A NOTE ON LIE NILPOTENT GROUP RINGS

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Let $KG$ be the group algebra of a group $G$ over a field $K$ of characteristic $p > 0$. It is proved that the following statements are equivalent: $KG$ is Lie nilpotent of class $\leq p$, $KG$ is strongly Lie nilpotent of class $\leq p$ and $G'$ is a central subgroup of order $p$. Also, if $G$ is nilpotent and $G'$ is of order $p^n$ then $KG$ is strongly Lie nilpotent of class $\leq p^n$ and both $U(KG)/\zeta(U(KG))$ and $U(KG)'$ are of exponent $p^n$. Here $U(KG)$ is the group of units of $KG$. As an application it is shown that for all $n \leq p + 1$, $\gamma_n(L(KG)) = 0$ if and only if $\gamma_n(KG) = 0$.

Let $KG$ be the group algebra of a group $G$ over a field $K$ of characteristic $p > 0$ and $L(KG)$ be its associated Lie ring with Lie product defined by $[x, y] = xy - yx$ for all $x, y \in KG$.

The Lie lower central chain of $KG$ is defined by $\gamma_1(L(KG)) = L(KG)$; $\gamma_{n+1}(L(KG)) = [\gamma_n(L(KG)), L(KG)]$ for $n \geq 1$. The strong Lie lower central chain is defined by $\gamma_1(KG) = KG$; $\gamma_{n+1}(KG) = [\gamma_n(KG), KG]KG$ for $n \geq 1$.

$KG$ is said to be Lie nilpotent (strongly Lie nilpotent) of class $n$ if $n$ is the least positive integer such that $\gamma_{n+1}(L(KG)) = 0$ ($\gamma_{n+1}(KG) = 0$).

The main results in this note are the following two theorems.

**Theorem A.** Let $K$ be a field of characteristic $p > 0$ and let $G$ be a non-abelian group. Then the following statements are equivalent:

1. $\gamma_{p+1}(KG) = 0$;
2. $\gamma_{p+1}(L(KG)) = 0$;
3. $G'$ is a central subgroup of order $p$.

**Theorem B.** Let $K$ be a field of characteristic $p > 0$ and let $G$ be a nilpotent group such that $|G'| = p^n$, $n \geq 1$. Then

1. $\gamma_{p^n+1}(KG) = 0$
2. $U(KG)$ is a nilpotent group of class at most $p^n$, $U(KG)^{p^n} \subseteq \zeta(U(KG))$, (the centre of $U(KG)$), and $U(KG)'$ is of exponent $p^n$.

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We start with the following:

**LEMMA 1.** Let $g \in \zeta(G)$ be such that $\gamma_n(KG) \subseteq (g - 1)KG$. Then

1. $\gamma_{n+r}(KG) \subseteq (g - 1)\gamma_{r+1}(KG)$ for $r \geq 0$.
2. $\gamma_{n+m(n-1)}(KG) \subseteq (g - 1)^{m+1}KG$ for $m \geq 0$.

**PROOF:** (1) is by induction on $r$.

Suppose $\gamma_{n+1}(KG) \subseteq (g - 1)\gamma_{i+1}(KG)$, for all $0 < i \leq r - 1$. Then

$$\gamma_{n+r}(KG) = [\gamma_{n+r-1}(KG), KG]KG$$
$$\subseteq [(g - 1)\gamma_r(KG), KG]KG, \text{ by the induction hypothesis}$$
$$\subseteq (g - 1)[\gamma_r(KG), KG]KG \text{ since } g \in \zeta(G)$$
$$= (g - 1)\gamma_{r+1}(KG).$$

(2) is by induction on $m$.

Suppose $\gamma_{n+j(n-1)}(KG) \subseteq (g - 1)^{j+1}KG$, for all $0 < j \leq m - 1$. Now

$$\gamma_{n+m(n-1)}(KG) = \gamma_{n+(m-1)(n-1)+n-1}(KG)$$
$$\subseteq (g - 1)\gamma_{n+(m-1)(n-1)}(KG), \text{ by (1)}$$
$$\subseteq (g - 1)(g - 1)^mKG, \text{ by the induction hypothesis}$$
$$= (g - 1)^{m+1}KG, \text{ as desired.}$$

**PROOF OF THEOREM A:** (1) implies (2) is always true, since $\gamma_n(\mathcal{L}(KG)) \subseteq \gamma_n(KG)$ for all $n \geq 1$.

(2) implies (3). Suppose that $\gamma_{p+1}(\mathcal{L}(KG)) = 0$. Then by [3, Theorem V.4.4], $G$ is nilpotent and $G'$ is a finite $p$-group. Also, by [1],

$$\gamma_{p+1}(G) - 1 \subseteq \gamma_{p+1}(\mathcal{L}(KG))KG = 0$$

and hence $\gamma_{p+1}(G) = 1$.

Let $m_i$ be the number of generators for $\gamma_i(G)/\gamma_{i+1}(G)$. Then $m_2 < p/(p - 1)$ if $p \geq 3$ [2, Theorem 3] and $m_2 < 2$ if $p = 2$ [2, Theorem 2].

Clearly $m_2 = 0$ or 1. If $m_2 = 0$, then $\gamma_2(G) = \gamma_3(G)$ and so $G$ is abelian, a contradiction. Hence $m_2 = 1$, for all $p$, and $\gamma_2(G)/\gamma_3(G)$ is a non-trivial cyclic group.

Further for any $x, y \in G$, by [2, Lemma 4(a)],

$$(x, y)^p - 1 = ((x, y) - 1)^p \in \gamma_{p+1}(\mathcal{L}(KG))KG = 0.$$
Thus, \((x, y)^p = 1\) for all \(x, y \in G\), so \(\gamma_2(G)/\gamma_3(G)\) is a cyclic group of order \(p\).

Now, if \(p = 2\), then \(\gamma_{p+1}(L(KG)) = \gamma_3(L(KG)) = 0\) and thus \(\gamma_3(G) - 1 \subseteq \gamma_3(L(KG))KG = 0\) implies that \(\gamma_3(G) = 1\). And if \(p > 3\), then by [2, Theorem 3],
\[
m_2(p - 1) + 3m_3(p - 1)/2 + 2m_4(p - 1) + \ldots + (c - 2)m_c(p - 1) < p,
\]
where \(c\) is the nilpotency class of \(G\). Since \(c \geq 2\) and \(m_2 = 1\), the above inequality is possible only if \(c = 2\). But then \(\gamma_3(G) = 1\). Hence \(\gamma_3(G) = 1\) and \(G'\) is a cyclic central subgroup of order \(p\).

(3) implies (1). Let \(G' = \langle g \rangle\), \(o(g) = p\) and \(g \in \zeta(G)\). Then \(\gamma_2(KG) = (G'/1 - 1)KG = (g - 1)KG\). Now by Lemma 1, with \(n = 2\) and \(m = p - 1\),
\[
\gamma_{p+1}(KG) = \gamma_2 + (p - 1)(2 - 1)(KG) \subseteq (g - 1)^p KG = 0.
\]
Thus, \(\gamma_2(KG) = 0\). This proves the result. 

Let \(G\) be a group and \(K\) a field of char \(K = p > 0\) such that \(\gamma_n(L(KG)) = 0\). If \(p \geq n\), then by [4, Theorem 3.8(i)] \(G\) must be abelian. Thus if \(G\) is non-abelian and \(p \geq n\), then \(\gamma_n(L(KG)) \neq 0\).

The following corollary answers Problem 33, [3, p.231] for \(n \leq p + 1\).

**Corollary 3.** For all groups \(G\) and for all \(n \leq p + 1\), \(\gamma_n(L(KG)) = 0\) if and only if \(\gamma_n(KG) = 0\).

**Lemma 4.** Let \(K\) be a field of characteristic \(p > 0\) and let \(G\) be a nilpotent group such that \(|G'| = p^n\). Then \(I_K(G', G') = 0\), where \(I_K(G, G')\) is the augmentation ideal of \(G'\) in \(KG\).

The proof is by induction on \(n\) and the observation that \(I_K(G, G') = \gamma_2(KG)\).

**Proof of Theorem B:** We prove (1) by induction on \(n\). If \(n = 1\) then \(|G'| = p\) and, since \(G\) is nilpotent, \(G' \subseteq \zeta(G)\). Hence, \(G'\) is a central subgroup of order \(p\). By Theorem A, \(\gamma_{p+1}(KG) = 0\).

If \(c\) is the nilpotency class of \(G\), then \(\gamma_c(G)\) is central. Choose \(g\) in \(\gamma_c(G) \subseteq \zeta(G) \cap G'\) such that \(g\) has order \(p\). Let \(N = \langle g \rangle\). Then \(N\) is a central subgroup of \(G\) and \(|(G/N)|' = p^{n-1}\).

Now by the induction hypothesis, \(\gamma_{p^n-1+1}(K(G/N)) = 0\).

So \(\gamma_{p^n-1+1}(KG) \subseteq (N - 1)KG = (g - 1)KG\). By Lemma 1, \(\gamma_{p^n+1}(KG) \subseteq (g - 1)^p KG = 0\) as desired.

For (2), observe that \(\gamma_{p^n+1}(U(KG)) - 1 \subseteq \gamma_{p^n+1}(L(KG))KG \subseteq \gamma_{p^n+1}(KG) = 0\).

Thus, \(U(KG)\) is nilpotent of class at most \(p^n\). Hence for any \(\alpha, \beta \in U(KG)\), by [3, Lemma V.4.3], \([\beta, \alpha^{p^n}] = [\beta, \alpha, \alpha, \ldots, \alpha] = 0\). Thus \(U(KG)p^n \subseteq \zeta(U(KG))\). But \(U(K(G/G')) \cong U(KG)/1 + I_K(G, G')\). So, \(U(KG)' \subseteq 1 + I_K(G, G')\). The result now follows by Lemma 4.
REFERENCES


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