THREE NONNEGATIVE SOLUTIONS FOR SECOND-ORDER IMPULSIVE DIFFERENTIAL EQUATIONS WITH A THREE-POINT BOUNDARY VALUE PROBLEM

JIANLI LI and JIANHUA SHEN

(Received 9 September, 2006; revised 4 January, 2008)

Abstract

In this paper, by using the Leggett–Williams fixed point theorem, we prove the existence of three nonnegative solutions to second-order nonlinear impulsive differential equations with a three-point boundary value problem.

Keywords and phrases: differential equation, boundary value problem, nonnegative solution, fixed point theorem.

1. Introduction

Let \( 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1 \) be given. In this paper we present results which guarantee the existence of three nonnegative solutions to the second-order impulsive equation

\[
\begin{aligned}
&y''(t) + h(t)f(y(t)) = 0 \quad \text{for } t \in (0, 1) \setminus \{t_1, \ldots, t_m\}, \\
&\Delta y(t_k) = l_k(y(t_k^-)), \quad k = 1, \ldots, m, \\
&\Delta y'(t_k) = j_k(y(t_k^-)), \quad k = 1, \ldots, m, \\
y(0) = 0, \\
&\alpha y(\eta) = y(1),
\end{aligned}
\]

(1.1)

where \( 0 < \eta < 1, \ 0 < \alpha < 1/\eta, \ \Delta y(t_k) = y(t_k^+) - y(t_k^-), \) and \( y(t_k^+) \ \text{and} \ y(t_k^-) \) respectively denote the right limit and left limit of \( y(t) \) at \( t = t_k \). Also \( \Delta y'(t_k) = y'(t_k^+) - y'(t_k^-) \). We define the Banach space \( (r = 0 \text{ or } 2) \) in this paper.

\[1\text{Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, China; e-mail: ljianli@sina.com.}
\[2\text{Department of Mathematics, College of Huaihua, Huaihua, Hunan 418008, China.}
\]

\( \odot \) Australian Mathematical Society 2008, Serial-fee code 0334-2700/08
Let \( E (2.1) \)
with the norm
\[
\| y \|_{PC'} = \max \{ \| y \|, \| y' \|, \ldots, \| y^{(r)} \| \}.
\]
Here
\[
\| y \| = \sup \{ |y(t)|, \ t \in [0, 1] \}.
\]

By a solution to (1.1) we mean a function \( y \in PC^2[0, 1] \) which satisfies (1.1). In (1.1), as \( \alpha = 0 \), the existence of three nonnegative solutions was considered by Agarwal and O’Regan [3], and as \( I_k(y(t^+_k)) = J_k(y(t^-_k)) \equiv 0 \), \( k = 1, \ldots, m \), the positive solution was obtained by Ma [6]. In this paper, motivated by [3] and [6], we shall show the existence of three nonnegative solutions to (1.1) by the Leggett–Williams fixed point theorem [5]. Recently [1–4, 7, 8] this fixed point theorem has been used to establish multiplicity results for differential, integral and difference equations.

Now we present some preliminaries which will be needed in Section 3. First, \( E = (E, \| \cdot \|) \) is a Banach space and \( P \subset E \) is a cone. By a concave nonnegative continuous functional \( \psi \) on \( P \) we mean a continuous mapping \( \psi : P \to [0, \infty) \) with
\[
\psi(\lambda x + (1 - \lambda)y) \geq \lambda \psi(x) + (1 - \lambda)\psi(y) \quad \text{for all } x, y \in P \text{ and all } \lambda \in [0, 1].
\]
Let \( K, L, r > 0 \) be constants with \( P \) and \( \psi \) as defined above. Let
\[
P_K = \{ y \in P : \| y \| < K \} \quad \text{and} \quad P(\psi, r, L) = \{ y \in P : \psi(y) \geq r \text{ and } \| y \| \leq L \}.
\]

We now state the Leggett–Williams fixed point theorem [5].

**Theorem 1.1.** Let \( E = (E, \| \cdot \|) \) be a Banach space, \( P \subset E \) a cone of \( E \) and \( R > 0 \) a constant. Suppose there exists a concave nonnegative continuous functional \( \psi \) on \( P \) with \( \psi(y) \leq \| y \| \) for all \( y \in P \) and let \( A : \bar{P}_R \to \bar{P}_R \) be a continuous compact map. Assume there are numbers \( r, L, K \) with \( 0 < r < L < K \leq R \) such that:

(H1) \( \{ y \in P(\psi, L, K) : \psi(y) > L \} \neq \emptyset \) and \( \psi(Ay) > L \) for all \( y \in P(\psi, L, K) \);

(H2) \( \| Ay \| < r \) for all \( y \in \bar{P}_r \); and

(H3) \( \psi(Ay) > L \) for all \( y \in P(\psi, L, R) \) with \( \| Ay \| > K \).

Then \( A \) has at least three fixed points \( y_1, y_2 \) and \( y_3 \) in \( \bar{P}_R \). Furthermore
\[
y_1 \in P_r, \quad y_2 \in \{ y \in P(\psi, L, R) : \psi(y) > L \} \quad \text{and} \quad y_3 \in \bar{P}_R \setminus (P(\psi, L, R) \cup \bar{P}_r).
\]

**2. Some lemmas**

Consider the impulsive integral equation
\[
y(t) = \int_0^1 H(t, s)h(s)f(y(s)) \, ds + \sum_{k=1}^m W_k(t, y), \quad t \in [0, 1], \quad (2.1)
\]
where
\[
H(t, s) = \frac{1}{1 - \alpha \eta} t (1 - s) - U(t, s) - \frac{\alpha}{1 - \alpha \eta} V(t, s), \quad 0 \leq t, s \leq 1,
\]
\[
U(t, s) = \begin{cases} 
    t - s, & s \leq t, \\
    0, & t \leq s
\end{cases},
\]
\[
V(t, s) = \begin{cases} 
    t (\eta - s), & s \leq \eta, \\
    0, & \eta \leq s
\end{cases},
\]
and for \(k = 1, \ldots, m\)
\[
W_k(t, y) = \begin{cases} 
    \frac{1 - t - \alpha \eta + \alpha t}{1 - \alpha \eta} [I_k(y(t_k^-)) - t_k J_k(y(t_k^-))], & 0 < t_k < \min\{t, \eta\}, \\
    \frac{t}{1 - \alpha \eta} [-I_k(y) - (1 - t_k) J_k(y) + \alpha [I_k(y) + (\eta - t_k) J_k(y)]], & t \leq t_k < \max\{t, \eta\}, \\
    \frac{1}{1 - \alpha \eta} [(1 - \alpha \eta) (I_k(y) - t_k J_k(y)) - t [I_k(y) - (t_k - \alpha \eta) J_k(y)]], & \eta \leq t_k < \max\{t, \eta\}, \\
    \frac{t}{1 - \alpha \eta} [-I_k(y(t_k^-)) - (1 - t_k) J_k(y(t_k^-))], & \max\{t, \eta\} \leq t_k < 1.
\]

**Lemma 2.1.** We have that \(y \in PC[0, 1] \cap PC^2[0, 1]\) is a solution of (1.1) if and only if \(y \in PC[0, 1]\) is a solution of the integral equation (2.1).

**Proof.** Suppose that \(y \in PC[0, 1]\) is a solution of (2.1). Then for \(t \neq t_k\),
\[
y'(t) = \frac{1}{1 - \alpha \eta} \int_0^1 (1 - s) h(s) f(y(s)) \, ds - \frac{\alpha}{1 - \alpha \eta} \int_0^\eta (\eta - s) h(s) f(y(s)) \, ds
\]
\[
- \frac{1}{1 - \alpha \eta} \sum_{0 < t_k < 1} [J_k(y)(1 - t_k) + I_k(y)]
\]
\[
+ \frac{\alpha}{1 - \alpha \eta} \sum_{0 < t_k < \eta} [J_k(y)(\eta - t_k) + I_k(y)]
\]
\[
- \int_0^t h(s) f(y(s)) \, ds + \sum_{0 < t_k < t} J_k(y),
\]
\[
y''(t) = -h(t) f(y(t)),
\]
and for \(t = t_k\),
\[
\Delta y(t_k) = y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)),
\]
\[
\Delta y'(t_k) = y'(t_k^+) - y'(t_k^-) = J_k(y(t_k^-)),
\]
and
\[
y(0) = 0, \quad \alpha y(\eta) = y(1).
\]
So \(y\) is a solution of (1.1).
On the other hand, if $y$ is a solution of (2.1), then

$$y'(t) = y'(0) - \int_0^t h(s) f(y(s)) \, ds + \sum_{0 < t_k < t} J_k(y),$$

$$y(t) = y'(0) t - \int_0^t (t - s) h(s) f(y(s)) \, ds + \sum_{0 < t_k < t} [J_k(y)(t - t_k) + I_k(y)].$$

This and the boundary value condition $y(0) = 0$ and $\alpha y(\eta) = y(1)$ imply that

$$y'(0) = \frac{1}{1 - \alpha \eta} \int_0^1 (1 - s) h(s) f(y(s)) \, ds - \frac{\alpha}{1 - \alpha \eta} \int_0^{\eta} (\eta - s) h(s) f(y(s)) \, ds$$

$$- \frac{1}{1 - \alpha \eta} \sum_{0 < t_k < 1} [J_k(y)(1 - t_k) + I_k(y)]$$

$$+ \frac{\alpha}{1 - \alpha \eta} \sum_{0 < t_k < \eta} [J_k(y)(\eta - t_k) + I_k(y)].$$

Therefore

$$y(t) = \frac{t}{1 - \alpha \eta} \int_0^1 (1 - s) h(s) f(y(s)) \, ds - \frac{\alpha t}{1 - \alpha \eta} \int_0^{\eta} (\eta - s) h(s) f(y(s)) \, ds$$

$$- \frac{t}{1 - \alpha \eta} \sum_{0 < t_k < 1} [J_k(y)(1 - t_k) + I_k(y)]$$

$$+ \frac{\alpha t}{1 - \alpha \eta} \sum_{0 < t_k < \eta} [J_k(y)(\eta - t_k) + I_k(y)]$$

$$- \int_0^t (t - s) h(s) f(y(s)) \, ds + \sum_{0 < t_k < t} [J_k(y)(t - t_k) + I_k(y)].$$

For $0 \leq t \leq \eta,$

$$y(t) = \int_0^1 H(t, s) h(s) f(y(s)) \, ds + \frac{1 - t - \alpha \eta + \alpha t}{1 - \alpha \eta} \sum_{0 < t_k < t} [I_k(y) - t_k J_k(y)]$$

$$+ \frac{t}{1 - \alpha \eta} \sum_{t_k \leq s < \eta} \{-I_k(y) - (1 - t_k) J_k(y) + \alpha [I_k(y) + (\eta - t_k) J_k(y)]\}$$

$$+ \frac{t}{1 - \alpha \eta} \sum_{\eta \leq t_k < 1} \{-I_k(y) - (1 - t_k) J_k(y)\}.$$
For $\eta \leq t \leq 1$,

$$y(t) = \int_0^1 H(t, s) h(s) f(y(s)) \, ds + \frac{1 - t - \alpha \eta + \alpha t}{1 - \alpha \eta} \sum_{0 < t_k < \eta} [I_k(y) - t_k J_k(y)]$$

$$+ \frac{1}{1 - \alpha \eta} \sum_{\eta \leq t_k < t} \{(1 - \alpha \eta)(I_k(y) - t_k J_k(y)) - t(I_k(y) - (t_k - \alpha \eta)J_k(y))\}$$

$$+ \frac{t}{1 - \alpha \eta} \sum_{\eta \leq t_k < 1} [-I_k(y) - (1 - t_k)J_k(y)].$$

So

$$y(t) = \int_0^1 H(t, s) h(s) f(y(s)) \, ds + \sum_{k=1}^m W_k(t, y), \quad t \in [0, 1].$$

**Lemma 2.2.** We have that:

1. $H : [0, 1] \times [0, 1] \to [0, +\infty)$ is continuous; and
2. $H(t, s) \leq M_1 H(s, s)$ for all $t, s \in [0, 1],
   \quad H(t, s) \geq M_2 H(s, s)$ for $s \in [0, 1], \quad t \in [a_k, b_k],$
   where

   $$M_1 = \max \left\{ \frac{1 - \alpha \eta}{1 - \eta}, \frac{1 + \alpha \eta}{\eta} \right\},$$
   $$M_2 = \min \left\{ \frac{1}{4}, \frac{(1 - \alpha \eta)(1 - t_m)}{4}, \frac{(1 - \alpha \eta)(1 - t_m)}{4\alpha(1 - \eta)} \right\},$$
   $$a_k = \frac{3t_k + t_{k+1}}{4}, \quad b_k = \frac{t_k + 3t_{k+1}}{4} \quad \text{for } k \in \{0, 1, \ldots, m\}.$$  

**Proof.** Part (1) is part (1) of [9, Lemma 3.1]. Now we prove part (2). We divide the proof into the following six cases.

(i) If $0 \leq s \leq t \leq \eta$, then

$$\frac{H(t, s)}{H(s, s)} = \frac{1 - \alpha \eta + t(\alpha - 1)}{1 - \alpha \eta + s(\alpha - 1)} \leq \begin{cases} 1, & \alpha \geq 1, \\ \frac{1 - \alpha \eta}{\alpha(1 - \eta)}, & \alpha < 1, \end{cases}$$

$$\frac{H(t, s)}{H(s, s)} = \frac{1 - \alpha \eta + t(\alpha - 1)}{1 - \alpha \eta + s(\alpha - 1)} \geq (1 - \alpha \eta)(1 - t) \geq (1 - \alpha \eta) \frac{1 - t_m}{4}, \quad t \in [a_k, b_k].$$

(ii) If $0 \leq t \leq s \leq \eta$, then

$$\frac{H(t, s)}{H(s, s)} = \frac{t}{s} \leq 1,$$

$$\frac{H(t, s)}{H(s, s)} = \frac{t}{s} \geq \frac{t_1}{4}, \quad t \in [a_k, b_k].$$
(iii) If $0 \leq s \leq \eta \leq t \leq 1$, then

$$H(t, s)H(s, s) = \frac{1 - \alpha \eta + t(\alpha - 1)}{1 - \alpha \eta + s(\alpha - 1)} \leq \begin{cases} 1, & \alpha \geq 1, \\ \frac{1 - \alpha \eta}{\alpha(1 - \eta)}, & \alpha < 1, \end{cases}$$

$$H(t, s)H(s, s) \geq \begin{cases} \frac{1 - t \geq 1 - t_m}{4}, & \alpha \leq 1, \ t \in [a_k, b_k], \\ \frac{(1 - \alpha \eta)(1 - t)}{\alpha(1 - \eta)} \geq \frac{(1 - \alpha \eta)(1 - t_m)}{4 \alpha(1 - \eta)}, & \alpha > 1, \ t \in [a_k, b_k]. \end{cases}$$

(iv) If $0 \leq t \leq \eta \leq s \leq 1$, then

$$H(t, s)H(s, s) = \frac{t}{s} \leq 1,$$

$$H(t, s)H(s, s) = \frac{t \geq \frac{t_1}{4}}{s} \quad t \in [a_k, b_k].$$

(v) If $\eta \leq s \leq t \leq 1$, then

$$H(t, s)H(s, s) = \frac{s(1 - t) + \alpha \eta(t - s)}{s(1 - s)} \leq \frac{s(1 - s) + \alpha \eta(1 - s)}{s(1 - s)} = \frac{s + \alpha \eta}{s} \leq \frac{1 + \alpha \eta}{\eta},$$

$$H(t, s)H(s, s) = \frac{s(1 - t) + \alpha \eta(t - s)}{s(1 - s)} \geq \frac{1 - t}{1 - s} \geq \frac{1}{1 - \eta} \geq \frac{1 - t_m}{4}, \quad t \in [a_k, b_k].$$

(vi) If $\eta \leq t \leq s \leq 1$, then

$$H(t, s)H(s, s) = \frac{t}{s} \leq 1,$$

$$H(t, s)H(s, s) = \frac{t \geq \frac{t_1}{4}}{s} \quad t \in [a_k, b_k].$$

Thus

$$H(t, s) \leq M_1 H(s, s) \quad \text{for } t, \ s \in [0, 1],$$

$$H(t, s) \geq M_2 H(s, s) \quad \text{for } s \in [0, 1], \ t \in [a_k, b_k].$$

\[ \Box \]

**Remark 2.1.** Note that $M_1 > 1$.

### 3. Existence

We will use Theorem 1.1 to establish the existence of three nonnegative solutions to (1.1). The following conditions will be assumed:
\[ h \in C(0, 1) \text{ with } h > 0 \text{ on } (0, 1) \text{ and } h \in L^1[0, 1], \]
\[ f : [0, \infty) \to [0, \infty) \text{ is continuous and nondecreasing}, \]
\[ I_k, J_k : [0, \infty) \to R \text{ are continuous for } k = 1, \ldots, m, \]
\[ t_k J_k(v) \leq I_k(v) \leq (t_k - 1) J_k(v) \text{ for } v \geq 0 \text{ and } k = 1, \ldots, m, \]
\[ \begin{cases} I_k(v) \geq (t_k - \eta) J_k(v) & \text{for } v \geq 0, t_k < \eta \text{ and } k \in \{1, \ldots, m\}, \\ I_k(v) \leq (t_k - \alpha \eta) J_k(v) & \text{for } v \geq 0, t_k \geq \eta \text{ and } k \in \{1, \ldots, m\}, \end{cases} \]
\[ W_k(t, u) \leq \Omega_k(u(t_k)) \text{ for } t \in [0, 1] \text{ and } u \in C[0, 1] \text{ with } u \geq 0, \]
\[ \text{and with } \Omega_k \geq 0 \text{ continuous and nondecreasing on } [0, \infty), \]
\[ \exists r > 0 \text{ with } f(r) \sup_{t \in [0,1]} \int_0^1 H(t, s)h(s) \, ds + \sum_{i=1}^m \Omega_i(t) < r, \]
\[ \exists L > r \text{ with } f(L) \min_{k \in \{0, 1, \ldots, m\}} \min_{t \in [a_k, b_k]} \int_{a_k}^{b_k} H(t, s)h(s) \, ds > L, \]
\[ \begin{cases} c_0, 0 < c_0 < 1 \text{ with, for each } j \in \{1, 2, \ldots, m\}, \ W_j(t, y) \geq c_0 \Omega_j(y(t_j)), \end{cases} \]
\[ \text{and} \]
\[ \exists R \geq LM^{-1}M_1 \text{ with } f(R) \sup_{t \in [0,1]} \int_0^1 H(t, s)h(s) \, ds + \sum_{j=1}^m \Omega_j(R) \leq R, \]
where
\[ M = \min\{c_0, M_2\}. \]

**Theorem 3.1.** Suppose that (3.1)–(3.10) hold. Then (1.1) has at least three nonnegative solutions \( y_1, y_2 \) and \( y_3 \) in \( PC^2[0, 1] \) such that
\[ \|y_1\| < r, \quad y_2(t) > L \text{ for } t \in [a_k, b_k], k \in \{0, 1, \ldots, m\}, \]
and
\[ \|y_3\| > r \text{ with } \min_{k \in \{0, \ldots, m\}} \min_{t \in [a_k, b_k]} y_3(t) < L. \]

**Proof.** Let
\[ E = (PC[0, 1], \|\cdot\|) \quad \text{and} \quad P = \{u \in PC[0, 1], u(t) \geq 0 \text{ for } t \in [0, 1]\}. \]
Now let \( A : PC[0, 1] \to PC[0, 1] \) be defined by
\[ Ay(t) = \int_0^1 H(t, s)h(s)f(y(s)) \, ds + \sum_{k=1}^m W_k(t, y) \text{ for } t \in [0, 1]. \]
For \( y \geq 0 \) the conditions (3.1), (3.2), (3.4) and (3.5) imply that \( Ay(t) \geq 0 \text{ for } t \in [0, 1]. \) So \( A(P) \subset P. \) It is easy to show that \( A : P \to P \) is continuous and completely continuous [3].
For \( y \in P \), let
\[
\psi(y) = \min_{k \in \{0, 1, \ldots, m\}} \min_{t \in [a_k, b_k]} y(t).
\]

Then \( \psi \) is a nonnegative continuous concave functional on \( P \) with \( \psi(y) \leq \|y\| \) for \( y \in P \). Next choose and fix \( K \) so that
\[
LM_1M^{-1} \leq K \leq R. \tag{3.13}
\]

First, we prove that condition (H2) of Theorem 1.1 holds. To do this, let \( y \in \tilde{P}_r \), then \( 0 \leq y \leq r \). Conditions (3.2), (3.6) and (3.7) imply for \( t \in [0, 1] \) that
\[
Ay(t) \leq f(r) \sup_{t \in [0,1]} \int_0^1 H(t, s)h(s) \, ds + \sum_{k=1}^m \Omega_k(r) < r.
\]

So
\[
\|Ay\| < r.
\]

This shows that condition (H2) of Theorem 1.1 follows. Also \( A : \tilde{P}_R \to \tilde{P}_R \) since, if \( y \in \tilde{P}_R \), then
\[
\|Ay\| \leq f(R) \sup_{t \in [0,1]} \int_0^1 H(t, s)h(s) + \sum_{k=1}^m \Omega_k(R) \leq R.
\]

Next, we show that \( \{y \in P(\psi, L, K) : \psi(y) \geq L \} \neq \emptyset \) and \( \psi(Ay) > L \) for all \( y \in P(\psi, L, K) \). In fact, take \( u(t) \equiv (L + K)/2 \) for \( t \in [0, 1] \), then
\[
u \in \{y \in P(\psi, L, K) : \psi(y) > L\}.
\]

Moreover, for \( y \in P(\psi, L, K) \), then \( \psi(y) = \min_{k \in \{0, 1, \ldots, m\}} \min_{t \in [a_k, b_k]} y(t) \geq L \) and \( \|y\| \leq K \), so for each \( k \in \{0, 1, \ldots, m\} \), we have
\[
y(t) \in [L, K] \quad \text{for} \quad t \in [a_k, b_k].
\]

This together with (3.8) yields
\[
\psi(Ay) = \min_{k \in \{0, 1, \ldots, m\}} \min_{t \in [a_k, b_k]} \left( \int_0^1 H(t, s)h(s)f(y(s)) \, ds + \sum_{j=1}^m W_j(t, y) \right)
\geq \min_{k \in \{0, 1, \ldots, m\}} \min_{t \in [a_k, b_k]} \int_{a_k}^{b_k} H(t, s)h(s)f(y(s)) \, ds
\geq f(L) \min_{k \in \{0, 1, \ldots, m\}} \min_{t \in [a_k, b_k]} \int_{a_k}^{b_k} H(t, s)h(s) \, ds > L.
\]

So condition (H1) of Theorem 1.1 is satisfied.
Finally, we assert that if \( y \in P(\psi, L, R) \) and \( \|Ay\| > K \), then \( \psi(Ay) > L \). To see this, let \( y \in P(\psi, L, R) \) and \( \|Ay\| > K \). Now (3.6) and Lemma 2.2 imply that
\[
\|Ay\| \leq M_1 \int_0^1 H(s, s)h(s)f(y(s))\,ds + \sum_{j=1}^m \Omega_j(y(t_j)) < M_1 \left( \int_0^1 H(s, s)h(s)f(y(s))\,ds + \sum_{j=1}^m \Omega_j(y(t_j)) \right).
\]
(3.14)

Fix \( k \in \{0, 1, \ldots, m\} \) and notice that (3.9), (3.12), (3.14) and Lemma 2.2 yield
\[
\min_{t \in [a_k, b_k]} Ay(t) = \min_{t \in [a_k, b_k]} \left( \int_0^1 H(t, s)h(s)f(y(s))\,ds + \sum_{j=1}^m W_j(t, y) \right)
\geq M_2 \int_0^1 H(s, s)h(s)f(y(s))\,ds + c_0 \sum_{j=1}^m \Omega_j(y(t_j))
\geq M \left( \int_0^1 H(s, s)h(s)f(y(s))\,ds + \sum_{j=1}^m \Omega_j(y(t_j)) \right)
\geq \frac{M}{M_1}\|Ay\| > \frac{M}{M_1}K \geq L.
\]

So we get for each \( k \in \{0, 1, \ldots, m\} \) that
\[
\psi(Ay) = \min_{k \in \{0,1,\ldots,m\}} \min_{t \in [a_k, b_k]} Ay(t) > L.
\]

Thus condition (H3) of Theorem 1.1 holds. By Theorem 1.1, \( A \) has at least three fixed points, that is, (1.1) has at least three nonnegative solutions \( y_1, y_2 \) and \( y_3 \) such that
\[
\|y_1\| < r, \quad y_2(t) > L \quad \text{for} \quad t \in [a_k, b_k], \quad k \in \{0, 1, \ldots, m\},
\]
and
\[
\|y_3\| > r \quad \text{with} \quad \min_{k \in \{0,1,\ldots,m\}} \min_{t \in [a_k, b_k]} y_3(t) < L.
\]

The proof is complete.

We work through an example to illustrate our results.

**Example 3.1.** Consider the following impulsive boundary value problem:
\[
\begin{align*}
\frac{d^2y}{dt^2}(t) + \left[ (y(t) - 1)^{1/3} + 1 \right] &= 0, \quad t \in (0, 1), \quad t \neq \frac{1}{2}, \\
\Delta y(t_1) &= \frac{1}{2}y(t_1^-), \quad t_1 = \frac{1}{2}, \\
\Delta y'(t_1) &= -\frac{2}{3}y(t_1^-), \quad t_1 = \frac{1}{2}, \\
y(0) &= 0, \quad y(\frac{1}{2}) = y(1),
\end{align*}
\]
(3.15)
where \( h(t) \equiv 1, f(y) = (y - 1)^{1/3} + 1, \alpha = 1, \eta = \frac{2}{3} \). It is easy to see that conditions (3.1)–(3.5) hold. Let \( \Omega_1(u) = 2u/3, c_0 = \frac{1}{8} \); it follows that (3.6) and (3.9) hold. Since

\[
\sup_{t \in [0,1]} \int_0^1 H(t, s) h(s) \, ds = \frac{21}{64}, \quad \min_k \min_{t \in [a_k, b_k]} \int_{a_k}^{b_k} H(t, s) h(s) \, ds = \frac{1}{32},
\]

taking \( r = 1, L = 2 \) and \( R = 91 > LM^{-1}M_1 = 90, \) then (3.7), (3.8) and (3.10) hold. So all the conditions of Theorem 3.1 hold. By Theorem 3.1, (3.15) has at least three nonnegative solutions.

Acknowledgements

This work is supported by the NNSF of China (No. 10571050 and No. 60671066), a project supported by the Scientific Research Fund of Hunan Provincial Education Department (07B041) and the Program for Young Excellent Talents in Hunan Normal University.

References