NOTE ON EXTREME FORMS

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Let $f(x_1, \ldots, x_n) = \sum a_{ij}x_ix_j$ be a positive definite quadratic form of determinant $D = |a_{ij}|$, and let M be the minimum of f for integral $x_1, \ldots, x_n$ not all zero. The form f is said to be *extreme* if the ratio M^n/D does not increase when the coefficients a_{ij} of f suffer any sufficiently small variation.

All extreme forms in n variables are known for $n \leqslant 5$. Hofreiter **(2)** investigated the problem of finding all extreme forms in 6 variables and listed four forms; but, as is pointed out by Coxeter **(1)**, one of these (F_4) is certainly not extreme. Coxeter **(1)** actually finds independently four extreme forms (including three of the four listed by Hofreiter) and makes the reasonable suggestion that the list is now complete.

The main purpose of this note is to show that there is an extreme form in 6 variables not given by these authors, namely

$$f(x_1, \ldots, x_6) = \left(\sum_{i=1}^{6} x_i\right)^2 + \sum_{j=1}^{3} \phi(x_j, x_{3+j}), \tag{1}$$

where generally

$$\phi(x, y) = x^2 - xy + y^2,$$

for which

$$M = 2, \ D = \frac{13 \cdot 3^3}{2^6}.$$

The form (1) is the particular case $n = 2r = 6$ of the form

$$f(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} x_i\right)^2 + \sum_{j=1}^{r} \phi(x_j, x_{r+j}) + \sum_{k=2r+1}^{n} x_k^{\,2}, \tag{2}$$

where $n \geqslant 2r \geqslant 2$ (and the last sum is empty if $n = 2r$). I show here that *the form* (2) *is extreme if and only if*

$$4r - 2 \geqslant n \geqslant 2r \geqslant 6. \tag{3}$$

We first examine all integral sets $x_1, \ldots, x_n$, not all zero, for which

$$f \leqslant 2. \tag{4}$$

Noting that

$$\phi(x, y) \begin{cases} = 0 \text{ if } \quad (x, y) = (0, 0), \\ = 1 \text{ if } \pm(x, y) = (1, 0), \ (0, 1) \text{ or } (1, 1), \\ \geqslant 3 \text{ otherwise,} \end{cases} \tag{5}$$

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we see that (4) implies that

$$\phi(x_j, x_{r+j}) \leqslant 1 \qquad (j = 1, \ldots, r), \tag{6}$$

and that equality can hold in (6) for at most two values of j.

If, say, $\phi(x_1, x_{r+1}) = \phi(x_2, x_{r+2}) = 1, \phi(x_j, x_{r+j}) = 0$ $(j \geqslant 3)$, then

$$x_j = x_{r+j} = 0 \ (3 \leqslant j \leqslant r),$$

and (4) requires $2f = 2$, $x_k = 0$ $(k \geqslant 2r+1)$, $x_1 + x_2 + x_{r+1} + x_{r+2} = 0$. Using (5), we see that all possible sets $(x_1, x_2, x_{r+1}, x_{r+2})$ are $\pm(1, -1, 0, 0)$, $\pm(1, 0, 0, -1)$, $\pm(0, 1, -1, 0)$, $\pm(0, 0, 1, -1)$, $\pm(1, -1, 1, -1)$.

If only one $\phi(x_j, x_{r+j})$ is non-zero, say $\phi(x_1, x_{r+1})$, then $x_j = x_{r+j} = 0$ $(2 \leqslant j \leqslant r)$ and $x_{2r+1}^2 + \ldots + x_n^2 \leqslant 1$. There are thus two possibilities:

(i) $x_{2r+1} = \ldots = x_n = 0, \quad f = 2x_1^2 + 2x_{r+1}^2 + x_1 x_{r+1},$

and trivially $f \geqslant 2$, with equality only if

$$(x_1, x_{r+1}) = \pm(1, 0), \pm(0, 1);$$

(ii) $x_k = \pm 1$ for some $k \geqslant 2r+1$,

$$f = (x_1 + x_{r+1} + x_k)^2 + 2,$$

and so $f \geqslant 2$, with equality only if

$$(x_1, x_{r+1}, x_k) = \pm(1, 0, -1), \pm(0, 1, -1).$$

If finally all $\phi(x_j, x_{r+j})$ are zero, then

$$f = \left(\sum_{k=2r+1}^{n} x_k\right)^2 + \sum_{k=2r+1}^{n} x_k^2,$$

and it is easily seen that $f \geqslant 2$, with equality only if some x_k is ± 1 and the rest zero, or if some pair $(x_k, x_l) = \pm(1, -1)$ and the remaining x_k are zero.

We have thus shown that $M = 2$. Also, if with each pair $\pm(x_1^{(t)}, \ldots x_n^{(t)})$ of minimal vectors (i.e., sets with $f = 2$), we associate the linear form

$$\lambda_t = \lambda_t(y_1, \ldots, y_n) = \sum_{i=1}^{n} x_i^{(t)} y_i,$$

we see (using the obvious symmetries of f) that there are precisely

$$s = \tfrac{1}{2}n(n+1) + \tfrac{1}{2}r(r-3) \tag{7}$$

such linear forms, given by

$$\begin{array}{lr} y_i & (1 \leqslant i \leqslant n), \\ y_j - y_k & (1 \leqslant j < k \leqslant n, k \neq r+j;\ j = 1, \ldots, r), \\ y_l - y_m + y_{r+l} - y_{r+m} & (1 \leqslant l < m \leqslant r). \end{array} \tag{8}$$

If $r \geqslant 3$, it is easily verified that f is uniquely determined by the fact that it has minimum 2 and associated linear forms (8); thus f is *perfect* (in the sense of Voronoï **(3)**). If however $r < 3$, (7) gives $s < \frac{1}{2}n(n+1)$ and so f

cannot be perfect. We have therefore shown that f *is perfect if and only if* $r \geqslant 3$.

Now Voronoï **(3)** has shown that a form is extreme if and only if it is perfect and eutactic. We therefore consider next the problem of deciding when f is eutactic, that is to say, when its adjoint $F(y_1, \ldots, y_n)$ is expressible as

$$F(y_1, \ldots, y_n) = \sum_{t=1}^{s} \rho_t \lambda_t^2, \rho_t > 0 \qquad (t = 1, \ldots, s). \tag{9}$$

The labour of calculating F (and the determinant D of f) may be lightened by using the following method:

The form

$$g(z_1, \ldots, z_n) = (\alpha_1 z_1 + \ldots + \alpha_n z_n)^2 + \sum_{i=1}^{n} z_i^2 \tag{10}$$

is easily found to have determinant

$$D(g) = 1 + \sum_{i=1}^{n} \alpha_i^2, \tag{11}$$

and adjoint a multiple of

$$G(z_1, \ldots, z_n) = \sum c_{ij} z_i z_j \tag{12}$$

with

$$c_{ii} = 1 + \sum_{\substack{k=1 \\ k \neq i}}^{n} \alpha_k^2, \; c_{ij} = -\alpha_i \alpha_j \qquad (i \neq j).$$

Under the linear transformation T defined by

$$\left.\begin{aligned} x_j &= z_j + (1/\sqrt{3}) z_{r+j} \\ x_{r+j} &= (2/\sqrt{3}) z_{r+j} \end{aligned}\right\} \qquad (j = 1, \ldots, r),$$

$$x_k = z_k \qquad (k = 2r + 1, \ldots, n),$$

f in (2) is reduced to the form (10) with

$$\alpha_i = 1 \;\; (1 \leqslant i \leqslant r, 2r + 1 \leqslant i \leqslant n), \quad \alpha_i = \sqrt{3} \;\; (r + 1 \leqslant i \leqslant 2r), \tag{13}$$

so that

$$1 + \sum_{i=1}^{n} \alpha_i^2 = n + 2r + 1. \tag{14}$$

Since T has determinant $(2/\sqrt{3})^r$, it follows from (11) and (14) that f has determinant

$$D = (\tfrac{3}{4})^r (n + 2r + 1). \tag{15}$$

Finally, a straightforward multiplication of matrices now shows that $F(y_1, \ldots, y_n)$ is a multiple of $\sum b_{ij} y_i y_j$ with

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$$b_{ii} = \begin{cases} 4(n+2r-2), & 1 \leqslant i \leqslant 2r, \\ 3(n+2r), & i > 2r, \end{cases}$$

(16) $$b_{ij} = \begin{cases} -12, & 1 \leqslant i < j \leqslant 2r, j-i \neq r, \\ 2(n+2r-5), & 1 \leqslant i < j \leqslant 2r, j-i = r, \\ -6, & i \leqslant 2r, j > 2r, \\ -3, & j > i > 2r. \end{cases}$$

Corresponding to the enumeration (8) of the associated linear forms, we write (9) as

(17) $$F(y_1, \ldots, y_n) = \sum \rho_i y_i^2 + \sum \sigma_{jk}(y_j - y_k)^2 + \sum \tau_{lm}(y_l - y_m + y_{r+l} - y_{r+m})^2,$$

where the suffixes have the ranges given in (8), and solve (17) for the $s = \frac{1}{2}n(n+1) + \frac{1}{2}r(r-3)$ coefficients ρ_i, σ_{jk}, τ_{lm}.

First, we have immediately

(18) $$\sigma_{jk} = -b_{jk} = 3, \qquad k > j > 2r,$$

(19) $$\sigma_{jk} = -b_{jk} = 6, \qquad k > 2r \geqslant j.$$

The coefficient of $2y_j y_k$ for $1 \leqslant j < k \leqslant 2r$, $k - j \neq r$ is $-\sigma_{jk} - \tau_{lm}$, where $l = j$ or $j - r$, $m = k$ or $k - r$; hence we have

(20) $$\sigma_{jk} = \sigma_{j,r+k} = \sigma_{r+j,r+k} = 12 - \tau_{jk}, \qquad 1 \leqslant j < k \leqslant r.$$

The coefficient of $2y_j y_{r+j}$, for $1 \leqslant j \leqslant r$, is

(21) $$\tau_{1j} + \ldots + \tau_{j-1,j} + \tau_{j,j+1} + \ldots + \tau_{jr} = 2(n+2r-5) \qquad (j = 1, \ldots, r).$$

The coefficient of y_1^2 is

$$\begin{aligned} &\rho_1 + \sigma_{12} + \ldots + \sigma_{1n} + \tau_{12} + \ldots + \tau_{1r} \\ &\quad = \rho_1 + (12 - \tau_{12}) + \ldots + (12 - \tau_{1r}) + (12 - \tau_{12}) + \ldots + (12 - \tau_{1r}) \\ &\qquad\qquad + 6(n - 2r) + \tau_{12} + \ldots + \tau_{1r} \\ &\quad = \rho_1 + 12(2r-2) + 6(n-2r) - (\tau_{12} + \ldots + \tau_{1r}) \\ &\quad = \rho_1 + 4n + 8r - 14, \end{aligned}$$

using (19), (20) and (21); since $b_{11} = 4(n + 2r - 2)$, it follows that $\rho_1 = 6$. The same argument gives

(22) $$\rho_i = 6 \qquad 1 \leqslant i \leqslant 2r.$$

The coefficient of y_{2r+1}^2 is

$$\begin{aligned} &\rho_{2r+1} + \sigma_{1,2r+1} + \ldots + \sigma_{2r,2r+1} + \sigma_{2r+1,2r+2} + \ldots + \sigma_{2r+1,n} \\ &\qquad = \rho_{2r+1} + 6(2r) + 3(n - 2r - 1), \end{aligned}$$

using (18) and (19); since $b_{2r+1,2r+1} = 3(n + 2r)$, it follows that $\rho_{2r+1} = 3$. The same argument gives

(23) $$\rho_i = 3, \qquad 2r + 1 \leqslant i \leqslant n.$$

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Now (20) and (21) give

$$\begin{aligned}\sigma_{12} + \ldots + \sigma_{1r} &= (12 - \tau_{12}) + \ldots + (12 - \tau_{1r}) \\ &= 12(r-1) - 2(n + 2r - 5) \\ &= 2(4r - 1 - n);\end{aligned}$$

if the σ_{ij} are all strictly positive, this shows that $n < 4r - 1$. Thus f *is not eutactic if* $n \geqslant 4r - 1$.

If, however, $n \leqslant 4r - 2$ and $r \geqslant 3$, we can show that f is eutactic by taking the particular solution

$$\tau_{lm} = \frac{2(n + 2r - 5)}{r - 1} \qquad (1 \leqslant l < m \leqslant r) \tag{24}$$

of the r equations (21). Then (20) gives

$$\sigma_{jk} = 12 - \frac{2(n + 2r - 5)}{r - 1} = \frac{2(4r - 1 - n)}{r - 1} > 0 \tag{25}$$

for all relevant j, k, and we have exhibited a solution of (17) in which all the coefficients ρ_i, σ_{jk}, τ_{lm} are positive.

We have now established our assertion that f is extreme if and only if (3) holds. In particular, we have shown that the senary form (1), for which $n = 2r = 6$, is extreme.

The form (2) gives some information on the possible structure of perfect, eutactic and extreme forms, as well as extending Coxeter's table (**1**, p. 439) of extreme forms for each $n \geqslant 6$.

Thus Coxeter remarks (**1**, p. 396): "For every known perfect form in less than nine variables there is a solution [of (9)] with the ρ's all equal." However, for the form (2), there is no such solution for any $n \geqslant 2r \geqslant 6$. This assertion is clear if $n > 2r$, from (22) and (23); if $n = 2r$, it follows from (24) and (25), since equality of the τ's and σ's would require

$$4r - 5 = 2r - 1, \quad r = 2.$$

Coxeter also remarks (**1**, p. 392): "We do not know whether every perfect form is extreme." The form (2), however, is perfect and non-eutactic (and so not extreme) for any $r \geqslant 3$, $n \geqslant 4r - 1$.

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