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A NOTE ON LIE NILPOTENT GROUP RINGS

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Let KG be the group algebra of a group G over a field K of characteristic $p > 0$. It is proved that the following statements are equivalent: KG is Lie nilpotent of class $\leqslant p$, KG is strongly Lie nilpotent of class $\leqslant p$ and G' is a central subgroup of order p. Also, if G is nilpotent and G' is of order p^n then KG is strongly Lie nilpotent of class $\leqslant p^n$ and both $U(KG)/\zeta(U(KG))$ and $U(KG)'$ are of exponent p^n. Here $U(KG)$ is the group of units of KG. As an application it is shown that for all $n \leqslant p+1$, $\gamma_n(\mathcal{L}(KG)) = 0$ if and only if $\gamma_n(KG) = 0$.

Let KG be the group algebra of a group G over a field K of characteristic $p > 0$ and $\mathcal{L}(KG)$ be its associated Lie ring with Lie product defined by $[x, y] = xy - yx$ for all $x, y \in KG$.

The Lie lower central chain of KG is defined by $\gamma_1(\mathcal{L}(KG)) = \mathcal{L}(KG)$; $\gamma_{n+1}(\mathcal{L}(KG)) = [\gamma_n(\mathcal{L}(KG)), \mathcal{L}(KG)]$ for $n \geqslant 1$. The strong Lie lower central chain is defined by $\gamma_1(KG) = KG$; $\gamma_{n+1}(KG) = [\gamma_n(KG), KG]KG$ for $n \geqslant 1$.

KG is said to be Lie nilpotent (strongly Lie nilpotent) of class n if n is the least positive integer such that $\gamma_{n+1}(\mathcal{L}(KG)) = 0$ $(\gamma_{n+1}(KG) = 0)$.

The main results in this note are the following two theorems.

THEOREM A. *Let K be a field of characteristic $p > 0$ and let G be a non-abelian group. Then the following statements are equivalent:*

(1) $\gamma_{p+1}(KG) = 0$;
(2) $\gamma_{p+1}(\mathcal{L}(KG)) = 0$;
(3) *G' is a central subgroup of order p.*

THEOREM B. *Let K be a field of characteristic $p > 0$ and let G be a nilpotent group such that $|G'| = p^n$, $n \geqslant 1$. Then*

(1) $\gamma_{p^n+1}(KG) = 0$
(2) *$U(KG)$ is a nilpotent group of class at most p^n, $U(KG)^{p^n} \subseteq \zeta(U(KG))$, (the centre of $U(KG)$), and $U(KG)'$ is of exponent p^n.*

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Theorem A and [4] together answer Problem 33 [**3**, p.231] for $n \leqslant p+1$ (see Corollary 3).

We start with the following:

Lemma 1. *Let $g \in \zeta(G)$ be such that $\gamma_n(KG) \subseteq (g-1)KG$. Then*

(1) $\gamma_{n+r}(KG) \subseteq (g-1)\gamma_{r+1}(KG)$ *for* $r \geqslant 0$.

(2) $\gamma_{n+m(n-1)}(KG) \subseteq (g-1)^{m+1}KG$ *for* $m \geqslant 0$.

Proof: (1) is by induction on r.

Suppose $\gamma_{n+1}(KG) \subseteq (g-1)\gamma_{i+1}(KG)$, for all $0 < i \leqslant r-1$. Then

$$\begin{aligned}\gamma_{n+r}(KG) &= [\gamma_{n+r-1}(KG),\, KG]KG\\ &\subseteq [(g-1)\gamma_r(KG),\, KG]KG, \text{ by the induction hypothesis}\\ &\subseteq (g-1)[\gamma_r(KG),\, KG]KG \text{ since } g \in \zeta(G)\\ &= (g-1)\gamma_{r+1}(KG).\end{aligned}$$

(2) is by induction on m.

Suppose $\gamma_{n+j(n-1)}(KG) \subseteq (g-1)^{j+1}KG$, for all $0 < j \leqslant m-1$. Now

$$\begin{aligned}\gamma_{n+m(n-1)}(KG) &= \gamma_{n+(m-1)(n-1)+n-1}(KG)\\ &\subseteq (g-1)\gamma_{n+(m-1)(n-1)}(KG), \text{ by (1)}\\ &\subseteq (g-1)(g-1)^m KG, \text{ by the induction hypothesis}\\ &= (g-1)^{m+1}KG, \text{ as desired.}\end{aligned}$$

□

Proof of Theorem A: (1) implies (2) is always true, since $\gamma_n(\mathcal{L}(KG)) \subseteq \gamma_n(KG)$ for all $n \geqslant 1$.

(2) implies (3). Suppose that $\gamma_{p+1}(\mathcal{L}(KG)) = 0$. Then by [**3**, Theorem V.4.4], G is nilpotent and G' is a finite p-group. Also, by [**1**],

$$\gamma_{p+1}(G) - 1 \subseteq \gamma_{p+1}(\mathcal{L}(KG))KG = 0$$

and hence $\gamma_{p+1}(G) = 1$.

Let m_i be the number of generators for $\gamma_i(G)/\gamma_{i+1}(G)$. Then $m_2 < p/(p-1)$ if $p \geqslant 3$ [**2**, Theorem 3] and $m_2 < 2$ if $p = 2$ [**2**, Theorem 2].

Clearly $m_2 = 0$ or 1. If $m_2 = 0$, then $\gamma_2(G) = \gamma_3(G)$ and so G is abelian, a contradiction. Hence $m_2 = 1$, for all p, and $\gamma_2(G)/\gamma_3(G)$ is a non-trivial cyclic group.

Further for any $x, y \in G$, by [**2**, Lemma 4(a)],

$$(x,\, y)^p - 1 = ((x,\, y) - 1)^p \in \gamma_{p+1}(\mathcal{L}(KG))KG = 0.$$

Thus, $(x, y)^p = 1$ for all $x, y \in G$, so $\gamma_2(G)/\gamma_3(G)$ is a cyclic group of order p.

Now, if $p = 2$, then $\gamma_{p+1}(\mathcal{L}(KG)) = \gamma_3(\mathcal{L}(KG)) = 0$ and thus $\gamma_3(G) - 1 \subseteq \gamma_3(\mathcal{L}(KG))KG = 0$ implies that $\gamma_3(G) = 1$. And if $p \geqslant 3$, then by [**2**, Theorem 3], $m_2(p-1) + 3m_3(p-1)/2 + 2m_4(p-1) + \ldots + (c-2)m_c(p-1) < p$, where c is the nilpotency class of G. Since $c \geqslant 2$ and $m_2 = 1$, the above inequality is possible only if $c = 2$. But then $\gamma_3(G) = 1$. Hence $\gamma_3(G) = 1$ and G' is a cyclic central subgroup of order p.

(3) implies (1). Let $G' = \langle g \rangle$, $o(g) = p$ and $g \in \zeta(G)$. Then $\gamma_2(KG) = (G'-1)KG = (g-1)KG$. Now by Lemma 1, with $n = 2$ and $m = p-1$, $\gamma_{p+1}(KG) = \gamma_{2+(p-1)(2-1)}(KG) \subseteq (g-1)^p KG = 0$. Thus, $\gamma_{p+1}(KG) = 0$. This proves the result. ∎

Let G be a group and K a field of char $K = p > 0$ such that $\gamma_n(\mathcal{L}(KG)) = 0$. If $p \geqslant n$, then by [**4**, Theorem 3.8(i)] G must be abelian. Thus if G is non-abelian and $p \geqslant n$, then $\gamma_n(\mathcal{L}(KG)) \neq 0$.

The following corollary answers Problem 33, [**3**, p.231] for $n \leqslant p+1$.

Corollary 3. *For all groups G and for all $n \leqslant p+1$, $\gamma_n(\mathcal{L}(KG)) = 0$ if and only if $\gamma_n(KG) = 0$.*

Lemma 4. *Let K be a field of characteristic $p > 0$ and let G be a nilpotent group such that $|G'| = p^n$. Then $I_K(G, G')^{p^n} = 0$, where $I_K(G, G')$ is the augmentation ideal of G' in KG.*

The proof is by induction on n and the observation that $I_K(G, G') = \gamma_2(KG)$.

Proof of Theorem B: We prove (1) by induction on n. If $n = 1$ then $|G'| = p$ and, since G is nilpotent, $G' \subseteq \zeta(G)$. Hence, G' is a central subgroup of order p. By Theorem A, $\gamma_{p+1}(KG) = 0$.

If c is the nilpotency class of G, then $\gamma_c(G)$ is central. Choose g in $\gamma_c(G) \subseteq \zeta(G) \cap G'$ such that g has order p. Let $N = \langle g \rangle$. Then N is a central subgroup of G and $|(G/N)'| = p^{n-1}$.

Now by the induction hypothesis, $\gamma_{p^{n-1}+1}(K(G/N)) = 0$.

So $\gamma_{p^{n-1}+1}(KG) \subseteq (N-1)KG = (g-1)KG$. By Lemma 1, $\gamma_{p^n+1}(KG) \subseteq (g-1)^p KG = 0$ as desired.

For (2), observe that

$$\gamma_{p^n+1}(U(KG)) - 1 \subseteq \gamma_{p^n+1}(\mathcal{L}(KG))KG \subseteq \gamma_{p^n+1}(KG) = 0.$$

Thus, $U(KG)$ is nilpotent of class at most p^n. Hence for any $\alpha, \beta \in U(KG)$, by [**3**, Lemma V.4.3], $[\beta, \alpha^{p^n}] = [\beta, \alpha, \alpha, \ldots, \alpha] = 0$. Thus $U(KG)^{p^n} \subseteq \zeta(U(KG))$. But $U(K(G/G')) \cong U(KG)/1 + I_K(G, G')$. So, $U(KG)' \subseteq 1 + I_K(G, G')$. The result now follows by Lemma 4. ∎

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