NORMAL OPERATORS ON THE BANACH SPACE $L^p(-\infty, \infty)$. PART I

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1. Introduction. Let $\mathfrak{B}\bar{R}^2$ be the Boolean algebra of all finite unions of subcells of the plane. Denote by $\mathscr{E}_p$ the algebra of all linear bounded transformations of $L^p(-\infty, \infty)$ into itself. Suppose for a moment that $p = 2$, and let $\mathscr{R}_p$ be an involutive abelian subalgebra of $\mathscr{E}_p$: if $\mathscr{R}_p$ is also a Banach space and if $T_p \in \mathscr{R}_p$, then:

(i) *The family of all homomorphic mappings of* $\mathfrak{B}\bar{R}^2$ *into the algebra* $\mathscr{R}_p$ *contains a member* $E_p{}^T$ *such that*

$$T_p = \int \lambda \cdot E_p^T(d\lambda). \tag{1}$$

Suppose, henceforth, that $1 < p < \infty$. The main result of this article (Theorem 6.14) shows that property (i) remains valid for a suitable algebra $\mathscr{R}_p$.

Let $\mathfrak{D}$ be the class of all bounded functions whose real and imaginary parts are piecewise monotone. In § 2 will be defined an isomorphism $f \rightarrow [\wedge f]_p$ whose domain includes $\mathfrak{D}$ and whose range $(t)_p$ is a normed involutive abelian subalgebra of $\mathscr{E}_p$. Theorem 6.14 will show that a member T_p of $(t)_p$ has the property (i) whenever $T_p = [\wedge f]_p$ for some f in $\mathfrak{D}$. The relation (1) involves a Riemann–Stieltjes integral defined in the strong operator-topology of $\mathscr{E}_p$ (see 6.11). The set-function $E_p{}^T$ need not be countably additive: we do not restrict ourselves to "*spectral resolutions*" in the sense of Dunford **(1)**. The values of $E_p{}^T$ are self-adjoint **(4**, p. 22**)**, idempotent members of $(t)_p$.

It is easily seen that the Hilbert transformation and the Dirichlet operators all have the property (i). For less trivial examples, let $\mathscr{M}^1$ be the set of all bounded Radon measures; if $A \in \mathscr{M}^1$, then the convolution operator A_{*p} is defined as the mapping $x \rightarrow A*x$ of $L^p(-\infty, \infty)$ into itself. In the special case $A \in L^1(-\infty, \infty)$, the operator A_{*p} is defined for all x in $L^p(-\infty, \infty)$ by the relation

$$A_{*p}x(\theta) = \int_{-\infty}^{\infty} A(\theta - \beta)x(\beta)d\beta.$$

In case the Fourier transform of A belongs to $\mathfrak{D}$, then the operator $T_p = A_{*p}$ satisfies property (i). Consequently, all the classical convolution operators (Picard, Poisson, Weierstrass, Stieltjes, Fejér, etc.) have property (i). Explicit determination of $E_p{}^T$ is readily inferred from § 6; in the case $p = 2$ our results

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coincide with the ones given by Dunford **(2**, p. 63**)** for operators T of this type. The completion of the algebra $\{A_{*p}: A \in \mathscr{M}^1\}$ is an (A^*)-subalgebra of $(t)_p$ (see 3.2).

Let ∇_p be the operator defined by the relation

$$\nabla_p x = \frac{i}{2\pi} \text{ (derivative of x)}$$

for all x in a suitable subset of $L^p(-\infty, \infty)$; this unbounded operator also has the property (i). Although details regarding such operators will be reserved for a subsequent article (see 7.0), it may be pertinent to remark here that a relation of the type

$$T_p = \int_{-\infty}^{\infty} f(\theta) E_p^{\nabla}(d\theta)$$

holds for any T_p in $(t)_p$ such that $T_p = [\mathbf{\wedge} f]_p$ for some function f of locally bounded variation. For example, take α in $(-\infty, \infty)$, and let R_p be the translator defined by $R_p x(\theta) = x(\theta - \alpha)$ for all x in $L^p(-\infty, \infty)$; then

$$R_p = \int_{-\infty}^{\infty} e^{2\pi i \alpha\theta} E_p^{\nabla}(d\theta).$$

2. The basic function-algebra. Let $\mathfrak{F}_+$ denote the set of all complex-valued measurable functions defined on $(-\infty, \infty)$. Note that $\mathfrak{F}_+$ is an algebra with multiplication $f \cdot g = \{\theta \rightarrow f(\theta) g(\theta)\}$. The customary identification of equivalent functions is implied henceforth.

Let L^+ be the intersection of the family $\{L^p(-\infty, \infty): 1 < p < \infty\}$. The Fourier transform Ψz of a function z in L^+ is defined as the function f such that $||f - f_n||_2 \rightarrow 0$, where $n \rightarrow \infty$ and

$$f_n(\theta) = \int_{-n}^{n} e^{2\pi i\theta\beta} z(\beta) d\beta \qquad (-\infty < \theta < \infty).$$

We denote by (t^+) the set of all linear mappings of L^+ into itself. If $T \in (t^+)$, then

$$|T|_p = \sup\{||Tx||_p : x \in L^+ \text{ and } ||x||_p \leqslant 1\}.$$

Let $\mathscr{E}$ denote the set of all T in (t^+) such that $|T|_p \neq \infty$ whenever $1 < p < \infty$. If $G \in \mathfrak{F}_+$, then $t(g)$ is defined as the set of all T in $\mathscr{E}$ such that

$$\Psi(Tx) = g \cdot \Psi x \qquad \text{for all } x \text{ in } L^+. \tag{2}$$

2.1. *Definition.* Let $\mathfrak{F}$ denote the algebra of all bounded members of $\mathfrak{F}_+$. Our basic operator-algebra is the set

$$(t) = \bigcup \{t(g): g \in \mathfrak{F}\}.$$

If $T \in (t)$, then $\mathbf{v}T$ will denote the unique g in $\mathfrak{F}$ such that $T \in t(g)$. The set $\{\mathbf{v}T: T \in (t)\}$ is denoted by $\mathfrak{F}_{\mathbf{v}}$.

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2.2. *Remarks.* The definition of $\mathbf{v}T$ is justified by the fact that $g = \mathbf{O}$ whenever $g \cdot \Psi x = \mathbf{O}$ for all x in L^+. Note that $\mathfrak{F}_{\mathbf{v}}$ is the set of all g in $\mathfrak{F}$ such that $\emptyset \neq t(g)$. It is easily checked that (t) is an abelian subalgebra of $\mathscr{E}$ and that $\{T \to \mathbf{v}T\}$ maps (t) isomorphically onto $\mathfrak{F}_{\mathbf{v}}$; in particular

(3) $$\mathbf{v}(T^{(0)}T^{(1)}) = (\mathbf{v}T^{(0)}) \cdot (\mathbf{v}T^{(1)}) \quad \text{when} \quad T^{(n)} \in (t).$$

2.3. *Notation.* If $x \in L^p(-\infty, \infty)$, let $x^{\cdot} = \{\theta \to x(-\theta)\}$, while $\bar{x} = \{\theta \to \overline{x(\theta)}\}$ and $\sim x = \{\theta \to \overline{x(-\theta)}\}$.

2.4. *Remarks.* If $T \in \mathscr{E}$ we define $\sim T$ as the operator $\{x \in L^+ \to \sim T \sim x\}$; observe that $|T|_p = |\sim T|_p$ (this follows from $||x||_p = ||\sim x||_p$). If $T \in t(g)$, then it is easily checked that $\sim T \in t(\bar{g})$. Therefore, the mapping $\{T \to \sim T\}$ of (t) into itself is an *involution* (**10**, p. 108).

2.5. The following terminology is found in Hille (**4**, p. 22): a member T of $\mathscr{E}$ is "*self-adjoint*" if $T = \sim T$. It is clear that T will be self-adjoint if and only if the function $\mathbf{v}T$ is real-valued.

3. The basic operator-algebra. From now on, p is a fixed number $(1 < p < \infty)$. Let $\mathscr{E}_p$ denote the Banach space of all bounded linear transformations of $L^p(-\infty, \infty)$ into itself. Since L^+ is dense in $L^p(-\infty, \infty)$, each T in $\mathscr{E}$ has a unique, continuous extension T_p in $\mathscr{E}_p$. Consequently, the algebra (t) is isomorphic to $(t)_p = \{T_p: T \in (t)\}$ under the mapping $\{T \to T_p\}$. Note that $|T_p|_p = |T|_p$. From 2.4 it follows that $(t)_p$ is a normed involutive subalgebra (**10**, p. 110) of $\mathscr{E}_p$. in the sense that $|T_p|_p = |\sim T_p|_p$. Note further that $(t)_p$ contains the identity operator $\mathbf{I}_p = \{x \in L^p(-\infty, \infty) \to x\}$, and the completion $(t)_p{}^*$ of $(t)_p$ is a (*)-algebra in the sense of (**4**, p. 22). The title of this article was suggested by the fact that all members of $(t)_p$ are "normal" (**4**, p. 22).

3.1. *Application.* Let $\mathscr{M}^1$ be the algebra of all bounded Radon measures on $(-\infty, \infty)$. If $A \in \mathscr{M}^1$, then $A*$ is defined as the mapping $\{x \to A*x\}$ of L^+ into itself (where $A*x$ = convolution of A and x; see (**9**)). In 3.2 it will be shown that the completion of $\mathscr{A}_p = \{A*_p: A \in \mathscr{M}^1\}$ is an (A^*)-subalgebra of $(t)_p{}^*$ (see (**4**, Definition 1.15.3)). It is known that $A* \in \mathscr{E}$. If $\Psi(dA)$ is the function g defined by

$$g(\theta) = \int_{-\infty}^{\infty} e^{2\pi i\theta\beta} dA(\beta) \qquad (-\infty < \theta < \infty),$$

then $\Psi(A*x) = \Psi(dA) \cdot \Psi x$ (this can be seen from (**9**, p. 133, (II)), where $\Psi(dA) \cdot$ is denoted (YA)). But $\Psi(dA) \in \mathfrak{F}$, whence $A* \in (t)$ and $\mathbf{v}A* = \Psi(dA)$. Consequently:

3.2. *If* $T_p = A*_p$ *and* $A \in \mathscr{M}^1$, *then* $T_p \in (t)_p$ *and* $\mathbf{v}T = \Psi(dA)$. Thus $\mathscr{A}_p \subset (t)_p$. To show that the completion of $\mathscr{A}_p$ is an (A^*)-algebra, suppose that $T_p = A*_p$ is self-adjoint; from 2.5, 3.2, and (**9**, (i)) it follows that the spectrum of T_p is real.

3.3. *Definitions.* If $f \in \mathfrak{F}_{\mathbf{V}}$, we denote by $[\wedge f]$ the inverse image of f under the mapping $\{T \to \mathbf{v}T\}$; in other words, $[\wedge f]$ is the member T of (t) such that $f = \mathbf{v}T$. If $p' = p/(p-1)$ and $L^p = L^p(-\infty, \infty)$, then

$$\langle x, y\rangle = \int_{-\infty}^{\infty} x \cdot y \qquad \text{and} \qquad (x|y) = \langle x, \bar{y}\rangle$$

whenever $(x, y) \in L^p \times L^{p'}$. Suppose $1 < u \leqslant 2$ and set $w = u/(u-1)$. If $z \in L^u$, then $Y_u(z)$ is defined as the function y such that $||y - y_n||_w \to 0$, where $n \to \infty$ and

$$y_n(\theta) = \int_{-n}^{n} e^{-2\pi i\theta\beta} z(\beta)\, d\beta \qquad (-\infty < \theta < \infty).$$

3.4. *Remark.* Let L^0 denote the set of all step functions on $(-\infty, \infty)$ having compact support. Suppose $x \in L^0$; it is easily seen that $\Psi x \in L^+$ and $Y_u(\Psi x) = x$ whenever $1 < u \leqslant 2$.

3.5. LEMMA. *Suppose* $1 < u \leqslant 2$. *If* $g \in \mathfrak{F}_{\mathbf{V}}$, *then*

$$[\wedge g]x = Y_u(g \cdot \Psi x) = Y_2(g \cdot \Psi x) \qquad \textit{when } x \in L^0.$$

Proof. From $x \in L^0$ it follows that $\Psi x \in L^+$ (see 3.4): therefore $g \cdot \Psi x \in L^u \cap L^2$. Thus $Y_u(g \cdot \Psi x) = Y_2(g \cdot \Psi x) = Y_2\Psi([g\wedge]x)$; the last equality being obtained by setting $T = [\wedge g]$ in (2). The conclusion now follows from 3.4.

3.6. LEMMA. *If* $T_p \in (t)_p$ *and* $q = p/(p-1)$, *then*

$$\langle T_p x^{\cdot}, y^{\cdot}\rangle = \langle x, T_q y\rangle \qquad \textit{when } (x, y) \in L^p \times L^q.$$

Proof. Set $B(x, y) = \langle T_p x^{\cdot}, y^{\cdot}\rangle$ and $B'(x, y) = \langle x, T_q y\rangle$. Both B and B' are continuous bilinear functionals on $L^p \times L^q$. Since the space L^0 is dense in both L^p and L^q (see 3.4), it will therefore suffice to show that B and B' coincide on $L^0 \times L^0$. To that effect, we will need the Parseval formula in the following two equivalent forms:

$$\langle x_1, x_2^{\cdot}\rangle = \langle \Psi x_1, \Psi x_2\rangle \qquad ((x_1, x_2) \in L^2 \times L^2), \tag{4}$$

$$\langle \Psi y_1, y_2\rangle = \langle y_1^{\cdot}, Y_2 y_2\rangle \qquad ((y_1, y_2) \in L^2 \times L^2) \tag{4'}$$

(see (**11**, Theorem 49 or 75); recall that $L^p = L^p(-\infty, \infty)$). Set $g = \mathbf{v}T$, and suppose that $(x, y) \in L^0 \times L^0$. From (4) and (2), therefore, we have:

$$\langle Tx^{\cdot}, y^{\cdot}\rangle = \langle g \cdot \Psi x^{\cdot}, \Psi y\rangle = \langle \Psi x^{\cdot}, g\Psi \cdot y^{\cdot}\rangle.$$

We now apply (4′) with $y_1 = x^{\cdot}$ and $y_2 = g \cdot \Psi y^{\cdot}$:

$$\langle Tx^{\cdot}, y^{\cdot}\rangle = \langle x, Y_2(g \cdot \Psi y)\rangle = \langle x, Ty\rangle;$$

the last equality comes from 3.5 and $T = [\wedge g]$.

3.7. *Remark.* The positive sesquilinear Hermitean form $\{(x, y) \to (x|y)\}$ on $L^+ \times L^+$ (see 3.3) makes L^+ into an inner-product space. From 3.6 it can easily be derived that $\sim T$ is the Hilbert adjoint of T:

$$(Tx|y) = (x|\sim Ty) \quad \text{when } x \in L^+ \text{ and } y \in L^+$$

We will make no use of these properties.

3.8. *Definition.* Suppose $-\infty < \alpha < \infty$. If $\phi \in \mathfrak{F}$, then $\tau_\alpha\phi$ will denote the function g defined for all θ in $(-\infty, \infty)$ by the relation $g(\theta) = \phi(\theta - \alpha)$.

3.9. THEOREM. *Suppose* $-\infty < \alpha < \infty$. *If* $\phi \in \mathfrak{F}_{\mathbf{V}}$ *then* $\tau_\alpha\phi \in \mathfrak{F}_{\mathbf{V}}$.

Proof. Let Ψ_α be the function $\{\theta \to e^{2\pi i\theta\alpha}\}$. Set $T^{(1)} = [\Lambda\phi]$, and let T be the operator defined by the relation

$$Tx = \bar{\Psi}_\alpha \cdot T^{(1)}(\Psi_\alpha \cdot x) \qquad \text{(all } x \text{ in } L^+).$$

Note that $|T|_p = |T^{(1)}|_p$, and therefore $T \in \mathscr{E}$. Since $g = \tau_\alpha\phi \in \mathfrak{F}$, it will suffice to show that (2) holds; but this follows easily from a repeated application of the relation $\tau_\alpha(\Psi\phi) = \Psi(\bar{\Psi}_\alpha \cdot \phi)$.

4. Two lattices of projectors.

The Hilbert transformation H is defined for all x in L^+ by the relation

$$(Hx)(\theta) = \int_{-\infty}^{\infty} \frac{1}{\pi(\beta - \theta)} x(\beta)d\beta \qquad (-\infty < \theta < \infty),$$

the integral being taken in the Cauchy principal value sense. It is well known that $H \in \mathscr{E}$. The fact that $H \in t(-i\cdot\text{sgn})$ is explicitly stated in (**12**, p. 22) and (**3**, p. 8); it can be extracted from (**11**, pp. 120–125). Thus $H \in (t)$ and $H = -i\cdot\text{sgn} \in \mathfrak{F}_{\mathbf{V}}$. Since $\mathfrak{F}_{\mathbf{V}}$ is a linear space containing the function $I^0 = \{\theta \to 1\}$, it follows that $g_0 = 2^{-1}(I^0 + \text{sgn}) \in \mathfrak{F}_{\mathbf{V}}$.

Suppose that α and β belong to the closed interval $[-\infty, \infty]$. Let $I_\#^0(\alpha, \beta)$ denote the characteristic function of the open interval (α, β), and set $\phi_\alpha = I_\#^0(\alpha, \infty)$. Recall that $g_0 = 2^{-1}(I^0 + \text{sgn}) \in \mathfrak{F}_{\mathbf{V}}$, and note that $g_0 = I_\#^0(0, \infty)$. From 3.9 it can therefore be inferred that $\tau_\alpha g_0 = \phi_\alpha \in \mathfrak{F}_{\mathbf{V}}$.

4.1. *Remark.* We now know that $\mathfrak{F}_{\mathbf{V}}$ contains the function $I_\#^0(\alpha, \infty)$ whenever $\alpha \in [-\infty, \infty]$. Again using the fact that $\mathfrak{F}_{\mathbf{V}}$ is an algebra containing I^0, we deduce that $\mathfrak{F}_{\mathbf{V}}$ contains any function of the form $I_\#^0(\alpha, \beta)$, where $-\infty \leqslant \alpha \leqslant \beta \leqslant \infty$.

4.2. *Notation.* Let V denote the set of all complex-valued functions defined on $(-\infty, \infty)$ such that $|f|_v \neq \infty$, where $|f|_v$ denotes the total variation of f on $(-\infty, \infty)$. We will write

$$||f||_\infty = \sup\{|f(\theta)|: -\infty < \theta < \infty\},$$

and

$$||f||_0 = ||f||_\infty + |f|_v.$$

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4.3. LEMMA. *If $L^1 \cap V$ denotes the set of all g in $L^1(-\infty, \infty)$ such that $g \in V$, then $L^1 \cap V \subset \mathfrak{F}_{\mathbf{V}}$. Moreover, there exists a number $c_p > 0$ with the property that, if $g \in L^1 \cap V$, then*

(5) $$|[\wedge g]|_p \leqslant 2^{-1}c_p|g|_v.$$

Proof. An operator Tg corresponds to g so that $||(Tg)x||_p \leqslant 2^{-1}c_p|g|_v||x||_p$ for all x in L^0 (see (**8**, 3.3 and 3.7), where $g = a$). Since L^0 is dense in L^+, it follows that Tg has an extension T_+ with $T_+ \in (t^+)$ and $|T_+|_p \leqslant 2^{-1}c_p|g|_v$, whence $T_+ \in \mathscr{E}$. Since $g \in \mathfrak{F}$, it remains to show that $T_+ \in t(g)$. From (**8**, 7.2 (14)) it follows that

$$\Psi(B_2(x, g)) = g \cdot \Psi x \qquad \text{(when } x \in L^0\text{)}.$$

From the definition (**8**, §5) of $B_p(x, g)$ it results immediately that $B_2(x, g) = (Tg)x$ when $x \in L^0$; consequently $B_2(x, g) = T_+x$ when $x \in L^+$. Thus $T_+ \in t(g)$, which concludes the proof.

4.4. *Remark.* Let "$\leqslant$" be the relation defined on $\mathscr{E}$ by:

$$T^{(1)} \leqslant T^{(2)} \Leftrightarrow T^{(1)} = T^{(1)}T^{(2)}.$$

A family $\mathscr{P}$ will be called an "$\mathscr{E}$-tower" if $(\mathscr{P}, \leqslant)$ forms a lattice of self-adjoint (see 2.5), idempotent members of $\mathscr{E}$ satisfying the following two conditions:

(ii) *The order-type of $(\mathscr{P}, \leqslant)$ is the order-type of some closed subinterval of $[-\infty, \infty]$;*

(iii) *If $P \in \mathscr{P}$, then $\mathbf{O} \in \mathscr{P}$ and $\mathbf{O} \leqslant P \leqslant \mathbf{I} \in \mathscr{P}$.*

4.5. Both families $\{[\wedge I_{\#}^0(\alpha, \infty)]: \alpha \in [-\infty, \infty]\}$ and $\{[\wedge I_{\#}^0(-n, n)]: 0 \leqslant n \leqslant \infty\}$ are $\mathscr{E}$-towers; in Part II it will be shown that they are the spectral resolutions pertaining to two unbounded operators.

Set $\psi_n = I_{\#}^0(-n, n)$. We here examine more closely the $\mathscr{E}$-tower $\{[\wedge\psi_n]: 0 \leqslant n \leqslant \infty\}$. Suppose $0 < n < \infty$, and let χ_n be the function defined by

$$\chi_n(\theta) = (\sin 2\pi n\theta)/\pi\theta \qquad (-\infty < \theta < \infty).$$

The Dirichlet operator $J^{(n)}$ is defined for all x in L^+ by the relation

$$(J^{(n)}x)(\theta) = \int_{-\infty}^{\infty} \chi_n(\theta - \beta)x(\beta)\, d\beta.$$

It is well known that $J^{(n)} \in \mathscr{E}$ (see (**6**)), and from (**11**, Theorem 65) we see that $\Psi(J^{(n)}x) = \Psi(\chi_n * x) = (\Psi\chi_n)\cdot(\Psi x)$. But $\Psi\chi_n = \psi_n$; therefore $J^{(n)} = [\wedge\psi_n]$.

4.6. LEMMA. *If $f \in V$ and $\psi_n = I_{\#}^0(-n, n)$, then*

$$|[\wedge f]|_p \leqslant 2^{-1}c_p \sup\{|\psi_n \cdot f|_v: 0 < n < \infty\}.$$

Proof. Clearly $h_n = \psi_n \cdot f \in L^1 \cap V$; from 4.3 therefore

(6) $$|[\wedge h_n]|_p \leqslant 2^{-1}c_p \sup\{|\psi_n \cdot f|_v: 0 < n < \infty\} = k_p'.$$

Suppose $x \in L^+$, and note that

$$\lim \| [\Lambda f]x - J^{(n)}([\Lambda f]x)\|_p = 0 \qquad (n \to \infty) \tag{7}$$

(see, for example, **(6** (1b′′)**)** or **(8**, 5.2**)**). In 4.5 we saw that $J^{(n)} = [\Lambda\psi_n]$; therefore $J^{(n)} \circ [\Lambda f] = [\Lambda(\psi_n \cdot f)] = [\Lambda h_n]$ (from (3)). Accordingly, (6) now states that $\|J^{(n)}([\Lambda f]x)\|_p \leqslant k_p'\|x\|_p$, which (from (7)) gives the conclusion $\|[\Lambda f]x\|_p \leqslant k_p'\|x\|_p$.

4.7. Theorem. *If $f \in V$, then $f \in \mathfrak{F}_{\mathbf{V}}$ and*

$$|[\Lambda f]|_p \leqslant c_p\|f\|_0. \tag{8}$$

Proof. Suppose $0 < n < \infty$ throughout, and set $\mathfrak{a}_n = (-n, n)$, while $\mathfrak{a}_n^- = (-\infty, -n]$ and $\mathfrak{a}_n^+ = [n, \infty)$. Note first that $h_n = I_\#^0\mathfrak{a}_n$ vanishes outside of $\mathfrak{a}_n$, so that $|h_n|_v \leqslant 2\|f\|_\infty + |f|_v$. In the notation of 4.6, we can write $h_n = \psi_n \cdot f$; consequently, the relation (8) follows from 4.6. It remains to show that $f \in \mathfrak{F}_{\mathbf{V}}$. Define $f^{(n)} = h_n + g^{(n)}$, where

$$g^{(n)} = f(-n)I_\#^0(\mathfrak{a}_n^-) + f(n)I_\#^0(\mathfrak{a}_n^+).$$

Since $g^{(n)}$ is a linear combination of members of $\mathfrak{F}_{\mathbf{V}}$ (see 4.1), it follows that $g^{(n)} \in \mathfrak{F}_{\mathbf{V}}$. Since $h_n \in L^1 \cap V$ and 4.3, this in turn necessitates that $f^{(n)} \in \mathfrak{F}_{\mathbf{V}}$. Set $T^{(n)} = [\Lambda f^{(n)}]$ and apply (8):

$$|T^{(n)} - T^{(m)}|_p \leqslant c_p\|f^{(n)} - f^{(m)}\|_0 \qquad (m > 0). \tag{9}$$

Let $v(g; \mathfrak{a})$ denote the total variation of g on $\mathfrak{a}$; observe that $v(f - f^{(n)}; \mathfrak{a}) = v(f; \mathfrak{a})$ when $\mathfrak{a} = \mathfrak{a}_n^-$ or $\mathfrak{a} = \mathfrak{a}_n^+$. Moreover, $f - f^{(n)}$ vanishes on $\mathfrak{a}_n$, and therefore

$$\|f - f^{(n)}\|_\infty \leqslant |f - f^{(n)}|_v = v(f; \mathfrak{a}_n^-) + v(f; \mathfrak{a}_n^+).$$

Since $f \in V$, this inequality implies that

$$0 = \lim\|f - f^{(n)}\|_0 = \lim\|f - f^{(n)}\|_\infty \qquad (n \to \infty). \tag{10}$$

From (9) and (10) it can be inferred that the sequence $\{T_p^{(n)}\}_n$ is a Cauchy sequence in $\mathcal{E}_p$, and it accordingly converges (when $n \to \infty$) to a member T_p of $\mathcal{E}_p$. Therefore, $p \in (1, \infty)$ and $x \in L^+$ implies that $0 = \lim\|T_p x - T^{(n)}x\|_p$ $(n \to \infty)$; but this in turn implies that $\{T^{(n)}x\}_n$ converges in measure to $T_p x$. Since measure-limits are uniquely defined, the outcome can be stated as follows: $p \in (1, \infty)$ and $x \in L^+$ implies that $T_2 x = T_p x \in L^p$. From this we infer that $T_2 \in \mathcal{E}$ (see § 2).

The proof is now concluded by showing that $T_2 \in t(f)$. Suppose $x \in L^+$, set $\phi = \Psi(T_2 x) - f \cdot \Psi x$ and note that

$$\|\phi\|_2 \leqslant |T_2 - T^{(n)}|_2\|x\|_2 + \|f - f^{(n)}\|_\infty\|x\|_2.$$

From (10) it follows that $\phi = \mathbf{O} = \Psi(T_2 x) - f \cdot \Psi x$. This shows that $T_2 \in t(f)$, whence $f \in \mathfrak{F}_{\mathbf{V}}$.

4.8. Corollary. $V \subset \mathfrak{F}_{\mathbf{V}}$.

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5. Two convergence theorems. Let F be a function defined on a set S. If $(S, \gg)$ is a directed set, then the net $(F, \gg)$ is also denoted $\{F(s): s \in S, \gg\}$ (our terminology and notation come from (**5**, p. 65)). If F maps into a set $\mathfrak{X}$, then $(F, \gg)$ is called a *net in* $\mathfrak{X}$. If $(F, \gg)$ is a net in a Hausdorff space $\mathfrak{X}$, then we write

$$x = \mathfrak{X} \lim\{F(s): s \in S, \gg\}$$

to indicate that $(F, \gg)$ converges to a point x in $\mathfrak{X}$ (see (**5**, p. 68)). Let $\mathscr{T}_p$ denote the strong operator-topology of the algebra $\mathscr{E}_p$ which was defined in § 3. For example, suppose that $F(s) \in \mathscr{E}$ (for all s in S) and $T \in \mathscr{E}$; then $F(s)$ and T admit continuous extensions $F(s)_p$ and T_p, respectively (see § 3; $F(s)_p \in \mathscr{E}_p$ and $T_p \in \mathscr{E}_p$). Accordingly, the statement

$$T_p = \mathscr{T}_p \lim\{F(s)_p: s \in S, \gg\} \tag{11}$$

means that the net $\{F(s)_p: s \in S, \gg\}$ converges to T_p in the strong operator-topology of $\mathscr{E}_p$ (see (**4**, p. 53)).

5.1. *Definition.* Let $(F, \gg)$ be a net in $\mathscr{E}$. If $T \in \mathscr{E}$, then

$$T = \mathscr{T} \lim\{F(s): s \in S, \gg\}$$

is written to mean that relation (11) occurs whenever $1 < p < \infty$.

5.2. *Remark.* If $\{f(s): s \in S, \gg\}$ is a net in $[0, \infty)$, then

$$\infty \neq \limsup\{f(s): s \in S, \gg\}$$

if and only if there exists a number $N_0 > 0$ and an element s_0 of S such that $f(s) \leqslant N_0$ whenever $s \in S$ and $s \gg s_0$.

5.3. THEOREM. *Suppose $g \in \mathfrak{F}_{\mathbf{V}}$, and let $\{G(s): s \in S, \gg\}$ be a net in V. Set $\mathfrak{X}_p = L^p(-\infty, \infty)$ and suppose further that the relation*

$$[\wedge g]x = \mathfrak{X}_2 \lim\{[\wedge G(s)]x: s \in S, \gg\} \tag{12}$$

holds for all x in L^0. If

$$\infty \neq \limsup\{\|G(s)\|_0: s \in S, \gg\}, \tag{13}$$

then

$$[\wedge g] = \mathscr{T} \lim\{[\wedge G(s)]: s \in S, \gg\}.$$

Proof. Suppose $1 < p < \infty$. We must prove (11) for $T = [\wedge g]$ and $F(s) = [\wedge G(s)]$; that is, we must show that

$$T_p x = \mathfrak{X}_p \lim\{F(s)_p x: s \in S, \gg\} \tag{14}$$

for all x in $\mathfrak{X}_p$. From (13), 5.2, and 4.7 follows the existence of a number N_0 and an element s_0 of S such that, if $s \in S$ and $s \gg s_0$, then

$$|F(s)_q|_q \leqslant N_0 c_q \tag{iv}$$

whenever $1 < q < \infty$. It will be convenient to describe (iv) by saying that the net $\{F(s)_q: s \in S, \gg\}$ is e.u.b. (eventually uniformly bounded) in $\mathscr{E}_q$. Consequently, the net $\{F(s)_p: s \in S, \gg\}$ is e.u.b. in $\mathscr{E}_p$. It is easily verified that the Banach–Steinhaus theorem (**4**, p. 41) applies not only to uniformly bounded sequences in $\mathscr{E}_p$, but also to e.u.b. nets in $\mathscr{E}_p$. Let us suppose for a moment that (14) holds for all x in L^0; since L^0 is dense in $\mathfrak{X}_p$, the Banach–Steinhaus theorem implies that (14) holds for all x in $\mathfrak{X}_p$, and the theorem is proved.

Suppose $x \in L^0$, and set $y(s) = Tx - F(s)x$; in view of our preceding remark, it will suffice to show that

$$0 = \lim\{||y(s)||_p: s \in S, \gg\}. \tag{v}$$

If $p = 2$, there is nothing to prove, since (v) is then our hypothesis (12). If $p \neq 2$ there clearly exists a number q with $1 < q < \infty$ such that p lies between 2 and q; there exists therefore a number m such that

$$\frac{1}{p} = \frac{1}{2}m + \frac{1}{q}(1 - m) \qquad \text{and} \qquad 0 < m < 1.$$

From the logarithmic convexity of the norm we see that

$$||y(s)||_p \leqslant (||y(s)||_2)^m \cdot (||Tx - F(s)x||_q)^{1-m}.$$

Accordingly, we can infer from (iv) that, if $s \gg s_0$, then

$$||y(s)||_p \leqslant (||y(s)||_2)^m \cdot ([|T|_q + N_0 c_q] \,.\, ||x||_q)^{1-m}.$$

Consequently, (v) results from the hypothesis (12).

5.4. Corollary. *Suppose $g \in \mathfrak{F}_V$ and let $\{G(s): s \in S, \gg\}$ be a net in V satisfying* (13). *If*

$$0 = \lim\{||g - G(s)||_\infty: s \in S, \gg\}, \tag{15}$$

then

$$[\wedge g] = \mathscr{T} \lim \{[\wedge G(s)]: s \in S, \gg\}. \tag{16}$$

Proof. In view of 5.3, it will suffice to establish (12). Take x in L^0; from 3.5 it follows that

$$||[\wedge g]x - [\wedge G(s)]x||_2 = ||Y_2([g - G(s)] \cdot \Psi x)||_2.$$

But $[g - G(s)] \cdot \Psi x$ is in L^+ (see 3.4). Since Y_2 is an isometric mapping, we see that

$$||[\wedge g]x - [\wedge G(s)]x||_2 \leqslant ||g - G(s)||_\infty \,.\, ||\Psi x||_2. \tag{17}$$

The conclusion (12) now results from (15), (17), and $\infty \neq ||\Psi x||_2$.

6. The main result. From now on, $R = (-\infty, \infty)$ and $\bar{R} = [-\infty, \infty] = R \cup \{-\infty, \infty\}$; if α and β lie in $\bar{R}$, then $(\alpha, \beta] = \{\theta \in R: \alpha < \theta \leqslant \beta\}$. The space $\bar{R}^2 = \bar{R} \times \bar{R}$ consists of all points $\lambda = (\lambda_1, \lambda_2)$ such that $\lambda_1 \in \bar{R}$ and

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$\lambda_2 \in \bar{R}$. The usual embedding $\{\alpha \to (\alpha, 0)\}$ of $\bar{R}$ into $\bar{R}^2$ will be assumed. Accordingly, $\bar{R} \subset \bar{R}^2$; if α and β belong to $\bar{R}^2$, then $(\alpha, \beta]$ is the Cartesian product $(\alpha_1, \beta_1] \times (\alpha_2, \beta_2]$, with the exception $(\alpha, \beta] = (\alpha_1, \beta_1] \times \{0\} = (\alpha_1, \beta_1]$ in the case $\alpha = \alpha_1$ and $\beta = \beta_1$.

6.1. *Definitions.* If $Q \subset \bar{R}^2$, then $\mathfrak{B}Q$ will denote the family of all finite unions of members of $\mathfrak{A}Q = \{(\alpha, \beta]: (\alpha, \beta) \in Q \times Q\}$.

6.2. The Boolean algebra $\mathfrak{C}_\blacktriangle$ will consist of all symmetric differences $B \dotplus N = (B \cup N) - (B \cap N)$, where $B \in \mathfrak{B}\bar{R}$ and N is a subset of R having zero measure.

6.3. The following notations will be used consistently. If $g \in \mathfrak{F}$, then $g_1 =$ (real part of g) and $g_2 =$ (imaginary part of g). If $\sigma \in \mathfrak{B}\bar{R}^2$, then $(g \in \sigma) = \{\theta \in R: g(\theta) \in \sigma\}$, except that $(g \in \sigma) = (g_1 \in \sigma)$ whenever $g = g_1$.

6.4. The set $\mathfrak{F}_\blacktriangle$ will consist of all functions g in $\mathfrak{F}$ such that $(g \in \sigma) \in \mathfrak{C}_\blacktriangle$ whenever $\sigma \in \mathfrak{A}\bar{R}^2$.

6.5. If $T \in (t)$ and $g = \vee T \in \mathfrak{F}_\blacktriangle$, then the set-function E^T is defined for all σ in $\mathfrak{B}\bar{R}^2$ by the relation

$$E^T(\sigma) = [\wedge I_{\#}{}^0(g \in \sigma)].$$

Recall that $\psi = I_{\#}{}^0(g \in \sigma)$ is a function such that $\psi(\theta) = 1$ whenever $\theta \in (g \in \sigma)$, while $\psi(\theta) = 0$ otherwise. Note that $\psi \in V$; in this connection, it should also be mentioned that $\mathfrak{A}\bar{R}$, $\mathfrak{B}\bar{R}$, and $\mathfrak{C}_\blacktriangle$ are Boolean set-algebras. Since the verification of these facts is routine, it will be omitted. Both $\emptyset$ and R^2 belong to $\mathfrak{B}\bar{R}^2$; it is clear that

$$E^T(\emptyset) = \mathbf{O} \qquad \text{and} \qquad E^T(R^2 - \sigma) = \mathbf{I} - E^T(\sigma)$$

whenever $\sigma \in \mathfrak{B}\bar{R}^2$. In fact, E^T is an isomorphism into (t) of the Boolean set-algebra $\mathfrak{B}\bar{R}^2$; if σ' and σ'' are in $\mathfrak{B}\bar{R}^2$, then

$$E^T(\sigma' \cup \sigma'') = E^T(\sigma') \vee E^T(\sigma'')$$

and

$$E^T(\sigma' \cap \sigma'') = E^T(\sigma') \wedge E^T(\sigma'')$$

(the operations "$\vee$" and "$\wedge$" are defined in (**1**, p. 219)).

6.6. *Orientation.* The following is aimed at defining two-dimensional Stieltjes integrals of commonplace type. In order to implement a later proof (6.14), an order-preserving notation for range partitions will first be described.

6.7. Let $\mathfrak{Z}$ be the family of all strictly monotone-increasing functions Z whose domain $D(Z)$ is a finite set of consecutive integers, and whose range $\{Z_\nu: \nu \in D(Z)\}$ is a subset of $\bar{R}$. If $Z \in \mathfrak{Z}$, we denote by Z^* the set $\{\nu \in D(Z):$

$\nu > \min D(Z)\}$ and write $Z(\nu] = (Z_{\nu-1}, Z_\nu]$ whenever $\nu \in Z^*$. In case $Q_\iota \subset \bar{R}$, then $\mathfrak{Z}Q_\iota$ will denote the family of all Z in $\mathfrak{Z}$ such that

$$Q_\iota \subset \cup \{Z(\nu]: \nu \in Z^*\}.$$

6.8. *Definition.* Suppose $g \in \mathfrak{F}$, and denote by $[g]$ the closed cell $[-\lambda, \lambda]$, where $\lambda_\iota = ||g_\iota||_\infty$ for $\iota = 1, 2$ (see 4.2). The family $S[g]$ consists of all ordered pairs $(Z, \mathfrak{z})$ whose first member $Z = (Z_1, Z_2)$ lies in $\mathfrak{Z}[g_1] \times \mathfrak{Z}[g_2]$, and such that $\mathfrak{z}$ is a function on $Z^* = Z_1^* \times Z_2^*$ whose values $\mathfrak{z}(\nu)$ lie in $Z^\nu = Z_1(\nu_1] \times Z_2(\nu_2]$ whenever $\nu = (\nu_1, \nu_2) \in Z^*$.

6.9. *Definition.* Suppose $T \in (t)$ and $\mathbf{v}T \in \mathfrak{F}_\blacktriangle$. If $s = (Z, \mathfrak{z}) \in S[\mathbf{v}T]$, then we write

$$(E^T: s) = \sum_{\nu \in \omega} \mathfrak{z}(\nu) E^T(Z^\nu) \qquad (\omega = Z^*).$$

6.10. THEOREM. *Suppose $T \in (t)$ and $\mathbf{v}T \in \mathfrak{F}_\blacktriangle$. If there exists a number $k_0 > 0$ such that $|\mathbf{v}(E^T: s)|_v \leqslant k_0||\mathbf{v}(E^T: s)||_\infty$ whenever $s \in S[\mathbf{v}T]$, then the following Stieltjes integral exists:*

$$\int \lambda \cdot E^T(d\lambda) = \mathscr{T} \lim \{(E^T: s): s \in S[\mathbf{v}T], \gg\}. \tag{18}$$

Moreover,

$$T = \int \lambda \cdot E^T(d\lambda). \tag{1}$$

6.11. *Remarks.* The set $S[\mathbf{v}T]$ is directed by the partial ordering "$\gg$" (see **(5**, p. 79**)** and 6.12). The meaning of the relation (1) will now be explicitly formulated. If $1 < p < \infty$, then the net

$$\left\{ \left\| T_p x - \sum_{\nu \in \omega} \mathfrak{z}(\nu) E^T(Z^\nu)_p x \right\|_p : (Z, \mathfrak{z}) \in S[\mathbf{v}T], \gg \right\}$$

converges to zero for all x in $L^p(R)$ (compare (18) with 5.1). Consequently, (1) implies that the net

$$\left\{ \sum_{\nu \in \omega} \mathfrak{z}(\nu) E^T(Z^\nu)_p : (Z, \mathfrak{z}) \in S[\mathbf{v}T], \gg \right\}$$

converges to T_p in the weak operator-topology (this again comes from (18) and 5.1); T_p is therefore a "scaled" member of $\mathscr{E}_p$ (see **(7**, p. 450**)**).

6.12. *Proof of* 6.10. If $Q \subset R^2$, let $|Q|$ denote the diameter of Q. Set $g = \mathbf{v}T$ and $S = S[g]$. Suppose $s = (Z, \mathfrak{z}) \in S$. We define $||s|| = \max\{|Z^\nu|: \nu \in Z^*\}$. The partial ordering is defined by: $s' \gg s \Leftrightarrow ||s'|| \leqslant ||s||$ whenever $s' \in S$. Set $G(s) = \mathbf{v}(E^T: s)$; from 6.9 and 6.5 we note that

$$G(s) = \sum_{\nu \in \omega} \mathfrak{z}(\nu) I^0_\#(g \in Z^\nu) \qquad (\omega = Z^*). \tag{19}$$

Clearly $G(s) \in V$ (see 6.5). It is easily seen that

$$||g - G(s)||_\infty \leqslant ||s||. \tag{20}$$

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But $\infty \neq ||g||_\infty$ and therefore $\infty \neq \lim\sup\{||G(s)||_\infty : s \in S, \gg\}$ (see 5.2), from which our hypothesis $||G(s)||_0 \leqslant (k_0 + 1)||G(s)||_\infty$ yields the relation (13) of 5.3. Since (20) implies (15) in 5.4, the net $\{G(s): s \in S, \gg\}$ satisfies all the conditions of 5.4. The conclusion now results from (16), $T = [\wedge g]$ and $(E^T: s) = [\wedge G(s)]$.

6.13. *Definition.* A function f is "piecewise monotone" if there exists a member Z of $\mathfrak{Z}\bar{R}$ such that f is monotone on $Z(\nu]$ for all ν in Z^* (see 6.7).

6.14. THEOREM. *Let g be a bounded function whose real and imaginary parts are piecewise monotone. Then $g \in \mathfrak{F}_\blacktriangle$ and $[\wedge g]$ is a member T of (t) such that*

$$T = \int \lambda \cdot E^T(d\lambda) \tag{1}$$

in the sense of 6.10–6.11.

COROLLARY. *Suppose that A is a bounded Radon measure on R, and let g be the Fourier transform of A. If T_p is the convolution operator $A *_p$, then T satisfies* (1) *whenever g satisfies the hypothesis of* 6.14.

Proof. Observe that $g = \Psi(dA)$ in the notation of 3.1; from 3.2 therefore $\vee T = g$, and the conclusion now comes from 6.14.

6.15. *Remark.* Suppose $J \in \mathfrak{A}\bar{R}$, and let f belong to the set $\mathfrak{G}(J)$ of all real-valued functions that are monotone increasing on J. If $\sigma = (\alpha, \infty)$ or $\sigma = [\alpha, \infty)$, then $J \cap (f \in \sigma)$ is a connected subset of $\bar{R}$; therefore $J \cap (f \in \sigma) \in \mathfrak{C}_\blacktriangle$.

6.16. Consider now the case $\sigma = (\alpha, \beta] \in \mathfrak{A}\bar{R}$; then $J \cap (f \in \sigma) \in \mathfrak{C}_\blacktriangle$. This can be seen by noting that $(f \in \sigma)$ is the set-theoretic difference $J \cap (f \in \sigma_1) - J \cap (f \in \sigma_2)$, where $\sigma_1 = (\alpha, \infty)$ and $\sigma_2 = (\beta, \infty)$; since $\mathfrak{C}_\blacktriangle$ is a Boolean ring, the conclusion follows from 6.15.

6.17. *Definition.* If $J \in \mathfrak{A}\bar{R}$, then $\mathfrak{M}(J)$ will be the set of all bounded functions whose real and imaginary parts are both monotone on J.

6.18. LEMMA. *If $J \in \mathfrak{A}\bar{R}$ and $g \in \mathfrak{M}(J)$, then $J \cap (g \in \sigma) \in \mathfrak{C}_\blacktriangle$ whenever $\sigma \in \mathfrak{A}\bar{R}^2$.*

Proof. Since $\sigma \in \mathfrak{A}\bar{R}^2$, we can write $\sigma = \sigma_1 \times \sigma_2$, where $\{\sigma_1, \sigma_2\} \subset \mathfrak{A}\bar{R}$, so that $J \cap (g \in \sigma) = J \cap (g_1 \in \sigma_1) \cap (g_2 \in \sigma_2)$. Set $\iota = 1, 2$. The proof will therefore be concluded by establishing that $J \cap (g_\iota \in \sigma_\iota) \in \mathfrak{C}_\blacktriangle$. Since this was proved in 6.16 for the case $g_\iota \in \mathfrak{G}(J)$, it will suffice to consider the case where g_ι is decreasing on J. But then $f = -g_\iota \in \mathfrak{G}(J)$, and the arguments in 6.16 (together with 6.15), give the conclusion $J \cap (g_\iota \in \sigma_\iota) \in \mathfrak{C}_\blacktriangle$.

6.19. *Definition.* If $Q \subset R^2$, then $\mathfrak{U}Q$ will denote the set of all mappings F of Q into R^2 such that, if $\lambda' = (\lambda_1', \lambda_2') \in Q$ and $\lambda'' = (\lambda_1'', \lambda_2'') \in Q$, then $\lambda_\iota' \leqslant \lambda_\iota''$ implies $F_\iota(\lambda') \leqslant F_\iota(\lambda'')$ whenever $\iota = 1$ and also when $\iota = 2$.

6.20. LEMMA. *Suppose $J \in \mathfrak{A}\bar{R}$ and $g \in \mathfrak{M}(J)$. If $F \in \mathfrak{U}[g]$ then $(F \circ g) \in \mathfrak{M}(J)$.*

Proof. The composition $(F \circ g)$ is the function h such that $h(\theta) = F(g(\theta))$ for all θ in R. In case $\theta' \leqslant \theta''$ and $g_1(\theta') \leqslant g_1(\theta'')$, set $\lambda' = g(\theta')$ and $\lambda'' = g(\theta'')$; then $\lambda_1' \leqslant \lambda_1''$ and $F_1(g(\theta')) \leqslant F_1(g(\theta''))$. Therefore $h_1 \in \mathfrak{G}(J)$. The remaining cases can be similarly derived.

6.21. *Remark.* Let $h \in \mathfrak{F}$ and $J = (\alpha, \beta] \in \mathfrak{A}\bar{R}$. Denote by $v(h; J)$ the total variation of h on $[\alpha, \beta] \cap R$. If $h \in \mathfrak{M}(J)$ (see 6.17), it is easily verified that $v(h; J) \leqslant 8||h||_\infty$.

Proof of 6.14. Set $\iota = 1, 2$. By hypothesis there exist two members Π_1 and Π_2 of $\mathfrak{Z}\bar{R}$ (see 6.7) such that g_ι is monotone on each $\Pi_\iota(\kappa_\iota]$ when $\kappa_\iota \in \Pi_\iota^*$. For any $\kappa = (\kappa_1, \kappa_2)$ in $\Pi^* = \Pi_1^* \times \Pi_2^*$, we write $\Pi^\kappa = \Pi_1(\kappa_1] \cap \Pi_2(\kappa_2]$. Note that $\Pi^\kappa \in \mathfrak{A}\bar{R}$ and $g \in \mathfrak{M}(\Pi^\kappa)$.

Observe first that $g \in V$, and therefore $g \in \mathfrak{F}_{\mathbf{V}}$ (by 4.8). Thus $\wedge g = T \in (t)$ and $\mathbf{v}T = g$. The property $g \in \mathfrak{F}_\blacktriangle$ is proved as follows. Take any σ in $\mathfrak{A}\bar{R}^2$, and note that $(g \in \sigma) = \bigcup\{\Pi^\kappa \cap (g \in \sigma) : \kappa \in \Pi^*\}$; since $\mathfrak{C}_\blacktriangle$ is a Boolean ring, the conclusion $(g \in \sigma) \in \mathfrak{C}_\blacktriangle$ is now inferred from 6.18.

Next, take any $s = (Z, \mathfrak{z})$ in $S[g]$, set $G(s) = \mathbf{v}(E^T : s)$ and note that

$$|G(s)|_v \leqslant \sum_{\kappa=1}^{m} v(G(s); \Pi^\kappa), \tag{21}$$

where $\{1, 2, 3, \ldots, m\} = \Pi^*$. From Definition 6.8, there exist functions Z_ι in $\mathfrak{Z}$ such that $\mathfrak{z}(\nu) \in Z_\nu = Z_1(\nu_1] \times Z_2(\nu_2]$ for all $\nu = (\nu_1, \nu_2)$ in $Z_1^* \times Z_2^*$ (the index-sets Z^* are defined in 6.7). If $\lambda \in [g]$, denote by $\nu[\lambda]$ the ν in Z^* such that $\lambda \in Z^\nu$, and let F be the function defined by $F(\lambda) = \mathfrak{z}(\nu[\lambda])$ for all λ in $[g]$. From the isotonicity of the correspondences set up in 6.7 it now follows that $F \in \mathfrak{U}[g]$ (see 6.19). On the other hand, it is easily checked that $G(s) = (F \circ g)$ (see (19)). From 6.20 therefore: $G(s) \in \mathfrak{M}(J)$ whenever $J \in \mathfrak{A}\bar{R}$.

Suppose $\kappa \in \Pi^*$. Since $G(s) \in \mathfrak{M}(\Pi^\kappa)$, it results from 6.21 that $v(G(s); \Pi^\kappa) \leqslant 8||G(s)||_\infty$, and from (21) therefore: $|G(s)|_v \leqslant 8m||G(s)||_\infty$. In view of 6.10, the proof of 6.14 is completed.

7.0. Added in proof. Part II of this article has appeared in the Journal of Math. and Mechanics, Vol. 10 (1961), 111–134.

7.1. *Remark.* (added March 9, 1961). The set V (defined in 4.2) is strictly included in the set V_β of all functions having generalized higher β-variation; it can be proved that $V_\beta \subset \mathfrak{F}_{\mathbf{V}}$. This last assertion is clearly stronger than our Corollary 4.8; it is implicit in a remark on p. 242 of an article by I. I. Hirschman, Jr. "*On multiplier transformations*", Duke Math. J., 26 (1959), 221–242. At the time the present article was written, I was unaware of Professor Hirschman's remark.

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