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ON SPLITTING AN INFINITE RECURSIVELY ENUMERABLE CLASS

J. B. FLORENCE

1. Introduction. By an *RE* (*recursively enumerable*) *sequence* we mean a sequence $V(0)$, $V(1)$, . . . of uniformly *RE* sets. $\mathscr{R}$ denotes the class of all *RE* sets. If $\mathscr{C} \subseteq \mathscr{R}$, $\mathscr{C}$ is an *RE class* if either $\mathscr{C} = \emptyset$ or $\mathscr{C} = \{V(x)|x \in N\}$ for some *RE* sequence $V(0)$, $V(1)$, ...

Let $\mathscr{C}$ be an infinite *RE* class. For $n \geqq 2$, we say that $\mathscr{C}$ has an *n-split* if $\mathscr{C}$ is the union of n disjoint infinite *RE* classes. $\mathscr{C}$ has a 1-*split* if $\mathscr{C}$ has an infinite *RE* subclass $\mathscr{D}$ such that $\mathscr{C} - \mathscr{D}$ is infinite. $\mathscr{C}$ has a 0-*split* if $\mathscr{C}$ has a proper infinite *RE* subclass. Clearly if $m > n$ and $\mathscr{C}$ has an m-split, then $\mathscr{C}$ has an n-split. Young [**3**] proved the existence of infinite *RE* classes with no 0-split (for a generalisation see [**1**]). If $k \geqq 1$, an *RE* class $\mathscr{C}$ is said to be k-*RE* (Pour-El and Putnam, [**2**]) if there is an *RE* sequence $V(0)$, $V(1)$, . . . which enumerates $\mathscr{C}$ with at most k occurrences of each set.

The main results of this paper are as follows:

(a) For each $n \geqq 0$ there is an *RE* class $\mathscr{C}$, whose members form a partition of N into finite sets, such that $\mathscr{C}$ has an n-split but no $(n + 1)$-split. Note that any infinite *RE* class which contains a finite set must have a 0-split.

(b) If $\mathscr{C}$ is an infinite *RE* class of finite sets of bounded cardinality, then $\mathscr{C}$ has an n-split for all n.

(c) A k-*RE* class must have a 1-split, but there is a 2-*RE* class of finite sets with no 2-split.

(d) Every infinite *RE* class $\mathscr{C}$ can be split in the sense that there is an *RE* class $\mathscr{D}$ such that both $\mathscr{C} \cap \mathscr{D}$ and $\mathscr{C} - \mathscr{D}$ are infinite.

N denotes the set of natural numbers and Sub N denotes the class of finite subsets of N. If $A \subseteq N$, $\bar{A}$ denotes the complement of A. We will assume a fixed listing of $N \times N$, that defined by the pairing function

$$\langle i, j\rangle = 2^i(2j + 1) - 1.$$

Thus the ordered pair corresponding to n is $(\pi_1(n), \pi_2(n))$ where

$$\pi_1(n) = \text{the exponent of 2 in } n + 1$$
$$\pi_2(n) = (n + 1)/2^{\pi_1(n)+1} - \tfrac{1}{2}.$$

2. No *RE* class is cohesive. If $\mathscr{C} \subseteq \mathscr{R}$, let $\mathscr{H}(\mathscr{C})$ = the class of all finite subsets of members of $\mathscr{C}$. Theorems 7 and 8 of Pour-El and Putnam [**2**] are special cases of the following result.

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THEOREM 1. *Let* $\mathscr{C}_1 = \{V(x)|x \in A\}$ *where* $V(0), V(1), \ldots$ *is an RE sequence and* $A \in \Sigma_2$, *and let* $\mathscr{C}_2$ *be an RE class such that* $\mathscr{H}(\mathscr{C}_1) \subseteq \mathscr{H}(\mathscr{C}_2)$. *Then* $\mathscr{C}_1 \cup \mathscr{C}_2$ *is an RE class.*

Proof. If $\mathscr{C}_2 = \emptyset$, then $\mathscr{C}_1 = \emptyset$ and $\mathscr{C}_1 \cup \mathscr{C}_2 = \emptyset$, Thus we can assume that $\mathscr{C}_2 = \{U(x)|x \in N\}$ where $U(0), U(1), \ldots$ is an *RE* sequence. Since $A \in \Sigma_2$, there is a recursive relation R such that $x \in A \Leftrightarrow \exists y \forall z R(x, y, z)$. We will define an *RE* sequence $T(0), T(1), \ldots$ in such a way that

$$T(\langle x, y\rangle) = V(x) \text{ if } \forall z R(x, y, z) \text{ and}$$
$$T(\langle x, y\rangle) = \text{a member of } \mathscr{C}_2 \text{ if } \exists z \neg R(x, y, z).$$

Then $\mathscr{C}_1 \cup \mathscr{C}_2 = \{T(x)|x \in N\} \cup \mathscr{C}_2$ and $\mathscr{C}_1 \cup \mathscr{C}_2$ is an *RE* class.

Instructions for enumerating $T(n)$, $n = \langle x, y\rangle$. Consider the following procedure:

Stage 0. Set $m(0) = 0$. Enumerate $V(0), V(1), \ldots$ until a number, $i(1)$ say, is put into $V(x)$. Then enumerate $U(0), U(1), \ldots$ until $m(1)$ is found with $i(1) \in U(m(1))$. Put $i(1)$ into $T(n)$.

Stage 1. Enumerate $V(0), V(1), \ldots$ until a new number, $i(2)$ say, is put into $V(x)$. Then enumerate $U(0), U(1), \ldots$ until $m(2)$ is found with $\{i(1), i(2)\} \subseteq U(m(2))$. Put $i(2)$ into $T(n)$.

....

Note that each stage consists of two searches, either of which may be non-terminating; then of course the next stage will never be reached.

To enumerate $T(n)$ alternate between following the above procedure and checking that $R(x, y, 0), R(x, y, 1), \ldots$. If z is ever found such that $\neg R(x, y, z)$, and at that time $m(0), \ldots, m(k)$ are defined, make $T(n) = U(m(k))$.

Case 1. $\exists z \neg R(x, y, z)$. Then $T(n) = U(m)$ for some m, i.e., $T(n) \in \mathscr{C}_2$.

Case 2. $\forall z R(x, y, z)$. Then $V(x) \in \mathscr{C}_1$ and so for every finite subset F of $V(x)$ there is m with $F \subseteq U(m)$. It follows that $T(n) = V(x)$.

THEOREM 2. *If* $\mathscr{C}$ *is an infinite RE class, then there is an RE class* $\mathscr{D}$ *such that both* $\mathscr{C} \cap \mathscr{D}$ *and* $\mathscr{C} - \mathscr{D}$ *are infinite.*

Proof. Suppose $\mathscr{C} = \{V(x)|x \in N\}$ where $V(0), V(1), \ldots$ is an *RE* sequence. Let $V^s(x)$ denote the finite set of numbers in $V(x)$ at the beginning of step s in some enumeration of $V(0), V(1), \ldots$.

Let $A = \{x|\forall y < x(V(x) \neq V(y))\}$. Then $\mathscr{C} = \{V(x)|x \in A\}$ and if x and y are two different members of A, $V(x) \neq V(y)$. Since $\mathscr{C}$ is infinite, A is infinite. We have $x \in A \Leftrightarrow \forall y < x \exists z \exists s \forall t$

$$[t \geqq s \Rightarrow z \in (V^t(x) - V^t(y)) \cup (V^t(y) - V^t(x))]$$

and it follows that $A \in \Sigma_2$. By the equivalence $A \in \Sigma_2 \Leftrightarrow A$ is *RE* in $0'$, $A = A_1 \cup A_2$, where A_1 and A_2 are disjoint infinite Σ_2 sets.

Let $\mathscr{D}_1 = \{V(x)|x \in A_1\}$, $\mathscr{D}_2 = \{V(x)|x \in A_2\}$. Then $\mathscr{D}_1$ and $\mathscr{D}_2$ are infinite, $\mathscr{C} = \mathscr{D}_1 \cup \mathscr{D}_2$ and $\mathscr{D}_1 \cap \mathscr{D}_2 = \emptyset$. By Theorem 1, $\mathscr{D}_1 \cup \{N\}$ and

$\mathscr{D}_2 \cup \{N\}$ are RE classes.

Now if we take

$$\mathscr{D} = \mathscr{D}_1 \cup \{N\} \text{ if } N \notin \mathscr{D}_2$$

and

$$\mathscr{D} = \mathscr{D}_2 \text{ if } N \in \mathscr{D}_2,$$

$\mathscr{D}$ is an RE class such that $\mathscr{C} \cap \mathscr{D}$ and $\mathscr{C} - \mathscr{D}$ are infinite.

3. RE **classes which partition** N **into finite sets.** Suppose A is an infinite set whose members in increasing order are $a_0, a_1, \ldots$. We define sets $I_n(A)$ as follows:

$$I_0(A) = \{0, \ldots, a_0\}$$

and

$$I_{n+1}(A) = \{a_n + 1, \ldots, a_{n+1}\}.$$

We also define $\mathscr{I}(A) = \{I_n(A) | n \in N\}$.

THEOREM 3. *If A is infinite, then $\mathscr{I}(A)$ is an RE class* $\Leftrightarrow A \in \Pi_1$.

Proof. First suppose $\mathscr{I}(A) = \{V(x) | x \in N\}$ where $V(0), V(1), \ldots$ is an RE sequence. Let $V^s(x)$ be the finite set of numbers in $V(x)$ at the beginning of step s in some enumeration of $V(0), V(1), \ldots$. Since $\mathscr{I}(A)$ partitions N, there is a recursive function g with $x \in V(g(x))$. Now

$$x \in A \Leftrightarrow \forall s \forall y [y \in V^s(g(x)) \Rightarrow y \leqq x]$$

and so $A \in \Pi_1$.

Now suppose $A \in \Pi_1$. Then $\bar{A}$ is RE. We have: if $x \in I_n(A)$, then

$$y \in I_n(A) \Leftrightarrow \forall z \text{ with } x \leqq z < y \text{ or } y \leqq z < x\, [z \in \bar{A}].$$

It follows that there is an RE sequence $V(0), V(1), \ldots$ such that $V(x) = I_n(A)$ if $x \in I_n(A)$. Hence $\mathscr{I}(A) (= \{V(x) | x \in N\})$ is an RE class.

THEOREM 4. *If $A = \bar{M}$, where M is a maximal set, then $\mathscr{I}(A)$ has no* 1-*split.*

Proof. Suppose $\mathscr{D}$ is an RE subclass of $\mathscr{I}(A)$ such that both $\mathscr{D}$ and $\mathscr{I}(A) - \mathscr{D}$ are infinite. Let B = the union of the members of $\mathscr{D}$. Then B is an RE set such that both $B \cap A$ and $\bar{B} \cap A$ are infinite. Thus A is not cohesive, contradicting the maximality of $\bar{A}$.

THEOREM 5. *If $A = \bar{S}$, where S is a simple set which is not hypersimple, then $\mathscr{I}(A)$ has a* 1-*split but no* 2-*split.*

Proof. Since A is not hyperimmune, there is a recursive function f such that $f(n) \geqq a_n$ where $a_0, a_1, \ldots$ are the members of A in increasing order. It follows that there is an infinite recursive set H such that for all n, $I_n(H) \cap A \neq \emptyset$.

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For if we define

$$h_0 = f(0)$$

and

$$h_{n+1} = f(h_n + 1)$$

we have

$$a_0 \leqq h_0 \text{ and } h_n < h_n + 1 \leqq a_{h_n+1} \leqq f(h_n + 1) = h_{n+1}.$$

Now if we let $B = \{b_0, b_1, \ldots\}$ where $b_n =$ the first member of $I_{2n}(H)$, B is an infinite recursive set such that $B \cap I_n(A) = \emptyset$ for infinitely many n. Let $\mathscr{D}$ be the RE class consisting of all members of $\mathscr{I}(A)$ which intersect B. Then $\mathscr{D}$ and $\mathscr{I}(A) - \mathscr{D}$ are infinite and $\mathscr{I}(A)$ has a 1-split.

Suppose $\mathscr{I}(A)$ has a 2-split, so that $\mathscr{I}(A) = \mathscr{C}_1 \cup \mathscr{C}_2$ where $\mathscr{C}_1$, $\mathscr{C}_2$ are disjoint infinite RE classes. Then B_1, the union of the members of $\mathscr{C}_1$, and B_2, the union of the members of $\mathscr{C}_2$, are complementary infinite recursive sets, and the set

$$\{x | (x \in B_1 \wedge x + 1 \in B_2) \vee (x \in B_2 \wedge x + 1 \in B_1)\}$$

is an infinite recursive subset of A. Thus A is not immune, contradicting the simplicity of $\bar{A}$.

Remark. It is easily proved that if A is an infinite Π_1 set, then $\mathscr{I}(A)$ has no 2-split $\Leftrightarrow \bar{A}$ is simple.

THEOREM 6. *For each $n \geqq 2$ there is an infinite RE class $\mathscr{C}$, whose members form a partition of N into finite sets, such that $\mathscr{C}$ has an n-split but no $(n + 1)$-split.*

Proof. Let $A = \bar{S}$ where S is a simple set. Let $E_1, \ldots, E_n$ be disjoint infinite recursive sets with union N. For $1 \leqq i \leqq n$, suppose f_i is a one-one recursive function from N onto E_i. Then, since $\mathscr{I}(A)$ has no 2-split (as in Theorem 5), $\mathscr{C}_i = \{f_i(I_n(A)) | n \in N\}$ is an infinite RE class such that

(1) the members of $\mathscr{C}_i$ form a partition of E_i into finite sets;

(2) $\mathscr{C}_i$ has no 2-split.

Let $\mathscr{C} = \mathscr{C}_1 \cup \ldots \cup \mathscr{C}_n$. Then $\mathscr{C}$ is an infinite RE class, whose members form a partition of N into finite sets. $\mathscr{C}$ can be split into n disjoint infinite RE classes, namely $\mathscr{C}_1, \ldots, \mathscr{C}_n$. Suppose $\mathscr{C}$ can be split into $n + 1$ disjoint infinite RE classes $\mathscr{D}_1, \ldots, \mathscr{D}_{n+1}$. For $1 \leqq i \leqq n$ and $1 \leqq j \leqq n + 1$, $\mathscr{C}_i \cap \mathscr{D}_j = \{W | W \in \mathscr{D}_j \wedge W \cap E_i \neq \emptyset\}$ is an RE class.

Let x be the number of ordered pairs (i, j) with $1 \leqq i \leqq n$, $1 \leqq j \leqq n + 1$ such that $\mathscr{C}_i \cap \mathscr{D}_j$ is infinite. For each j, since $\mathscr{D}_j$ is infinite and

$$\mathscr{D}_j = (\mathscr{C}_1 \cap \mathscr{D}_j) \cup \ldots \cup (\mathscr{C}_n \cap \mathscr{D}_j),$$

$\mathscr{C}_i \cap \mathscr{D}_j$ is infinite for some i. Hence $x \geqq n + 1$. For each i, since $\mathscr{C}_i$ cannot be split into two disjoint infinite RE classes and

$$\mathscr{C}_i = (\mathscr{C}_i \cap \mathscr{D}_1) \cup \ldots \cup (\mathscr{C}_i \cap \mathscr{D}_{n+1}),$$

$\mathscr{C}_i \cap \mathscr{D}_j$ is infinite for at most one j. Hence $x \leqq n$.

We now have $n + 1 \leqq x \leqq n$, a contradiction.

4. RE classes of finite sets of bounded cardinality.

THEOREM 7. *If $\mathscr{C}$ is an infinite RE class of finite sets of bounded cardinality, then $\mathscr{C}$ has a 2-split (and therefore, $\mathscr{C}$ has an n-split for all $n \in N$).*

Proof. Let $m = \max \{x | \mathscr{C}$ has a set of cardinality $x\}$. If $\mathscr{D}$ is an RE class and F is a finite set, then $\mathscr{D} - \{F\}$ is an RE class; hence we can assume that $\mathscr{C}$ has infinitely many sets of cardinality m, and that $\emptyset \notin \mathscr{C}$.

Suppose $\mathscr{C} = \{A(x) | x \in N\}$ where $A(0), A(1), \ldots$ is an RE sequence. Enumerate $A(0), A(1), \ldots$ in such a way that at each step exactly one of the sets $A(x)$ acquires a new member. We give instructions for simultaneously enumerating RE sequences $B(0), B(1), \ldots$ and $C(0), C(1), \ldots$. We will then let $\mathscr{C}_0 = \{B(x) | x \in N\}$ and $\mathscr{C}_1 = \{C(x) | x \in N\}$.

Notation. $A'(x)$ will denote the set of numbers currently in $A(x)$ (at any stage of the construction). Similarly for $B'(x)$, $C'(x)$, $\mathscr{C}_0'$, $\mathscr{C}_1'$.

When $A(x)$ acquires its first member, $A(x)$ will be assigned a unique *follower* which will be one of the B's or one of the C's. When (and if) $A(x)$ acquires its second and subsequent members, the follower of $A(x)$ may be *released*. $A(x)$ will then be assigned a new follower unless $A'(x)$ now has m members.

$B(x)$ is said to be *covered* (at a given stage) if $\exists y[B'(x) \subseteq B'(y)$ and cardinality $B'(y) = m]$. The instruction "*reject* $B(x)$" (used only where $B(x)$ is covered) means: "if y is the smallest number such that $B'(x) \subseteq B'(y)$ and cardinality $B'(y) = m$, put all the numbers in $B'(y)$ into $B(x)$." *Covered* and *reject* for $C(x)$ are defined analogously.

Instructions for step s, s even.

Suppose $A(x)$ acquires a new member k at step s.

(i) (Taking care of $A(x)$).

Case 1. $A(x)$ has a follower which is not covered. Put k into the follower of $A(x)$.

Case 2. Otherwise. If $A(x)$ has a follower, reject and release it.

Sub-case (a). $A'(x)$ has m members. Do nothing.

Sub-case (b). $0 < \text{card } A'(x) < m$ and $A'(x) \in \mathscr{C}_0'$. Pick the smallest z such that $B'(z) = \emptyset$, put all the members of $A'(x)$ into $B(z)$, and make $B(z)$ the follower of $A(x)$.

Sub-case (c). $0 < \text{card } A'(x) < m$ and $A'(x) \notin \mathscr{C}_0' \cup \mathscr{C}_1'$ and $A'(x)$ is contained in a member of $\mathscr{C}_0'$ of cardinality m. Proceed as in (b).

Sub-case (d). Otherwise. Pick the smallest z such that $C'(z) = \emptyset$, put all the members of $A'(x)$ into $C(z)$, and make $C(z)$ the follower of $A(x)$.

(ii) (Increasing the cardinality of $\mathscr{C}_0$).

List the sets in $\mathscr{C}$ of cardinality m until a set $A \notin \mathscr{C}_0' \cup \mathscr{C}_1'$ is found.

Case 1. $C'(z) \subseteq A$ for some z such that $C'(z) \neq \emptyset$ and $C(z)$ is not covered. Pick the smallest y such that $C'(y) = \emptyset$ and put all the members of A into $C(y)$.

Case 2. Otherwise. Pick the smallest y such that $B'(y) = \emptyset$ and put all the members of A into $B(y)$.

In Case 1, repeat the operation. It is clear that the process of (ii) will end with an occurrence of Case 2.

Instructions for step s, s odd.

As above, except that in (ii) the roles of $\mathscr{C}_0$ and $\mathscr{C}_1$ are exchanged.

This completes the instructions.

That $\mathscr{C}_0$, $\mathscr{C}_1$ form a 2-split of C will follow from Lemmas 2, 3, 4, and 7 below.

LEMMA 1. *For each z, $B(z)$ has one and only one of the following "histories" (similarly for $C(z)$).*

(1) *$B(z)$ becomes a follower of $A(x)$, for some x, never to be released. In this case $B(z) = A(x)$.*

(2) *$B(z)$ becomes a follower of $A(x)$, for some x, but is later released. In this case* card $B(z) = m$.

(3) *$B(z)$ acquires its members by (ii). In this case* card $B(z) = m$ *and* $B(z) \in \mathscr{C}$.

Proof. That (1), (2) and (3) are mutually exclusive follows from the fact that only empty sets are chosen to become followers or to be used in (ii).

That (1), (2) and (3) are exhaustive follows from the fact that $\mathscr{C}$ contains infinitely many sets of cardinality m.

LEMMA 2. *$\mathscr{C}_0$, $\mathscr{C}_1$ are infinite.*

Proof. That $\mathscr{C}_0$ is infinite is ensured by (ii) at even steps. In fact, $\mathscr{C}_0$ will contain infinitely many sets of cardinality m. Note that by Lemma 1, no member of $\mathscr{C}_0$ has more than m members. Thus if we ever have $A \in \mathscr{C}_0'$, card $A = m$ then $A \in \mathscr{C}_0$.

Similarly for $\mathscr{C}_1$ at odd steps.

LEMMA 3. $\mathscr{C}_0 \cup \mathscr{C}_1 \subseteq \mathscr{C}$.

Proof. We prove that $B(z) \in \mathscr{C}$ for all z (the argument for $C(z)$ is similar). First suppose $B(z)$ has fewer than m members. Then (1) of Lemma 1 occurs and $B(z) \in \mathscr{C}$. Now suppose $B(z)$ has m members. We may assume that when $B(z)$ acquires its m-th member, this is the first time in the construction that this set has appeared in $\mathscr{C}_0$. In cases (1) and (3) of Lemma 1, $B(z) \in \mathscr{C}$. But by assumption, case (2) cannot occur.

LEMMA 4. $\mathscr{C} \subseteq \mathscr{C}_0 \cup \mathscr{C}_1$.

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Proof. We prove that $A(x) \in \mathscr{C}_0 \cup \mathscr{C}_1$ for all x. If $A(x)$ has m members, this is taken care of by (ii). Suppose then that $A(x)$ has fewer than m members. When $A(x)$ acquires its first member, Sub-case (b), (c), or (d) will apply. Thereafter $A(x)$ will always have a follower. The final follower of $A(x)$ will have the same members as $A(x)$.

LEMMA 5. *For all z, at no stage of the construction do we have the situation:* $0 <$ card $B'(z)$ *and* $B(z)$ *is not covered.*

Proof. It may be seen on examining the instructions that there is no possible way in which this situation can arise for the first time.

LEMMA 6. *For all y and z, at no stage of the construction do we have the situation:* $0 <$ card $C'(y)$, $C(y)$ *is not covered*, card $B(z) = m$ *and* $C'(y) \subseteq B'(z)$.

Proof. As above, the proof is a matter of verifying that the instructions are such that there is no way in which this situation can arise for the first time. In particular this is where Case 1 of the procedure in (ii) comes in (to ensure that in enlarging $\mathscr{C}_0$ we do not obtain the situation of the Lemma). We note also that some of the possibilities are eliminated using Lemma 5.

LEMMA 7. $\mathscr{C}_0 \cap \mathscr{C}_1 = \emptyset$.

Proof. Suppose that $\mathscr{C}_0 \cap \mathscr{C}_1 \neq \emptyset$. Since all the sets are finite, $\mathscr{C}_0' \cap \mathscr{C}_1' \neq \emptyset$ at some finite stage of the construction. Consider the first time that we have $\mathscr{C}_0' \cap \mathscr{C}_1' \neq \emptyset$, and suppose that at this time $B'(z) = C'(y)$. This can only have arisen by an occurrence of Case 1 of (i), and by Lemma 5, $C(y)$ (and not $B(z)$) is the follower of $A(x)$ which acquires the number k making $B'(z) = C'(y)$. Hence at the previous stage $0 <$ card $C'(y)$, $C(y)$ is not covered and $C'(y) \subseteq B'(z)$. By Lemma 5 there is t with card $B(t) = m$, $B'(z) \subseteq B'(t)$. This contradicts Lemma 6.

5. *k-RE* classes.

THEOREM 8. *If an infinite RE class* $\mathscr{C}$ *is k-RE for some positive integer k, then* $\mathscr{C}$ *has a* 1-*split.*

Proof. Let $V(0)$, $V(1)$, ... be a k-enumeration of $\mathscr{C}$. Let $A_1, \ldots, A_{k+1}$ be disjoint infinite recursive sets with union N. Then for $1 \leqq i \leqq k+1$, $\mathscr{C}_i = \{W | W = V(x) \text{ for some } x \in A_i\}$ is an RE class. We claim that one of the classes $\mathscr{C}_i$ yields a 1-split of $\mathscr{C}$. $\mathscr{C}_i$ is infinite since A_i is infinite and $V(0)$, $V(1)$, ... is a k-enumeration.

It remains only to show that $\mathscr{C} - \mathscr{C}_i$ is infinite for some i. Now

$$\mathscr{C} = (\mathscr{C} - (\mathscr{C}_1 \cap \ldots \cap \mathscr{C}_{k+1})) \cup (\mathscr{C}_1 \cap \ldots \cap \mathscr{C}_{k+1})$$
$$= (\mathscr{C} - \mathscr{C}_1) \cup \ldots \cup (\mathscr{C} - \mathscr{C}_{k+1}) \cup (\mathscr{C}_1 \cap \ldots \cap \mathscr{C}_{k+1}).$$

Since $V(0)$, $V(1)$, ... is a k-enumeration, $\mathscr{C}_1 \cap \ldots \cap \mathscr{C}_{k+1} = \emptyset$. Hence $\mathscr{C} - \mathscr{C}_i$ is infinite for some i as required.

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THEOREM 9. *If $\mathscr{C}$ is an infinite 2-RE class of disjoint sets, then $\mathscr{C}$ has a 2-split.*

Proof. We may assume that $\mathscr{C}$ is not 1-*RE*. Thus let $V(0)$, $V(1)$, ... be a 2-enumeration of $\mathscr{C}$ in which infinitely many sets are repeated. Then it is clear that we can find numbers

$$x_0 < x_1 < y_0 < y_1 < x_2 < x_3 < y_2 < y_3 < \ldots$$

such that for all n

$$V(x_{2n}) = V(x_{2n+1}) \text{ and } V(y_{2n}) = V(y_{2n+1})$$

(using the fact that if $V(n) \cap V(m) \neq \emptyset$ then $V(n) = V(m)$).
Let

$$\mathscr{C}_1 = \{V(x_n)|n \in N\} \text{ and}$$
$$\mathscr{C}_2 = \{V(y_n)|n \in N\} \cup \{V(x)|x \neq \text{all the } x_n \text{ and all the } y_n\},$$

and $\mathscr{C}_1$, $\mathscr{C}_2$ form a 2-split of $\mathscr{C}$.

By Theorem 9, if $\mathscr{C}$ is an infinite 2-*RE* class of disjoint sets, then $\mathscr{C}$ has a 2-split. Now $\mathscr{C}$ is a class of disjoint sets if and only if

$$\forall A \in \mathscr{C} \forall x \in A \forall B \in \mathscr{C}[A \neq B \Rightarrow x \notin B].$$

Our example of a 2-*RE* class with no 2-split will satisfy the weaker condition

$$\forall A \in \mathscr{C} \exists x \in A \forall B \in \mathscr{C}[A \neq B \Rightarrow x \notin B].$$

THEOREM 10. *There exists an infinite RE class $\mathscr{C}$ of finite sets, satisfying*

$$\forall A \in \mathscr{C} \exists x \in A \forall B \in \mathscr{C}[A \neq B \Rightarrow x \notin B].$$

which is 2-RE but has no 2-split.

Proof. Let $R: N \rightarrow \{1, 2\} \times N \times N \times N$ be a recursive function such that if we let

$$P_n(x) = \{y|(1, n, x, y) \in \text{range } R\}$$
$$Q_n(x) = \{y|(2, n, x, y) \in \text{range } R\}$$
$$\mathscr{P}_n = \{P_n(x)|x \in N\}$$
$$\mathscr{Q}_n = \{Q_n(x)|x \in N\}$$

then $(\mathscr{P}_0, \mathscr{Q}_0)$, $(\mathscr{P}_1, \mathscr{Q}_1)$, ... gives all ordered pairs of non-empty *RE* classes.
Let

$$P_n^s(x) = \{y|(1, n, x, y) \in \{R(0), \ldots, R(s)\}\}$$
$$Q_n^s(x) = \{y|(2, n, x, y) \in \{R(0), \ldots, R(s)\}\}$$
$$\mathscr{P}_n^s = \{P_n^s(x)|x \in N\}$$
$$\mathscr{Q}_n^s = \{Q_n^s(x)|x \in N\}$$

then $\mathscr{P}_n^s$, $\mathscr{Q}_n^s$ are the *step s approximations* to $\mathscr{P}_n$, $\mathscr{Q}_n$ respectively.

Condition I_n is the requirement that $(\mathscr{P}_n, \mathscr{Q}_n)$ not yield a 2-split of $\mathscr{C}$.

For all $x, s \in N$ we will define $V^s(x) \in \text{Sub } N$, $G^s(x) \in \text{Sub } N$ (the set of *associates of* I_x *at step* s) and $H^s(x) \in \text{Sub } N$ (the set of *active associates of* I_x *at step* s). $H^s(x)$ will have 0, 2, 3, or 4 members. The definition is by induction on s. Let $V^0(x) = \{2x\}$, $G^0(x) = \emptyset$ and $H^0(x) = \emptyset$. For each s we will have $V^s(x) = \{2x\}$, $G^s(x) = \emptyset$ and $H^s(x) = \emptyset$ for almost all x. It will be clear that $V^s(x)$, $G^s(x)$ and $H^s(x)$ can be found effectively from x and s.

Before giving the details of the induction step we attempt to motivate the construction. $\mathscr{C}$ will be $\{V(x)|x \in N\}$ where $V(x) = \{y|(\exists s)(y \in V^s(x))\}$. First note that if we can ensure that $\mathscr{C}$ consists of finite sets, and maintain

$$\forall x, y, z[x < y < z \Rightarrow \neg\, (V^s(x) = V^s(y) = V^s(z))] \text{ for all } s,$$

then $\mathscr{C}$ will be 2-*RE*. The basic strategy for satisfying I_n is as follows. Suppose that at some step $V^s(i) \in \mathscr{P}_n{}^s$ and $V^s(j) \in \mathscr{Q}_n{}^s$. Then we "combine" $V(i)$ and $V(j)$ by putting every number in $V^s(i)$ into $V(j)$ and vice versa. If we can ensure that the only set in $\mathscr{C}$ which contains either $V^s(i)$ or $V^s(j)$ is $V^s(i) \cup V^s(j)$, and $\mathscr{P}_n \cup \mathscr{Q}_n \subseteq \mathscr{C}$, then $\mathscr{P}_n \cap \mathscr{Q}_n \neq \emptyset$. The priority method, together with the "separating" of sets previously combined, is used to resolve conflicts among the conditions.

For the induction step, suppose $\pi_1(s) = n$. First we define $H^{s+1}(n)$ and $V^{s+1}(x)$ for all x. There are three cases. Case 1 has sub-cases (1.1) and (1.2); (1.2) has sub-sub-cases (1.21), (1.22), (1.23), and (1.24). Case 2 has sub-cases (2.1) and (2.2).

Case 1. $H^s(n) = \emptyset$.

(1.1) $\forall x, y[V^s(x) \neq V^s(y) \wedge V^s(x) \in \mathscr{P}_n{}^s \wedge V^s(y) \in \mathscr{Q}_n{}^s. \Rightarrow . \{x, y\} \cap G^s(m) \neq \emptyset$ for some $m < n]$. Then

$$H^{s+1}(n) = H^s(n)$$
$$V^{s+1}(x) = V^s(x) \text{ for all } x.$$

(1.2) Otherwise. Choose i and j such that $V^s(i) \neq V^s(j)$, $V^s(i) \in \mathscr{P}_n{}^s$, $V^s(j) \in \mathscr{Q}_n{}^s$ and $\{i, j\} \cap G^s(m) = \emptyset$ for all $m < n$.

(1.21) $\forall x[x \neq i \Rightarrow V^s(x) \neq V^s(i)] \wedge \forall x[x \neq j \Rightarrow V^s(x) \neq V^s(j)]$. Then

$$H^{s+1}(n) = \{i, j\}$$
$$V^{s+1}(x) = V^s(x) \text{ for all } x \notin \{i, j\}$$
$$V^{s+1}(i) = V^{s+1}(j) = V^s(i) \cup V^s(j)$$

(*combining* $V(i)$ with $V(j)$).

(1.22) $\exists x[x \neq i \wedge V^s(x) = V^s(i)] \wedge \forall x[x \neq j \Rightarrow V^s(x) \neq V^s(j)]$.

Choose $i_1 \neq i$ such that $V^s(i_1) = V^s(i)$. Then

$$H^{s+1}(n) = \{i, i_1, j\}$$
$$V^{s+1}(x) = V^s(x) \text{ for all } x \notin \{i, i_1\}$$
$$V^{s+1}(i) = V^s(i) \cup \{a\}$$
$$V^{s+1}(i_1) = V^s(i_1) \cup \{b\}$$

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where a and b are two distinct odd numbers not in any of the sets $V^s(x)$ (*separating* $V(i)$ from $V(i_1)$).

(1.23) $\forall x[x \neq i \Rightarrow V^s(x) \neq V^s(i)] \wedge \exists x[x \neq j \wedge V^s(x) = V^s(j)]$.

Similarly.

(1.24) $\exists x[x \neq i \wedge V^s(x) = V^s(i)] \wedge \exists x[x \neq j \wedge V^s(x) = V^s(j)]$.

Choose $i_1 \neq i$ and $j_1 \neq j$ such that $V^s(i_1) = V^s(i)$ and $V^s(j_1) = V^s(j)$. Then

$$H^{s+1}(n) = \{i, i_1, j, j_1\}$$
$$V^{s+1}(x) = V^s(x) \text{ for all } x \notin \{i, i_1, j, j_1\}$$
$$V^{s+1}(i) = V^s(i) \cup \{a\}$$
$$V^{s+1}(i_1) = V^s(i_1) \cup \{b\}$$
$$V^{s+1}(j) = V^s(j) \cup \{c\}$$
$$V^{s+1}(j_1) = V^s(j_1) \cup \{d\}.$$

where a, b, c, d are four distinct odd numbers not in any of the sets $V^s(x)$ (*separating* $V(i)$ from $V(i_1)$, and $V(j)$ from $V(j_1)$).

Case 2. $H^s(n)$ has 3 or 4 members.

(2.1) $\forall x, y[x \neq y \wedge \forall z(z \neq x \Rightarrow V^s(z) \neq V^s(x)) \wedge \forall z(z \neq y \Rightarrow V^s(z) \neq V^s(y)) \wedge V^s(x) \in \mathscr{P}_n{}^s \wedge V^s(y) \in \mathscr{Q}_n{}^s \cdot \Rightarrow \cdot \{x, y\} \nsubseteq H^s(n)]$. As in (1.1).

(2.2) Otherwise. Choose i and j such that $i \neq j$, $\forall z(z \neq i \Rightarrow V^s(z) \neq V^s(i))$, $\forall z(z \neq j \Rightarrow V^s(z) \neq V^s(j))$, $V^s(i) \in \mathscr{P}_n{}^s$, $V^s(j) \in \mathscr{Q}_n{}^s$ and $\{i, j\} \subseteq H^s(n)$. As in (1.21).

Case 3. $H^s(n)$ has 2 members.

As in (1.1).

Now we define $H^{s+1}(m)$ for $m \neq n$ and $G^{s+1}(m)$ for all m.

$$H^{s+1}(m) = \emptyset \text{ if } m \neq n \text{ and } H^{s+1}(n) \cap H^s(m) \neq \emptyset$$
$$H^{s+1}(m) = H^s(m) \text{ if } m \neq n \text{ and } H^{s+1}(n) \cap H^s(m) = \emptyset$$
$$G^{s+1}(n) = G^s(n) \cup H^{s+1}(n)$$
$$G^{s+1}(m) = G^s(m) - H^{s+1}(n) \text{ for } m \neq n.$$

This completes the definition of $V^s(x)$, $G^s(x)$ and $H^s(x)$. Since $V^s(x)$ can be found effectively from x and s, if we define $V(x) = \{y | (\exists s)(y \in V^s(x))\}$ then $V(0), V(1), \ldots$ is an *RE* sequence and $\mathscr{C} = \{V(x) | x \in N\}$ is an *RE* class.

We now prove nine lemmas. Lemmas 5 through 9 show that $\mathscr{C}$ satisfies the conditions of the Theorem.

LEMMA 1. *For all* s,
(a) $\forall x, y, z[x < y < z \Rightarrow \neg (V^s(x) = V^s(y) = V^s(z))]$
(b) $\forall x \exists y \forall z[V^s(x) \neq V^s(z) \Rightarrow y \in V^s(x) - V^s(z)]$.

Proof. By (simultaneous) induction on s. (a) and (b) are clear for $s = 0$, since $V^0(x) = \{2x\}$. For the induction step it suffices to verify that (a) $\wedge$ (*b*) is preserved by the operations of combining (applied to $V(x)$, $V(y)$ with

$x \neq y, \forall z(z \neq x \Rightarrow V^s(z) \neq V^s(x))$ and $\forall z(z \neq y \Rightarrow V^s(z) \neq V^s(y)))$ and separating (applied to $V(x)$, $V(y)$ with $x \neq y$ and $V^s(x) = V^s(y)$).

LEMMA 2. (a) $\forall s, m[H^s(m) \subseteq G^s(m)]$
(b) $\forall s, m, k[m \neq k \Rightarrow G^s(m) \cap G^s(k) = \emptyset]$.

Proof. (a) By induction on s. $H^0(m) = G^0(m) = \emptyset$. Suppose that $H^s(m) \subseteq G^s(m)$ for all m and we show that $H^{s+1}(m) \subseteq G^{s+1}(m)$ for all m. Assume $\pi_1(s) = n$. Since $G^{s+1}(n) = G^s(n) \cup H^{s+1}(n)$, $H^{s+1}(n) \subseteq G^{s+1}(n)$. If $m \neq n$ and $H^{s+1}(n) \cap H^s(m) \neq \emptyset$, then $H^{s+1}(m) = \emptyset \subseteq G^{s+1}(m)$. If $m \neq n$ and $H^{s+1}(n) \cap H^s(m) = \emptyset$, then $G^{s+1}(m) = G^s(m) - H^{s+1}(n) \supseteq H^s(m)$ (using the induction hypothesis) $= H^{s+1}(m)$.

(b) By induction on s. $G^0(m) = \emptyset$ for all m. Suppose that $G^s(m) \cap G^s(k) = \emptyset$ whenever $m \neq k$ and we show that $G^{s+1}(m) \cap G^{s+1}(k) = \emptyset$ whenever $m \neq k$. Assume $\pi_1(s) = n$. If $m \neq n$, $k \neq n$ then $G^{s+1}(m) = G^s(m) - H^{s+1}(n)$, $G^{s+1}(k) = G^s(k) - H^{s+1}(n)$ and $G^{s+1}(m) \cap G^{s+1}(k) = \emptyset$ by the induction hypothesis. It remains to show that if $m \neq n$, then $G^{s+1}(m) \cap G^{s+1}(n) = \emptyset$. We have $G^{s+1}(m) = G^s(m) - H^{s+1}(n)$ and $G^{s+1}(n) = G^s(n) \cup H^{s+1}(n)$. $G^s(m) \cap G^s(n) = \emptyset$ by hypothesis, and of course $(G^s(m) - H^{s+1}(n)) \cap H^{s+1}(n) = \emptyset$.

LEMMA 3. $\forall s, x, y[(x \neq y \wedge V^s(x) = V^s(y) \Rightarrow \exists n(H^s(n) = \{x, y\})]$.

Proof. Suppose $x \neq y$ and $V^s(x) = V^s(y)$. Let t be such that $t \leqq s$, $V^u(x) = V^u(y)$ for all u with $t \leqq u \leqq s$, and $V^{t-1}(x) \neq V^{t-1}(y)$. There must be such a t since $V^0(x) = \{2x\} \neq \{2y\} = V^0(y)$. Let $\pi_1(t-1) = n$. Since $V^{t-1}(x) \neq V^{t-1}(y)$ but $V^t(x) = V^t(y)$, only (1.21) or (2.2) can occur at step $t-1$, and by Lemma 1 (a) $\{x, y\}$ can only be $H^t(n)$. We claim that $H^s(n) = \{x, y\}$. Suppose otherwise. Then there is u, $t < u \leqq s$, such that $H^v(n) = \{x, y\}$ for all v with $t \leqq v < u$, and $H^u(n) \neq \{x, y\}$. Let $\pi_1(u-1) = k$. Since $H^{u-1}(n) = \{x, y\}$ and $H^u(n) \neq \{x, y\}$, $k \neq n$ (see Case 3) and $H^u(k) \cap \{x, y\} = H^u(k) \cap H^{u-1}(n) \neq \emptyset$. By Lemma 2 (both parts) $H^{u-1}(k) \cap H^{u-1}(n) = \emptyset$ and so $H^u(k) - H^{u-1}(k) \neq \emptyset$. Hence (1.2) occurs at step $u-1$. Suppose $x \in H^u(k)$ (the case $y \in H^u(k)$ is similar). It is then clear by examining (1.2) that either $\forall z(z \neq x \Rightarrow V^{u-1}(z) \neq V^{u-1}(x))$ or $\forall z(z \neq x \Rightarrow V^u(z) \neq V^u(x))$. But $V^{u-1}(x) = V^{u-1}(y)$ and $V^u(x) = V^u(y)$. This is a contradiction, and so $H^s(n) = \{x, y\}$ as claimed.

LEMMA 4. $\forall n \exists t \forall s[s \geqq t \Rightarrow (H^s(n) = H^t(n) \wedge G^s(n) = G^t(n))]$.

Proof. By induction on n. Suppose that for all $k < n$, $\exists t \forall s\ [s \geqq t \Rightarrow (H^s(k) = H^t(k) \wedge G^s(k) = G^t(k))]$. Then there is u such that for all $k < n$, $s \geqq u \Rightarrow (H^s(k) = H^u(k) \wedge G^s(k) = G^u(k))$. We claim that if $s \geqq u$ and $\pi_1(s) \neq n$, then $H^{s+1}(n) = H^s(n)$ and $G^{s+1}(n) = G^s(n)$.

Let $\pi_1(s) = k$. If either $H^{s+1}(n) \neq H^s(n)$ or $G^{s+1}(n) \neq G^s(n)$, then (using Lemma 2(a)) $H^{s+1}(k) \cap G^s(n) \neq \emptyset$ and (again using Lemma 2, both parts) $H^{s+1}(k) - H^s(k) \neq \emptyset$. It follows that (1.2) occurs at step s, and (using

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Lemmas 3 and 2) that $\{i, j\} \cap G^s(n) \neq \emptyset$, where $\{i, j\} \cap G^s(m) = \emptyset$ for all $m < k$. Hence $n > k$ and $H^{s+1}(k) \neq H^s(k)$, contradicting $s \geqq u$. The claim is proved.

Since $G^{s+1}(n) = G^s(n) \cup H^{s+1}(n)$ when $\pi_1(s) = n$, Lemma 2 (a) implies that if $\pi_1(s) = n$ and $H^{s+1}(n) = H^s(n)$, then $G^{s+1}(n) = G^s(n)$. Thus it is enough to show that there is $t \geqq u$ such that if $s \geqq t$ and $\pi_1(s) = n$, then $H^{s+1}(n) = H^s(n)$.

If $u \leqq s_1 < s_2$, $\pi_1(s_1) = \pi_1(s_2) = n$, $\pi_1(s) \neq n$ for all s with $s_1 < s < s_2$, Case x_1 occurs at step s_1 and Case x_2 occurs at step s_2, then $x_1 \leqq x_2 \leqq 3$ (by inspection of the cases). Thus there is $t \geqq u$ such that at all steps $s \geqq t$ with $\pi_1(s) = n$, the same Case (1, 2, or 3) occurs. It easily follows that t has the required property.

LEMMA 5. *$\mathscr{C}$ is a class of finite sets.*

Proof. We prove that for all x, $V(x)$ is finite. Note that $V^s(x) \subseteq V^{s+1}(x)$ for all x; also that if $V^s(x) \neq V^{s+1}(x)$ then $x \in H^{s+1}(n)$ and $H^{s+1}(n) \neq H^s(n)$, where $n = \pi_1(s)$. Hence if x does not belong to any $H^s(n)$, $V(x) = \{2x\}$.

Suppose $x \in G^s(n) - G^{s+1}(n)$, where $\pi_1(s) = k$. Then $n \neq k$ and since $G^{s+1}(n) = G^s(n) - H^{s+1}(k)$, $x \in H^{s+1}(k) \cap G^s(n)$. We can show just as in the proof of the "claim" in Lemma 4 that $n > k$. Thus $x \in G^{s+1}(k)$ with $n > k$.

It follows that if $V(x) \neq \{2x\}$, then there are t, n such that $x \in G^s(n)$ for all $s \geqq t$. Thus if $s \geqq t$ and $V^s(x) \neq V^{s+1}(x)$, $\pi_1(s) = n$, $x \in H^{s+1}(n)$ and $H^{s+1}(n) \neq H^s(n)$. By Lemma 4 there is u such that for all $s \geqq u$, $H^{s+1}(n) = H^s(n)$. Now if $s \geqq \max\{t, u\}$, $V^s(x) = V^{s+1}(x)$.

LEMMA 6. $\forall A \in \mathscr{C} \exists y \in A \forall B \in \mathscr{C}[A \neq B \Rightarrow y \notin B]$.

Proof. Let $A = V(x)$. Choose t (by Lemma 5) so that $V(x) = V^t(x)$, and choose y (by Lemma 1 (b)) so that $\forall z[V^t(x) \neq V^t(z) \Rightarrow y \in V^t(x) - V^t(z)]$. There are two cases.

(i) $V^t(x_1) = V^t(x)$ for some $x_1 \neq x$.

If $V^s(x_1) = V^s(x)$ but $V^{s+1}(x_1) \neq V^{s+1}(x)$, $V(x)$ must be separated from $V(x_1)$ at step s (inspection of cases, using Lemma 1 (a) in (1.22), (1.23), and (1.24)); but then $V^{s+1}(x) \neq V^s(x)$. It follows by the choice of t that $V^s(x_1) = V^s(x)$ for all $s \geqq t$ (and hence that $V(x_1) = V(x)$).

By Lemma 1 (a), $y \in V^t(z) \Leftrightarrow z \in \{x, x_1\}$. It suffices to prove that for all $s \geqq t$, $y \in V^s(z) \Leftrightarrow z \in \{x, x_1\}$ (for if $B \in \mathscr{C}$ with $A \neq B$, then $B = V(z)$ with $z \notin \{x, x_1\}$). Suppose otherwise, then there is $s \geqq t$ such that $y \in V^s(z) \Leftrightarrow z \in \{x, x_1\}$ but $y \in V^{s+1}(z)$ for some $z \notin \{x, x_1\}$. This can happen only if at step s either $V(x)$ or $V(x_1)$ is involved in a combining operation. But this is impossible, since $V^s(x_1) = V^s(x)$.

(ii) $\forall z(z \neq x \Rightarrow V^t(z) \neq V^t(x))$.

It suffices to prove that for all $s \geqq t$, $y \in V^s(z) \Leftrightarrow z = x$. Supposing otherwise, we find as above that $V(x)$ is involved in a combining operation at some step $s \geqq t$. But this would imply (by Lemma 1 (b)) that $V^{s+1}(x) \neq V^s(x)$, contradicting the choice of t.

LEMMA 7. *$\mathscr{C}$ is 2-RE, in fact $V(0), V(1), \ldots$ is a 2-enumeration of $\mathscr{C}$.*

Proof. Immediate from Lemma 1 (a) and Lemma 5.

LEMMA 8. *$\mathscr{C}$ is infinite.*

Proof. Immediate from Lemma 7.

LEMMA 9. *$\mathscr{C}$ has no 2-split.*

Proof. Suppose for reductio that $(\mathscr{P}_n, \mathscr{Q}_n)$ yields a 2-split of $\mathscr{C}$. Then $\mathscr{C} = \mathscr{P}_n \cup \mathscr{Q}_n$, $\mathscr{P}_n$ and $\mathscr{Q}_n$ are infinite, and $\mathscr{P}_n \cap \mathscr{Q}_n = \emptyset$.

Choose t as small as possible so that $H^{s+1}(n) = H^s(n)$ for all $s \geqq t$ (see Lemma 4).

Remark. If $V^{s+1}(x) \neq V^s(x)$ then $x \in H^{s+1}(k)$ and $H^{s+1}(k) \neq H^s(k)$, where $\pi_1(s) = k$; and so by Lemma 2, if $x \in H^t(n)$, $V^{s+1}(x) = V^s(x)$ for all $s \geqq t$.

There are three cases.

(i) $H^t(n) = \emptyset$. By Lemma 4, the set

$$G = \{g | g \in G^s(m) \text{ for some } m < n \text{ and some } s\}$$

is finite. Since $\mathscr{P}_n$ is infinite and $\mathrm{P}_n \subseteq \mathscr{C}$, there is $i \notin G$ with $V(i) \in \mathscr{P}_n$. Similarly there is $j \notin G$ with $V(j) \in \mathscr{Q}_n$. Since $\mathscr{P}_n \cap \mathscr{Q}_n = \emptyset$, $V(i) \neq V(j)$. By Lemma 5 choose s with $\pi_1(s) = n$ sufficiently large that $s \geqq t$, $V^s(i) = V(i)$, $V^s(j) = V(j)$, $V(i) \in \mathscr{P}_n{}^s$ and $V(j) \in \mathscr{Q}_n{}^s$. Clearly (1.2) occurs at step s and $H^{s+1}(n) \neq H^s(n)$, a contradiction.

(ii) $H^t(n) = \{i, j\}$, $\pi_1(t) = n$ and (1.21) or (2.2) occurs at step $t - 1$. Thus $V^t(i) = V^t(j) = V^{t-1}(i) \cup V^{t-1}(j)$. By the remark above, $V(i) = V^t(i)$ and $V(j) = V^t(j)$. By Lemma 1 (b) there are y_1, y_2 such that $\forall z(z \neq i \Rightarrow y_1 \in V^{t-1}(i) - V^{t-1}(z))$ and $\forall z(z \neq j \Rightarrow y_2 \in V^{t-1}(j) - V^{t-1}(z))$. Since $V(i) = V^t(i)$ and $\mathrm{V}(j) = V^t(j)$, neither $V(i)$ nor $V(j)$ is involved in a combining operation from step t on. Hence, $\forall z(z \notin \{i, j\} \Rightarrow y_1, y_2 \notin V(z))$. Thus the only set in $\mathscr{C}$ which contains either $V^{t-1}(i)$ or $V^{t-1}(j)$ is $V(i)$ $(= V(j))$. Now $V^{t-1}(i) \in \mathscr{P}_n{}^{t-1}$ and $V^{t-1}(j) \in \mathscr{Q}_n{}^{t-1}$ and so $\mathscr{P}_n \cup \mathscr{Q}_n \subseteq \mathscr{C}$ gives $V(i) \in \mathscr{P}_n \cap \mathscr{Q}_n$, contradicting $\mathscr{P}_n \cap \mathscr{Q}_n = \emptyset$.

(iii) $H^t(n) = \{i, i_1, j\}$ (or $\{i, j, j_1\}$ or $\{i, i_1, j, j_1\}$), $\pi_1(t) = n$ and (1.22) (or (1.23), or (1.24)) occurs at step $t - 1$. We deal only with (1.22); the arguments for (1.23) and (1.24) are similar. $V(i)$ and $V(i_1)$ are separated at step $t - 1$. By Lemma 1 (both parts) there is y such that $\forall z(y \in V^{t-1}(z) \Leftrightarrow z \in \{i, i_1\})$. By the remark above, $V^t(i) = V(i)$ and $V^t(i_1) = V(i_1)$, and so neither $V(i)$ nor $V(i_1)$ is involved in a combining operation from step t on. Hence, $\forall z$ $(y \in V(z) \Leftrightarrow z \in \{i, i_1\})$ and the only sets in $\mathscr{C}$ which contain $V^{t-1}(i)$ $(= V^{t-1}(i_1))$ are $V(i)$ and $V(i_1)$. Now $V^{t-1}(i) \in \mathscr{P}_n{}^{t-1}$ and so, since $\mathscr{P}_n \subseteq \mathscr{C}$, either $V(i) \in \mathscr{P}_n$ or $V(i_1) \in \mathscr{P}_n$. A similar argument shows that $V^t(j) =$

$V(j) \in \mathscr{Q}_n$. Also, for all $s \geqq t$ and all $x \in \{i, i_1, j\}$, we have $\forall z(z \neq x \Rightarrow V^s(z) \neq V^s(x))$. By Lemma 5 choose s with $\pi_1(s) = n$ sufficiently large that $s \geqq t$, $V(j) \in \mathscr{Q}_n{}^s$ and either $V(i) \in \mathscr{P}_n{}^s$ or $V(i_1) \in \mathscr{P}_n{}^s$. Clearly (2.2) occurs at step s and $H^{s+1}(n) \neq H^s(n)$, a contradiction.

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The University of Western Ontario,
London, Ontario