
In the 1820s, Gauss showed, very surprisingly, that the ‘inner geometry’ or ‘metric’ of a surface in 3-space, that is the measurement of lengths of curves on the surface, actually determines the curvature of the surface, in the sense of what is now called Gauss curvature.

Riemann, in the 1850s, turned this round and, instead of starting with a surface in 3-space, started with a metric and an abstract 2-dimensional structure, an ‘abstract surface’, and built from this a geometric object which would now be called a Riemannian manifold of dimension 2. This idea has had enormous repercussions in the ensuing 150 years. Riemannian geometry has become a vast subject, influencing, famously, the development of general relativity and, more recently, the classification of 3-manifolds by hyperbolic structures in the work of William Thurston and his many followers.

Marcel Berger’s book is an overview of this enormous subject. In a mere 824 pages, and even with 1310 references in the bibliography, it is of course not possible to cover everything, and the author does not pretend to do so. The book is a distillation of a lifetime’s experience of geometry, a guided tour by an expert guide, pointing out the most significant features, the most beautiful theorems, telling us why they are important and what is difficult about them, giving hints and insights and warnings.

Virtually everything is illustrated with clear and useful diagrams—for the author, geometry is clearly a visual subject—and this gives the text an approachable appearance, tempting the reader to ‘look at the pictures’ and then explore. The book has been developed from lectures given in the 1990s in various places (this shows occasionally as when, on page 44, he uses the phrase ‘the end of the last century’ to mean, evidently, the 1890s).

The book is therefore far from being a standard textbook: proofs are subordinated to ideas and theorems—417 numbered results—which are often introduced in an informal way, especially in the early chapters, and explained in detail later. The opening chapter is about curves and surfaces and is intended to motivate the later more formal work. The chapter is a wonderful tour, but definitely not for the beginner since a considerable mathematical sophistication is taken for granted. This is an account to come back to after learning the subject from a more conventional source—it is unlikely that, reading the account of the first and second fundamental forms, for example, a novice reader would actually be able to calculate one of them. On the other hand, reading the chapter does give a broad view. For example, straight after the discussion of fundamental forms, we get Carathéodory’s conjecture that every compact surface in Euclidean 3-space which is homeomorphic to a sphere has at least two umbilics, together with many references up to the year 1999. Similarly, after geodesics (curves of locally minimal length) have been introduced, we get the paragraph:

‘We look first at the sphere [of unit radius] and discover that all geodesics [great circles] are periodic with the same period equal to $2\pi$. Given a general surface, assuming moreover that it is strictly convex, even Poincaré could not prove there is at least one periodic geodesic on it. Birkhoff established it in 1917. Proof that there is an infinity for every surface had to wait until 1993; prior to that date it was not known even for strictly convex surfaces. We will come back to this more seriously in Chapter 10.’
REVIEWS

Indeed, Chapter 10 is a study of geodesics on riemannian manifolds, including work into the 1990s. (It also contains the interesting addition to Figure 10.1 ‘periodic geodesics and geodesics which get lost’, repeated with correct spelling in the caption of the figure. In such a large book, misprints and infelicities are inevitable; for example there are figures with lettering so tiny that it is almost impossible to read.)

Giving an overview of a panorama is not an easy task; here are some of the topics which are treated, after the initial motivating work on curves and surfaces in euclidean 2- and 3-space. Chapters 2 and 3 give an account of surface theory from the time of Gauss, with many interesting excursions including the Willmore conjecture on the integral of the squared mean curvature of a torus immersed in euclidean 3-space and, of course, the Gauss-Bonnet theorem on the integral of the Gauss curvature over a closed surface. Chapters 4 to 6 introduce riemannian manifolds, metric geometry and curvature in earnest, while Chapter 7 concentrates on geometric inequalities. Here is an example. The systole of a torus (with an arbitrary riemannian metric) is the smallest length of any curve which cannot be contracted to a point on the torus. Then a theorem of Loewner in 1949 states that the area of the torus is $\geq \sqrt{3}/2$ times the square of the systole. For an ordinary torus of revolution, obtained by rotating a circle $c$ of radius $r$ about an axis in its plane, so that the centre of $c$ traces a circle $C$ in space of radius $R > r$, the area is (using a well-known theorem of Pappus) the length of $c$ times the length of $C$, that is $4\pi r R$. In this case, the area of the torus is $\geq 1$ times the square of the systole; for an arbitrary metric the constant needs to be reduced to $\sqrt{3}/2$.

Chapters 8 and 9 give an account of the Laplacian of a function on a riemannian manifold, including the heat, wave and Schrödinger equations; Chapter 10 is on geodesics, Chapter 11 on choosing riemannian metrics to suit particular purposes, including Einstein manifolds. Chapter 12 investigates topological questions and the remainder of the book deals with holonomy groups and generalizations. There is a brief ‘technical’ chapter at the end which gives an introduction to tensors, connections and other basic ideas of differential geometry.

This is the sort of book one could dip into or refer to over a period of years. It is not a book from which to learn riemannian geometry, though it would be possible to derive inspiration from it while learning the subject at a more normal pace. It would be nice to carry it around to get out at odd moments on trains or aeroplanes, if it were not for the size: the book is more than two inches thick. Whatever happened to thin paper? I have several books with more pages than this but of half the thickness, much handier for carrying around. The book will have to stay on my desk.

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This book is inspired by the five regular polyhedra which have played a central role in Western culture from the time of the Greeks to the present day. Kepler surmised that they explained the configuration of the solar system as he knew it. Such speculative mysticism, however, is outside the scope of John Parker's monograph, although it is clear that he is inspired, as was Plato, by the beauty and symmetry of the solids. His book is addressed to those teachers who are looking for a series of activities to consolidate and extend the mathematical skills of their pupils. Three-dimensional geometry is famously difficult for students, since it calls upon a