of the family $I_0(\mathbb{N})$. If, moreover, the series $\sum_i\left(\sum_{j\in B} x_{ij}\right)$ is convergent, for any $B\in\mathcal{F}$, then $\sum_i\sum_j x_{ij} = \sum_j\sum_i x_{ij}$.

As a consequence of the results that appear in the former remark and by proceeding as in the proof of Theorem 2.1, the following result can be proved:

THEOREM 2.4. *Let X, Y be two Banach spaces and let $\mathcal{F}$ be a natural family with the property P_1. Let $(T_j)_j$ be a sequence in $K(X,Y)$ such that, for each $B\in\mathcal{F}$,*

(i) *$\sum_{j\in B} T_j$ is stronger operator topology convergent,*

(ii) *$\sum_{j\in B} T_j^*$ is stronger operator topology convergent, and*

(iii) *the operator T_B defined by $T_B(x) = \sum_{j\in B} T_j x$ is compact.*

Then $\sum_j T_j$ is convergent in $K(X,Y)$.

In the former theorem we can replace hypothesis (ii) by the two following hypothesis:

(ii′) the space X^* does not have a copy of l_∞, and

(ii″) the family $\mathcal{F}$ is such that any real and $\mathcal{F}$-convergent series $\sum_j a_j$ is unconditionally convergent.

3. SERIES WEAK UNCONDITIONALLY CAUCHY IN $C\mathcal{L}(X,Y)$

Let S be a subspace of l_∞ such that $c_0\subset S$ and let X be a Banach space. We denote by $X(S)$ ([**1**]) the space of the sequences $(x_i)_i\in X^{\mathbb{N}}$ such that $\sum_i a_i x_i$ converges

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for $(a_i)_i \in S$. It is clear that $X(S)$ is a Banach space with the norm

$$\|(x_i)_i\| = \sup\left\{\sum_{i=1}^{n}|f(x_i)| : n \in \mathbb{N}, f \in S_{X^*}\right\} = \sup\left\{\left\|\sum_{i=1}^{\infty} a_i x_i\right\| : (a_i)_i \in B_{C_{00}}\right\}.$$

With this notation, we have that $X(c_0)$ can be identified to the space of weak unconditionally Cauchy series in X (it can also be identified with the space $l_1^w(X)$ of the weakly 1- summing sequences [**6**]). The space $X(l_\infty)$ can be identified with the space of unconditionally convergent series.

We denote by $c(X)$ the space of the convergent sequences $(x_i)_i \in X^{\mathbb{N}}$, endowed with the sup norm. Similarly, $c_0(X)$ will denote the subspace of the sequences $(x_i)_i \in c(X)$ such that $\lim_i x_i = 0$.

Our next theorem provides a simple characterisation of weak unconditionally Cauchy series in $C\mathcal{L}(X,Y)$.

Theorem 3.1. *Let X, Y be two Banach spaces and let $\sum_j T_j$ be a series in $C\mathcal{L}(X,Y)$. Then, $\sum_j T_j$ is weak unconditionally Cauchy if and only if, for any sequence $(x_i)_i \in c(X)$ such that $\lim_i x_i = x_0$, $(T_j x_i)_j \in Y(c_0)$ for $i \in \mathbb{N}$ and the sequence $\big((T_j x_i)_j\big)_i$ converges to $(T_j x_0)_j$.*

Proof: Let us suppose that $\sum_j T_j$ is a weak unconditionally Cauchy series. Let us denote by $H = \|(T_j)_j\|$ the norm of $(T_j)_j$ in the space of weak unconditionally Cauchy series in $C\mathcal{L}(X,Y)$. For $x \in B_X$ and $g \in B_{Y^*}$, the map $\varphi_{xg} : C\mathcal{L}(X,Y) \to \mathbb{R}$ defined by $\varphi_{xg}(T) = g(Tx)$ is linear, continuous and satisfies $\|\varphi_{xg}\| \leqslant 1$. Therefore, $\sum_{j=1}^{\infty}|gT_j(x)| \leqslant H$.

If $M \subset \mathbb{N}$ is a finite set, then $\left\|\sum_{j\in M} gT_j\right\| \leqslant H$. Let $(x_i)_i \in c(X)$ and let $x_0 = \lim_i x_i$. It is clear that $\overline{y}^i = (T_j x_i)_j$ is an element of $Y(c_0)$ for every $i \in \mathbb{N}$. For any given $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $\|x_p - x_q\| \leqslant \varepsilon/H$ for $p, q \geqslant n_0$.

If $p, q \geqslant n_0$, we have that there exist $g \in S_{Y^*}$ and $n \in \mathbb{N}$ such that

$$\|\overline{y}^p - \overline{y}^q\| - \varepsilon/2 < \sum_{j=1}^{n}|gT_j(x_p) - gT_j(x_q)| \leqslant \|x_p - x_q\|\left\|\sum_{j=1}^{n} gT_j\right\| \leqslant \varepsilon.$$

Therefore, the sequence $(\overline{y}^i)_i$ is convergent in $Y(c_0)$ and $\lim_i \overline{y}^i = \overline{y}^0$ in $Y(c_0)$, where $\overline{y}^0 = (T_j x_0)_j$.

In order to prove sufficiency, let us suppose $\sum_j T_j$ is not a weak unconditionally Cauchy series. We can inductively obtain a sequence $(F_n)_n \subset \phi_0(\mathbb{N})$ such that $\sup F_n < \inf F_{n+1}$ and $\left\|\sum_{j\in F_n} T_j\right\| > 2^n$. For $n \in \mathbb{N}$ there exists a $x_n \in S_X$ such that $\left\|\sum_{j\in F_n} T_j x_n\right\| > 2^n$. Let $y_n = (1/2^n)x_n$, for $n \in \mathbb{N}$. Then $\lim \|y_n\| = 0$ and, by our hypothesis, $\big((T_j y_i)_j\big)_i$ is a sequence in $Y(c_0)$ that converges to zero. This contradicts that $\left\|\sum_{j\in F_n} T_j y_n\right\| > 1$ for $n \in \mathbb{N}$. ∎

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REMARK. (a) Let X, Y be two Banach spaces and let $\sum_j T_j$ be a series in $C\mathcal{L}(X,Y)$. From the proof of theorem 3.1, it can be deduced that:

(i) $\sum_j T_j$ is weak unconditionally Cauchy if and only if $\left((T_j x_i)_j\right)_i$ converges to zero in $Y(c_0)$, for any unconditionally convergent series $\sum_i x_i$ in X;

(ii) $\sum_j T_j$ is weak unconditionally Cauchy if and only if $\left((T_j x_i)_j\right)_i$ converges to zero in $Y(c_0)$, for any absolutely convergent series $\sum_i x_i$ in X.

(b) When $Y = \mathbb{R}$ and $\sum_j f_j$ is a series in X^*, we have that $\sum_j f_j$ is weak unconditionally Cauchy if and only if for any sequence $(x_i)_i \in c(X)$, $\left(\left(f_j(x_i)\right)_j\right)_i$ is convergent in $\mathbb{R}(c_0)$ (the space of unconditionally convergent series in $\mathbb{R}$).

Our next result provides another type of characterisation of weak unconditionally Cauchy series in $C\mathcal{L}(X,Y)$, when Y satisfies additional properties.

THEOREM 3.2. *Let X, Y be two Banach spaces and let us suppose that Y^* has the Dunford-Pettis property. Let $\sum_j T_j$ be a series in $C\mathcal{L}(X,Y)$. Then, $\sum_j T_j$ is weak unconditionally Cauchy if and only if for any two weakly convergent sequences $(x_i)_i$ in X and $(g_i)_i$ in Y^*, to x_0 and g_0 respectively, we have that $(g_i T_j x_i)_j \in c$ for $i \in \mathbb{N}$ and $\left((g_i T_j x_i)_j\right)_i$ converges to $(g_0 T_j x_0)_j$.*

PROOF: Let us suppose that $\sum_j T_j$ is weak unconditionally Cauchy. Let $(x_i)_i$, $(g_i)_i$ be two sequences, in X and Y^* respectively, such that $w\lim_i x_i = x_0$ and $w\lim_i g_i = g_0$. Let us consider the matrix $(g_i T_j x_i)_{i,j}$. We have that any row defines an unconditionally convergent series and the jth column converges to $g_0 T_j x_0$, for $j \in \mathbb{N}$. For $M \subset \mathbb{N}$, we consider the map $T_M : X \to Y^{**}$ defined by $T_M(x) = w^* \sum_{j\in M} T_j x$. This map is linear and continuous and we have that $w\lim_i\left(w^* \sum_{j\in M} T_j x_i\right) = w^* \sum_{j\in M} T_j x_0$. Since Y^* is Dunford-Pettis property and $w\lim g_i = g_0$, $\lim_i \sum_{j\in M} g_i T_j x_i = \sum_{j\in M} g_0 T_j x_0$. This proves that the matrix $(g_i T_j x_i)_{i,j}$ satisfies the conditions to be a basic matrix [4]. As a consequence, the sequence $\left((g_i T_j x_i)_j\right)_i$ converges in c to $(g_0 T_j x_0)_j$.

Conversely, let us suppose that the series $\sum_j T_j$ is not weak unconditionally Cauchy. We can obtain a sequence $(F_k)_k$ in $\phi_0(\mathbb{N})$ such that $\sup F_k < \inf F_{k+1}$ and $\left\|\sum_{j\in F_k} T_j\right\| > 2^{2k}$ for $k \in \mathbb{N}$. For $k \in \mathbb{N}$, there exist $x_k \in S_X$ and $g_k \in S_{Y^*}$ such that $\left\|\sum_{j\in F_k} g_k T_j x_k\right\| > 2^{2k}$ for $k \in \mathbb{N}$. We have that $\lim(1/2^k)g_k = 0$ and $\lim(1/2^k)x_k = 0$. We can easily obtain a contradiction, because of the convergence of the sequence $\left(\left((1/2^k)g_i T_j (1/2^k)x_i\right)_j\right)_i$. ∎

REMARK 3.4. Let X, Y be two Banach spaces. If X has the Dunford-Pettis property and does not have a copy of l_1, or Y is subsequentially complete, then $K(X,Y) = WK(X,Y)$ (the space of weakly compact maps) and the results of this section remain valid if we replace the term " compact operator" by the term "weakly compact operator". We do not know any result for weakly compact operators similar to Kalton's theorem.

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