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SOLVABILITY OF SINGULAR SECOND ORDER m-POINT BOUNDARY VALUE PROBLEMS OF DIRICHLET TYPE

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Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying the Carathéodory conditions and $t(1-t)e(t) \in L^1(0,1)$. Let $a_i \in \mathbb{R}$ and $\xi_i \in (0,1)$ for $i = 1, \ldots, m-2$ where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$. In this paper we study the existence of $C[0,1]$ solutions for the m-point boundary value problem

$$x'' = f\big(t, x(t), x'(t)\big) + e(t), \quad 0 < t < 1$$
$$x(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i).$$

The proof of our main result is based on the Leray-Schauder continuation theorem.

1. INTRODUCTION

In [4], Gupta, Ntouyas and Tsamatos considered the problem of proving the existence of a $C^1[0,1]$ solution of the m-point boundary value problem

$$x''(t) = f_1\big(t, x(t), x'(t)\big) + e_1(t), \quad 0 < t < 1 \tag{1.1}$$

$$x(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i) \tag{1.2}$$

where $\xi_i \in (0,1)$ for $i = 1, 2, \ldots, m-2$ satisfies $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, the $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, m-2$, have the same sign, $e_1 \in L^1[0,1]$, and $f_1 : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Carathéodory's conditions as well as a growth condition of the form

$$\big|f_1(t,u,v)\big| \leqslant p_1(t)|u| + q_1(t)|v| + r_1(t), \tag{1.3}$$

where $p_1,\ q_1,\ r_1 \in L^1[0,1]$.

This, of course, raises the following natural question: What would happen if f_1 and e_1 have a higher order singularity at $t = 0$ and $t = 1$? The results of Gupta, Ntouyas and Tsamatos do not apply to the case $t(1-t)e(t) \in L^1[0,1]$.

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The purpose of this paper is to investigate the existence of $C[0,1]$ solutions for the second order m-point boundary value problem

$$x''(t) = f\big(t, x(t), x'(t)\big) + e(t), \quad 0 < t < 1 \tag{1.4}$$

$$x(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i) \tag{1.5}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the *Carathéodory conditions* (that is, for each $(x,y) \in \mathbb{R}^2$, the function $f(\cdot, x, y)$ is measurable on $[0,1]$ and for almost every $t \in [0,1]$, the function $f(t,\cdot,\cdot)$ is continuous on $\mathbb{R}^2$). We make the following additional assumptions:

(H0) $a_i \in \mathbb{R}$ and $\xi_i \in (0,1)$ for $i = 1, 2, \dots, m-2$ where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and
$$\sum_{i=1}^{m-2} a_i \xi_i \neq 1;$$

(H1) There exist $q(t) \in L^1[0,1]$ and $p(t)$, $r(t) \in L^1_{\mathrm{loc}}(0,1)$ with $t(1-t)p(t)$, $t(1-t)r(t) \in L^1[0,1]$, such that

$$\big|f(t,u,v)\big| \leqslant p(t)|u| + q(t)|v| + r(t), \quad \text{almost everywhere } t \in [0,1],\ (u,v) \in \mathbb{R}^2 \tag{1.6}$$

where

$$L^1_{\mathrm{loc}}(0,1) = \big\{u \mid u|_{[c,d]} \in L^1[c,d] \text{ for every compact interval } [c,d] \subset (0,1)\big\};$$

(H2) The function $e : [0,1] \to \mathbb{R}$ satisfies $\int_0^1 t(1-t)\big|e(t)\big|\,dt < \infty$.

For results concerning the existence and multiplicity of solutions (or positive solutions) of singular nonlinear two-point boundary value problems, one may refer, to Agarwal and O'Regan [**1**], Asakawa [**2**], Baxley [**3**], O'Regan [**7**], Shi and Chen [**8**] and Taliaferro [**10**] and the references therein. The existence and multiplicity of solutions of non-singular multi-point boundary value problems have been studied by many authors; see, for example, Gupta [**4**], Ma [**4, 5**], Webb [**10**] and the references therein for more information on this problem. For recent results on singular multi-point boundary value problems, see Zhang and Wang [**11**].

2. Preliminary Lemmas

In this section, we always assume that (H0) holds.

We shall use the classical Banach spaces $C[0,1]$, $C^k[0,1]$, $L^1[0,1]$ and $L^\infty[0,1]$. We denote by $AC[a,b]$ the space of all absolutely continuous functions on $[a,b]$, and define $AC^k[a,b]$ by

$$AC^k[a,b] = \big\{u \in C^k[a,b] \mid u^{(k)} \in AC[a,b]\big\},$$

where $AC^0[a,b] = AC[a,b]$. Let I be an interval in R. We denote by $AC_{\rm loc}I$ and $L^1_{\rm loc}I$ the spaces of functions on I defined by

$$AC_{\rm loc}I = \{u \mid u|_{[c,d]} \in AC[c,d] \text{ for every compact interval } [c,d] \subset I\}$$

and

$$L^1_{\rm loc}I = \{u \mid u|_{[c,d]} \in L^1[c,d] \text{ for every compact interval } [c,d] \subset I\}.$$

Let E be the Banach space

$$E = \{y \in L^1_{\rm loc}(0,1) \mid t(1-t)y(t) \in L^1[0,1]\}$$

equipped with the norm

$$\|y\|_E = \int_0^1 t(1-t)\big|y(t)\big|\,dt$$

and let X be the Banach space

$$X = \{u \in C^1(0,1) \mid u \in C[0,1],\ \lim_{t\to 1}(1-t)u'(t) \text{ and } \lim_{t\to 0} tu'(t) \text{ exist}\}$$

equipped with the norm

$$\|u\|_X = \max\Big\{\|u\|_\infty,\ \big\|t(1-t)u'(t)\big\|_\infty\Big\}$$

where $\|\cdot\|_\infty$ denotes the sup norm.

Let $G(t,s)$ be the Green's function for the second-order boundary value problem

$$-u''(t) = 0, \qquad t \in (0,1) \tag{2.1}$$

$$u(0) = u(1) = 0 \tag{2.2}$$

which is explicitly given by

$$G(t,s) = \begin{cases} (1-t)s, & 0 \leqslant s \leqslant t \leqslant 1 \\ (1-s)t, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{2.3}$$

For each $y \in E$, we define the operator T by

$$(Ty)(t) = \int_0^1 G(t,s)y(s)\,ds + \frac{t}{1-\sum_{i=1}^{m-2} a_i\xi_i}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)y(s)\,ds. \tag{2.4}$$

Now since

$$\begin{aligned}
&\left|\int_0^1 G(t,s)y(s)\,ds + \frac{t}{1-\sum_{i=1}^{m-2} a_i\xi_i}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)y(s)\,ds\right| \\
&\quad\leqslant \int_0^1 G(t,s)\big|y(s)\big|\,ds + \frac{t}{\left|1-\sum_{i=1}^{m-2} a_i\xi_i\right|}\sum_{i=1}^{m-2} |a_i| \int_0^1 G(\xi_i,s)\big|y(s)\big|\,ds \\
&\quad\leqslant \int_0^t (1-t)s\big|y(s)\big|\,ds + \int_t^1 (1-s)t\big|y(s)\big|\,ds \\
&\qquad\quad + \frac{1}{\left|1-\sum_{i=1}^{m-2} a_i\xi_i\right|}\sum_{i=1}^{m-2} |a_i| \left[\int_0^{\xi_i} (1-\xi_i)s\big|y(s)\big|\,ds + \int_{\xi_i}^1 (1-s)\xi_i\big|y(s)\big|\,ds\right] \\
&\quad\leqslant \left(1+\frac{\sum_{i=1}^{m-2}|a_i|}{\left|1-\sum_{i=1}^{m-2} a_i\xi_i\right|}\right)\int_0^1 s(1-s)\big|y(s)\big|\,ds < \infty
\end{aligned}$$

we know from (H0) that $(Ty):[0,1]\to\mathbb{R}$ is well-defined.

REMARK 2.1. If all of the a_i's have the same sign, then $x(1)=\sum_{i=1}^{m-2} a_i x(\xi_i)$ implies

$$x(1) = \alpha x(\eta) \tag{2.5}$$

for some $\eta \in [\xi_1, \xi_{m-2}]$, where $\alpha = \sum_{i=1}^{m-2} a_i$. To study (1.4)–(2.5), Gupta, Ntouyas and Tsamatos defined the operator T by

$$(Ty)(t) = \int_0^t (t-s)y(s)\,ds + \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s)y(s)\,ds - \frac{t}{1-\alpha\eta}\int_0^1 (1-s)y(s)\,ds.$$

This form of T is not suitable for studying the multi-point boundary value problem (1.4)–(2.5), and accordingly (1.4)–(1.5), in singular case.

LEMMA 2.1. ([2, Lemma 2.1].) *Suppose that $\phi \in E$. Then*

(i) $\int_0^t s\phi(s)\,ds,\ \int_t^1 (1-s)\phi(s)\,ds \in L^1[0,1]$, *and*

$$\int_0^1\int_0^t s\phi(s)\,ds\,dt = \int_0^1\int_t^1 (1-s)\phi(s)\,ds\,dt = \int_0^1 s(1-s)\phi(s)\,ds$$

(ii) $\lim_{t\to 0} t\int_t^1 (1-s)\phi(s)\,ds = 0,\ \lim_{t\to 1}(1-t)\int_0^t s\phi(s)\,ds = 0.$

LEMMA 2.2. *Let $y \in E$. Then $Ty \in X$ and*

$$(Ty)''(t) + y(t) = 0, \qquad \text{almost everywhere } t \in (0,1). \tag{2.6}$$

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PROOF: For $y(t) \in E$, we have that $t(1-t)y(t) \in L^1[0,1]$. So for each $r \in (0,1)$, $ty(t) \in L^1[0,r]$ and $(1-t)y(t) \in L^1[r,1]$. Thus $(Ty)(t) \in AC_{\text{loc}}(0,1)$ since

$$(2.7)\quad (Ty)(t) = \int_0^t (1-t)sy(s)\,ds + \int_t^1 (1-s)ty(s)\,ds + \frac{t}{1-\sum_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \left[\int_0^{\xi_i} (1-\xi_i)sy(s)\,ds + \int_{\xi_i}^1 (1-s)\xi_i y(s)\,ds\right].$$

Moreover

$$(2.8)\qquad (Ty)'(t) = -\int_0^t sy(s)\,ds + \int_t^1 (1-s)y(s)\,ds + D_y.$$

where

$$(2.9)\qquad D_y := \frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)y(s)\,ds.$$

Now since

$$\begin{aligned}\int_0^1 \left|(Ty)'(t)\right| dt &= \int_0^1 \left| -\int_0^t sy(s)\,ds + \int_t^1 (1-s)y(s)\,ds + D_y \right| dt \\ &\leqslant \int_0^1 \int_0^t s\left|y(s)\right| ds\,dt + \int_0^1 \int_t^1 (1-s)\left|y(s)\right| ds\,dt + |D_y| \\ &= \int_0^1 \int_s^1 s\left|y(s)\right| dt\,ds + \int_0^1 \int_0^s (1-s)\left|y(s)\right| dt\,ds + |D_y| \\ &= 2\int_0^1 s(1-s)\left|y(s)\right| ds + |D_y| < \infty\end{aligned}$$

we have $Ty \in AC[0,1]$. Now (2.8) together with the fact that $sy(s) \in L^1[0,r]$ and $(1-s)y(s) \in L^1[r,1]$ for each $r \in (0,1)$ imply that $(Ty)'(t) \in AC_{\text{loc}}(0,1)$, and accordingly

$$(2.10)\qquad (Ty)''(t) = -y(t), \qquad \text{almost everywhere } t \in (0,1).$$

Now set

$$\gamma(t) := \left[t(1-t)(Ty)'(t)\right]', \qquad t \in [0,1].$$

First we show $\gamma \in L^1[0,1]$. If this is true then $t(1-t)(Ty)'(t) \in AC[0,1]$, and consequently, $\lim_{t\to 1}(1-t)(Ty)'(t)$ and $\lim_{t\to 0} t(Ty)'(t)$ exist.

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A simple computation (by interchanging the order of integration) yields

$$
\begin{aligned}
\int_0^1 |\gamma(t)|\,dt &= \int_0^1 \Big|\big[(1-t)(Ty)'(t) - t(Ty)'(t) + t(1-t)(Ty)''(t)\big]\Big|\,dt \\
&\leqslant \int_0^1 (1-t)\big|(Ty)'(t)\big|\,dt + \int_0^1 t\big|(Ty)'(t)\big|\,dt + \int_0^1 t(1-t)\big|(Ty)''(t)\big|\,dt \\
&\leqslant \int_0^1 \left[(1-t)\int_0^t s\big|y(s)\big|\,ds\right]dt + \int_0^1 \left[(1-t)\int_t^1 (1-s)\big|y(s)\big|\,ds\right]dt \\
&\quad + \int_0^1 (1-t)|D_y|\,dt + \int_0^1 \left[t\int_0^t s\big|y(s)\big|\,ds\right]dt \\
&\quad + \int_0^1 \left[t\int_t^1 (1-s)\big|y(s)\big|\,ds\right]dt + \int_0^1 t|D_y|\,dt + \int_0^1 t(1-t)\big|y(t)\big|\,dt \\
&= \int_0^1 \left[\int_s^1 (1-t)s\big|y(s)\big|\,dt\right]ds + \int_0^1 \left[\int_0^s (1-t)(1-s)\big|y(s)\big|\,dt\right]ds \\
&\quad + \int_0^1 \left[\int_s^1 ts\big|y(s)\big|\,dt\right]ds + \int_0^1 \left[\int_0^s t(1-s)\big|y(s)\big|\,dt\right]ds \\
&\quad + \int_0^1 t(1-t)\big|y(t)\big|\,dt + |D_y| \\
&\leqslant 5\int_0^1 s(1-s)\big|y(s)\big|\,ds + |D_y| < \infty.
\end{aligned}
$$

This completes the proof of the lemma. ∎

Lemma 2.3. *Let $y \in E$. Then*

$$
(Ty)(0) = 0 \text{ and } (Ty)(1) = \sum_{i=1}^{m-2} a_i (Ty)(\xi_i). \tag{2.11}
$$

Proof: By Lemma 2.2, $Ty \in X$. Thus we have from (2.7) that

$$
\begin{aligned}
(Ty)(0) &= \lim_{t\to 0}(Ty)(t) \\
&= \lim_{t\to 0}\int_0^t (1-t)sy(s)\,ds + \lim_{t\to 0}\int_t^1 (1-s)ty(s)\,ds \\
&\quad + \frac{0}{1-\sum_{i=1}^{m-2} a_i\xi_i}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)y(s)\,ds \\
&= 0.
\end{aligned}
$$

Again applying (2.7) and the fact that $Ty \in X$, we have

$$
\begin{aligned}
(Ty)(1) &= \lim_{t\to 1}(Ty)(t) \\
&= \lim_{t\to 1}\int_0^t (1-t)sy(s)\,ds + \lim_{t\to 1}\int_t^1 (1-s)ty(s)\,ds \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)y(s)\,ds.
\end{aligned} \tag{2.12}
$$

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Applying (ii) of Lemma 2.1 and using the fact that $(1-s)y(s) \in L^1[r,1]$ for some r, $0 < r < 1$, we conclude that

$$(Ty)(1) = \frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)y(s)\,ds. \tag{2.13}$$

Similarly

$$(Ty)(\xi_i) = \int_0^1 G(\xi_i, s)y(s)\,ds + \frac{\xi_i}{1-\sum_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)y(s)\,ds. \tag{2.14}$$

This together with (2.13) implies that $(Ty)(1) = \sum_{i=1}^{m-2} a_i(Ty)(\xi_i)$.

For $x \in X$, we define a nonlinear operator N by

$$(Nx)(t) = -f\big(t, x(t), x'(t)\big) - e(t), \qquad t \in (0,1). \tag{2.15}$$

From (H1) and (H2), we conclude that $N : X \to E$ is well-defined. In fact

$$\begin{aligned} \|Nx\|_E &= \big\|t(1-t)(Nx)(t)\big\|_{L^1} \\ &= \int_0^1 t(1-t)\Big|f\big(t, x(t), x'(t)\big) + e(t)\Big|\,dt \\ &\leqslant \int_0^1 \Big[t(1-t)p(t)\big|x(t)\big| + \big|q(t)\big|t(1-t)x'(t)\ \big| + t(1-t)\big|r(t)\big| \\ &\qquad\qquad + t(1-t)\big|e(t)\big|\Big]\,dt \\ &\leqslant \|p\|_E\|x\|_\infty + \|q\|_{L^1}\big\|t(1-t)x'(t)\big\|_\infty + \|r\|_E + \|e\|_E \\ &< \infty. \end{aligned} \tag{2.16}$$

□

Lemma 2.4 *$TN : X \to X$ is completely continuous.*

Proof: From the definitions of T and N and (H1) and (H2), it is easy to show that $TN : X \to X$ is continuous. Let $B \subset X$ be a bounded set. We need to show that $(TN)(B)$ is a relatively compact subset of X.

Let $\{x_n\} \subset B$ and set

$$w_n(t) = \big((TN)x_n\big)(t) \text{ and } z_n(t) = t(1-t)\big((TN)x_n\big)'(t). \tag{2.17}$$

We need show only that there exists a subsequence with

$$w_n \to w^* \qquad \text{in } C[0,1] \tag{2.18}$$

and

$$z_n \to z^* \qquad \text{in } C[0,1]. \tag{2.19}$$

(We note that (2.18) together with (2.19) and the fact that $z^*(t) = t(1-t)\big(w^*(t)\big)'$ for $t \in [0,1)$ implies that, after taking a subsequence if necessary, $\|w_n - w^*\|_X \to 0$.)

To prove (2.18), we recall that $N : X \to E$ and

$$\big|(Nx_n)(t)\big| \leqslant p(t)M + \frac{q(t)}{t(1-t)}M + r(t) + \big|e(t)\big| \\ := \chi(t) \tag{2.20}$$

where

$$M = \max\big\{\|x\|_X \mid x \in B\big\}. \tag{2.21}$$

Clearly, (H1) and (H2) imply that $\chi \in E$. Now for each n and for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$

$$\begin{aligned} \big|w_n(t_1) - w_n(t_2)\big| &= \left|\int_{t_2}^{t_1} \big((TN)x_n\big)'(\tau)d\tau\right| \\ &\leqslant \int_{t_2}^{t_1} \Big|\big((TN)x_n\big)'(\tau)\Big|d\tau \\ &= \int_{t_2}^{t_1} \left|-\int_0^\tau s(Nx_n)(s)\,ds + \int_\tau^1 (1-s)(Nx_n)(s)\,ds + D_{Nx_n}\right|d\tau \\ &\leqslant \int_{t_2}^{t_1} \left[\int_0^\tau s\big|(Nx_n)(s)\big|\,ds + \int_\tau^1 (1-s)\big|(Nx_n)(s)\big|\,ds + D\right]d\tau \\ &= \int_{t_2}^{t_1} \left[\int_0^\tau s\chi(s)\,ds + \int_\tau^1 (1-s)\chi(s)\,ds + D\right]d\tau \end{aligned} \tag{2.22}$$

where

$$D = \sup\big\{|D_y| \;|y \in B\big\}.$$

By (i) of Lemma 2.2, $\displaystyle\int_0^\tau s\chi(s)\,ds, \int_\tau^1 (1-s)\big|\chi(s)\big|\,ds \in L^1[0,1]$. Thus (2.22) shows that $\{w_n\}_{n=1}^\infty$ is equi-continuous on $[0,1]$. Therefore by the Arzela-Ascoli theorem, after taking a subsequence if necessary, (2.18) holds.

To prove (2.19), in view of the Arzela-Ascoli theorem, we need to verify that

(a) $\|z_n\|_\infty < M_1$ for some positive constant M_1, independent of n;

(b) $\big\{z_n(t)\big\}_{n=1}^\infty$ is equi-continuous on $[0,1]$.

Since (a) can be easily deduced from the definitions of T and N and the conditions (H1) and (H2), we only prove (b) here.

For $n \in \mathbb{N}$ and $t \in (0,1)$, we have from (H1) and (H2) that

$$
\begin{aligned}
|z_n'(t)| &= \Big|(1-t)\big((TN)x_n\big)'(t) - t\big((TN)x_n\big)'(t) + t(1-t)\big((TN)x_n\big)''(t)\Big| \\
&\leqslant (1-t)\Big|\big((TN)x_n\big)'(t)\Big| + t\Big|\big((TN)x_n\big)'(t)\Big| + t(1-t)\Big|\big((TN)x_n\big)''(t)\Big| \\
&\leqslant (1-t)\int_0^t s|Nx_n(s)|\,ds + (1-t)\int_t^1 (1-s)|Nx_n(s)|\,ds + (1-t)|D| \\
&\quad + t\int_0^t s|Nx_n(s)|\,ds + t\int_t^1 (1-s)\big|Nx_n(s)\big|\,ds + t|D| + t(1-t)\big|Nx_n(t)\big| \qquad (2.23)\\
&\leqslant (1-t)\int_0^t s\chi(s)\,ds + (1-t)\int_t^1 (1-s)\chi(s)\,ds + (1-t)|D| \\
&\quad + t\int_0^t s\chi(s)\,ds + t\int_t^1 (1-s)\chi(s)\,ds + t|D| + t(1-t)\chi(t) \\
&:= \psi_1(t).
\end{aligned}
$$

By (i) of Lemma 2.1,

$$
\psi_1 \in L^1[0,1]. \qquad (2.24)
$$

Now (2.23) is sufficient to ensure the validity of (b) since

$$
\big|z_n(t_1) - z_n(t_2)\big| = \left|\int_{t_2}^{t_1} z_n'(\tau)d\tau\right| \leqslant \int_{t_2}^{t_1} \big|z_n'(\tau)\big|d\tau \leqslant \int_{t_2}^{t_1} \psi_1(\tau)d\tau.
$$

∎

3. Main Result

Theorem 3.1. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the Carathéodory conditions. Assume that* (H0),(H1) *and* (H2) *hold. Then problem* (1.4)–(1.5) *has at least one solution in X provided*

$$
\|p\|_E\left(1 + \frac{\sum_{i=1}^{m-2}|a_i|}{\big|1 - \sum_{i=1}^{m-2} a_i\xi_i\big|}\right) + \|q\|_{L^1} < 1. \qquad (3.1)
$$

Remark 3.1. In [4], a key condition is that all a_i have same sign. We don't need the restriction on a_i in (H0).

Remark 3.2. Let us consider the three-point boundary value problem

$$
\begin{aligned}
&x'' = g(t,x,x') \\
&x'(0) = 0, \quad x(1) = x\Big(\frac{1}{3}\Big) - x\Big(\frac{2}{3}\Big)
\end{aligned} \qquad (3.2)
$$

where

$$
g(t,u,v) = \frac{\alpha}{t(1-t)}\sin(u+v)u + \beta v + \frac{1}{t(1-t)}\big[1 + \cos(u^{200} + v^{30})\big].
$$

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It is easy to see that

$$|g(t,u,v)| \leqslant p(t)|u| + q(t)|v| + r(t)$$

with $p(t) = \alpha/(t(1-t))$, $q(t) = \beta$ and $r(t) = 2/(t(1-t))$. Clearly, $\|p\|_E = |\alpha|$, $\|q\|_{L^1} = |\beta|$, $\|r\|_E = 2$, and

$$\frac{\sum_{i=1}^{m-2}|a_i|}{|1-\sum_{i=1}^{m-2}a_i|} = \frac{1+1}{|1-(1\times(1/3)-1\times(2/3))|} = \frac{3}{2}.$$

By Theorem 3.1, (3.2) has at least solution in X provided

$$\frac{5}{2}|\alpha| + |\beta| < 1.$$

Now we cannot apply the main results of [4] to deal with (3.2) since $p, r \notin L^1[0,1]$.

PROOF OF THEOREM 3.1. From Lemmas 2.2 and 2.3, we know that $u \in X$ is a solution of (1.4)–(1.5) if and only if

$$u = TNu. \tag{3.3}$$

By Lemma 2.4, we can apply the Leray-Schauder continuation theorem (see, for example, [**6**, Corollary IV. 7]) to obtain the existence of a solution for (3.3) in X.

To do this it suffices to verify that the set of all possible solutions of the family of equations

$$x''(t) = \lambda f(t, x(t), x'(t)) + \lambda e(t), \qquad t \in (0,1) \tag{3.4$_\lambda$}$$

$$x(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i) \tag{3.5}$$

is a priori bounded in X by a constant independent of $\lambda \in [0,1]$.

Let $u \in X$ be a solution of (3.4$_\lambda$)-(3.5) for some $\lambda \in [0,1]$. Then for $t \in [0,1]$, we have

$$\begin{aligned}
|u(t)| &= \left|\int_0^1 G(t,s)\lambda(Nu)(s)\,ds + \frac{t}{1-\sum_{i=1}^{m-2}a_i\xi_i}\sum_{i=1}^{m-2}a_i\int_0^1 G(\xi_i,s)\lambda(Nu)(s)\,ds\right| \\
&= \left|\int_0^1 G(t,s)u''(s)\,ds + \frac{t}{1-\sum_{i=1}^{m-2}a_i\xi_i}\sum_{i=1}^{m-2}a_i\int_0^1 G(\xi_i,s)u''(s)\,ds\right| \\
&\leqslant \left(1 + \frac{\sum_{i=1}^{m-2}|a_i|}{|1-\sum_{i=1}^{m-2}a_i\xi_i|}\right)\int_0^1 s(1-s)|u''(s)|\,ds \qquad (3.6)\\
&= \left(1 + \frac{\sum_{i=1}^{m-2}|a_i|}{|1-\sum_{i=1}^{m-2}a_i\xi_i|}\right)\|u''(s)\|_E
\end{aligned}$$

which implies that

$$\|u\|_\infty \leqslant \left(1 + \frac{\sum_{i=1}^{m-2}|a_i|}{|1-\sum_{i=1}^{m-2}a_i\xi_i|}\right)\|u''\|_E. \tag{3.7}$$

Similarly,

$$
\begin{aligned}
\big|t(1-t)u'(t)\big| &= \left|t(1-t)\left[-\int_0^t s(\lambda Nu)(s)\,ds + \int_t^1 (1-s)(\lambda Nu)(s)\,ds\right]\right| \\
&= \left|t(1-t)\left[-\int_0^t su''(s)\,ds + \int_t^1 (1-s)u''(s)\,ds\right]\right| \\
&\leqslant t(1-t)\int_0^t s\big|u''(s)\big|\,ds + t(1-t)\int_t^1 (1-s)\big|u''(s)\big|\,ds \qquad (3.8)\\
&\leqslant (1-t)\int_0^t s\big|u''(s)\big|\,ds + t\int_t^1 (1-s)\big|u''(s)\big|\,ds \\
&\leqslant \int_0^1 s(1-s)\big|u''(s)\big|\,ds
\end{aligned}
$$

which implies that

$$\big\|t(1-t)u'(t)\big\|_\infty \leqslant \|u''\|_E. \qquad (3.9)$$

Now from (3.4) , (3.7) and (3.9) it follows that

$$
\begin{aligned}
\big|t(1-t)u''(t)\big| &= \lambda t(1-t)\Big|f\big(t,u(t),u'(t)\big) + e(t)\Big| \\
&\leqslant t(1-t)\Big[p(t)\big|u(t)\big| + q(t)\big|u'(t)\big| + \big|r(t)\big| + \big|e(t)\big|\Big] \\
&\leqslant \|p\|_E\|u\|_\infty + \|q\|_{L^1}\big\|t(1-t)u'(t)\big\|_\infty + \|r\|_E + \|e\|_E \qquad (3.10)\\
&\leqslant \|p\|_E\left(1+\frac{\sum_{i=1}^{m-2}|a_i|}{\left|1-\sum_{i=1}^{m-2}a_i\xi_i\right|}\right)\|u''\|_E + \|q\|_{L^1}\|u''\|_E + \|r\|_E + \|e\|_E
\end{aligned}
$$

for $t \in (0,1)$. Thus

$$\|u''\|_E \leqslant \left[\|p\|_E\left(1+\frac{\sum_{i=1}^{m-2}|a_i|}{\left|1-\sum_{i=1}^{m-2}a_i\xi_i\right|}\right) + \|q\|_{L^1}\right]\|u''\|_E + \|r\|_E + \|e\|_E. \qquad (3.11)$$

It follows from the assumption (3.1) that there is a constant c, independent of $\lambda \in [0,1]$, such that

$$\|u''\|_E \leqslant c. \qquad (3.12)$$

This together (3.7) implies that

$$\|u\|_\infty \leqslant \left(1+\frac{\sum_{i=1}^{m-2}|a_i|}{\left|1-\sum_{i=1}^{m-2}a_i\xi_i\right|}\right)c. \qquad (3.13)$$

Similarly, (3.12) together with (3.9) imply that

$$\|t(1-t)u'(t)\|_\infty \leqslant c. \qquad (3.14)$$

Therefore

$$\|u\|_X \leqslant \max\left\{c,\ \left(1+\frac{\sum_{i=1}^{m-2}|a_i|}{\left|1-\sum_{i=1}^{m-2}a_i\xi_i\right|}\right)c\right\}. \qquad (3.15)$$

This completes the proof of the theorem. ∎

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