8

The loop representation of quantum gravity

8.1 Introduction

Having cast general relativity as a Hamiltonian theory of a connection, we are now in a position to apply the same techniques we used to construct a loop representation of Yang–Mills theories to the gravitational case. We should recall that we are dealing with a complex $SU(2)$ connection. However, we can use exactly the same formulae that we developed in chapter 5 since few of them depend on the reality of the connections. Whenever the presence of a complex connection introduces changes, we will discuss this explicitly .

As we have seen, we can introduce a loop representation either through a transform or through the quantization of a non-canonical algebra. The initial steps are exactly the same as those in the $SU(2)$ Yang–Mills case. The differences arise when we want to write the constraint equations. In the Yang–Mills case the only constraint was the Gauss law and one had to represent the Hamiltonian in terms of loops. In the case of gravity one has to impose the diffeomorphism and Hamiltonian constraints in terms of loops. In order to do so one can either use the transform or write them as suitable limits of the operators in the T algebra. We will outline both derivations for the sake of comparison. As we argued in the Yang–Mills case both derivations are formal and in a sense equivalent, although the difficulties are highlighted in slightly different ways in the two derivations.

The space of states of an $SU(2)$ theory in terms of the loop representation has been discussed in detail in chapter 3. It is formed by wavefunctions with support on the group of loops,

$$\Psi(\gamma), \tag{8.1}$$

that satisfy the basic Mandelstam identities,

$$\Psi(\gamma) = \Psi(\gamma^{-1}), \tag{8.2}$$

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$$\Psi(\gamma\circ\eta)=\Psi(\eta\circ\gamma), \tag{8.3}$$

$$\Psi(\gamma\circ\eta\circ\beta)+\Psi(\gamma\circ\eta\circ\beta^{-1})=\Psi(\eta\circ\gamma\circ\eta)+\Psi(\eta\circ\gamma\circ\beta^{-1}), \tag{8.4}$$

and by combination of these identities one can find an infinite number of linear relations among the wavefunctions.

In many papers on the subject, multiloops have been used to build the loop representation. As we discussed in chapter 3, for the $SU(2)$ case all expressions in terms of multiloops can be rewritten as single-loop expressions via Mandelstam identities. We will therefore restrict ourselves here to single-loop wavefunctions.

The outline of this chapter is as follows. In the next two sections we will derive the expression of the constraints of quantum gravity in the loop representation both as a limit of the T algebra and via the loop transform. We will then discuss the regularization of the Hamiltonian in terms of loops and briefly discuss the solution space. We will return to the issue of solution to the constraints in chapters 10 and 11.

8.2 Constraints in terms of the T algebra

We need to write the classical diffeomorphism and Hamiltonian constraints in terms of the T operators. It is quite simple to write the diffeomorphism constraint as a limit of a T^1 operator. Consider a one-parameter family $\gamma^{\delta}_{\hat{a}\hat{b}}(x)$ of closed curves in the $\hat{a}\hat{b}$ coordinate plane base-pointed at the point x such that in the limit $\delta\to 0$ the loops shrink to a point. The area element of the loop is given by

$$\sigma^{cd}(\gamma^{\delta}_{\hat{a}\hat{b}})=\delta^2\delta^{[c}_{\hat{a}}\delta^{d]}_{\hat{b}}. \tag{8.5}$$

The diffeomorphism constraint is given by the limit

$$C(\vec{N})=\lim_{\delta\to 0}\frac{1}{\delta^2}\int d^3x N^{\hat{a}}(x)T^{\hat{b}}(\gamma^{\delta}_{\hat{a}\hat{b}}(x)). \tag{8.6}$$

To prove this, notice that in this limit the holonomy is given by

$$\lim_{\delta\to 0}\mathbf{H}(\gamma^{\delta}_{ab}(x))=\mathbf{1}+\tfrac{1}{2}\sigma^{cd}(\gamma_{\hat{a}\hat{b}}(x))\mathbf{F}_{cd}(x). \tag{8.7}$$

When one takes the trace to construct the T^1, the contribution from the identity drops out because of the tracelessness of the triad and the leading contributions is $\mathrm{Tr}(\tilde{\mathbf{E}}^a(x)\mathbf{F}_{ab}(x))$, which corresponds with the usual expression of the vector constraint.

A remarkable fact is that the constraint algebra is consistently reproduced in the limit

$$\begin{array}{ccc} C^\delta(\vec{N}) & \rightarrow & \{C^\delta(\vec{N}), C^\delta(\vec{M})\} \\ \downarrow & & \downarrow \\ C(\vec{N}) & \rightarrow & \{C(\vec{N}), C(\vec{M})\} \end{array} \tag{8.8}$$

i.e., computing the Poisson bracket of two T^1s and shrinking the loops yields the same results as shrinking the loops and computing the Poisson algebra of the constraints [139].

To obtain the Hamiltonian constraint we will introduce a double limiting procedure, which in what follows will be useful as a regularization procedure for the quantum calculation. We will consider the point-split classical Hamiltonian,

$$\begin{aligned} C(\underset{\sim}{N}) = \lim_{\epsilon\to 0} C^\epsilon(\underset{\sim}{N}) = \lim_{\epsilon\to 0} \int d^3x \underset{\sim}{N}(x) \int d^3y f_\epsilon(x-y) \\ \times \mathrm{Tr}(\tilde{\mathbf{E}}^a(y)\mathbf{H}(\mu_y^x)\tilde{\mathbf{E}}^b(x)\mathbf{F}_{ab}(x)\mathbf{H}(\mu_x^y)), \end{aligned} \tag{8.9}$$

where we have introduced an arbitrary infinitesimal path μ_y^x. The introduction of this path is needed in order to have a gauge invariant point-split Hamiltonian. Since the T variables are gauge invariant it would be impossible to retrieve a non-invariant quantity from them. The contribution from the holonomy $\mathbf{H}(\mu_y^x)$ reduces to the identity in the limit.

We will present a shrinking loop procedure that will yield the split constraint $C^\epsilon(\underset{\sim}{N})$, and from there one recovers the usual constraint in the limit $\epsilon \to 0$. We introduce a one-parameter family of shrinking loops as before $\gamma^\delta_{\hat{a}\hat{b}}(x)$. The Hamiltonian constraint is given by

$$C^\epsilon(\underset{\sim}{N}) = \lim_{\delta\to 0} \frac{1}{\delta^2} \int d^3x \underset{\sim}{N}(x) \int d^3y f_\epsilon(x-y) T^{[\hat{a}\hat{b}]}(\mu_x^y, \mu_y^x \circ \gamma^\delta_{\hat{a}\hat{b}}(x)). \tag{8.10}$$

The proof follows similar lines as before: in the shrinking limit the holonomy yields two contributions; the one proportional to the identity vanishes due to the antisymmetrization in the $\hat{a}\hat{b}$ indices (if not one would get the metric $\mathrm{Tr}(\tilde{\mathbf{E}}^a\tilde{\mathbf{E}}^b)$ as leading contribution) and the term proportional to $\mathbf{F}_{ab}$ yields the constraint.

We therefore have classical expressions relating the constraints and the T operators. This allows us to find expressions for the constraints as quantum mechanical operators by promoting their definitions in terms of the T quantities to quantum mechanical operators. The quantum mechanical expressions for the T operators were introduced in chapter 5, choosing a factor ordering with the triads to the right. We recall here their expression

$$\hat{T}^0(\eta)\Psi(\gamma) \equiv \Psi(\gamma\circ\eta) + \Psi(\gamma\circ\eta^{-1}), \tag{8.11}$$

$$\hat{T}^a(\eta^x_x)\Psi(\gamma) \equiv \sum_{\epsilon=-1}^{1} \epsilon \oint dy^a \delta(x-y)\Psi(\gamma \circ \eta^\epsilon), \tag{8.12}$$

$$\begin{aligned}\hat{T}^{ab}(\eta^y_x, \eta^x_y)\Psi(\gamma) = & X^{ax}(\gamma)X^{by}(\gamma)[\Psi(\gamma^y_x \circ \bar{\eta}^x_y, \gamma^x_y \circ \bar{\eta}^y_x) \\ & +\Psi(\gamma^y_x \circ \eta^x_y, \gamma^x_y \circ \eta^y_x) + \Psi(\gamma^y_x \circ \bar{\eta}^x_y \circ \bar{\gamma}^y_x \circ \eta^x_y) \\ & +\Psi(\gamma^x_y \circ \bar{\eta}^y_x \circ \bar{\gamma}^x_y \circ \eta^y_x)].\end{aligned} \tag{8.13}$$

We now promote the relation (8.6) to an operatorial equation,

$$\begin{aligned}\hat{C}(\vec{N})\Psi(\gamma) = & \lim_{\delta\to 0} \frac{1}{\delta^2} \int d^3x N^{\hat{a}}(x) \hat{T}^{\hat{b}}(\gamma^\delta_{\hat{a}\hat{b}}(x))\Psi(\gamma) \\ = & \lim_{\delta\to 0} \frac{-1}{2\delta^2} \int d^3x N^{\hat{a}}(x) \sum_{\epsilon=-1}^{1} \epsilon \oint dy^{\hat{b}} \delta(x-y)\Psi(\gamma \circ (\gamma^\delta_{\hat{a}\hat{b}}(x))^\epsilon),\end{aligned} \tag{8.14}$$

and we notice that the introduction of the infinitesimal loop $\gamma^\delta_{\hat{a}\hat{b}}(x)$ with the two possible orientations given by the power ϵ corresponds to the action of the loop derivative. Since the loop derivative along the reversed loop introduces a minus sign the two contributions $\epsilon = \pm 1$ add up to give

$$\hat{C}(\vec{N})\Psi(\gamma) = -\int d^3x N^a(x) \oint dy^b \delta(x-y)\Delta_{ab}(\gamma^x_o)\Psi(\gamma) \tag{8.15}$$

and we see that the diffeomorphism constraint in the loop representation can be obtained in the limit of shrinking loops from the T^1 operator. As the derivation shows, the loop derivative arises because the action of the T^1 operator corresponds to the introduction of a small loop of precisely the same form as in the loop derivative.

The Hamiltonian constraint can be obtained through manipulations that are very similar to those of the diffeomorphism constraint. Since the final expression coincides exactly with the one we will obtain in the next section via the loop transform we do not give the explicit calculation. For details see reference [139]. We will just outline the first steps of the calculation to facilitate the comparison with the expression that we derive in the next secion. We need to compute

$$\begin{aligned}T^{ab}(\mu^y_x, \mu^x_y \circ {\gamma^\delta}_{ab}(x))\Psi(\gamma) = & X^{ax}(\gamma)X^{bx}(\gamma)[\Psi(\gamma^y_x \circ \mu^x_y, \bar{\gamma}^\delta_{ab}(x) \circ \mu^y_x \circ \gamma^x_y) \\ & +\Psi(\gamma^y_x \circ \mu^x_y {\gamma^\delta}_{ab}(x), \mu^y_x \gamma^x_y) + \Psi(\gamma^y_x \circ \mu^x_y \circ \bar{\gamma}^y_x \circ \mu^x_y \circ {\gamma^\delta}_{ab}(x)) \\ & +\Psi(\gamma^x_y \circ \bar{\gamma}^\delta_{ab}(x) \circ \mu^y_x \circ \bar{\gamma}^x_y \circ \mu^y_x)]\end{aligned} \tag{8.16}$$

and using the Mandelstam identities and recalling that we are only interested in the antisymmetric part of T^{ab} we get

$$\hat{C}^\epsilon(\underset{\sim}{N}) = \lim_{\delta\to 0} \frac{1}{\delta^2} \int d^3x \underset{\sim}{N}(x) \int d^3y f_\epsilon(x-y) X^{[a|x}(\gamma)X^{b]y}(\gamma)$$

$$\times \Big[2\Psi(\gamma^{\delta}{}_{ab}(x) \circ \gamma_x^y \circ \mu_y^x \circ \bar{\gamma}_x^y \circ \mu_y^x)$$
$$+2\Psi(\gamma^{\delta}{}_{ab}(x) \circ \bar{\gamma}x^y \circ \mu_y^x \circ \gamma_x^y \circ \mu_y^x)\Big] \tag{8.17}$$

which, taking into account the definition of the loop derivative, yields a regularized expression for the Hamiltonian constraint that we will present in an explicit fashion in the next section.

8.3 Constraints via the loop transform

To obtain the quantum version of the constraints via the loop transform, we proceed in the same way as we did for an $SU(2)$ Yang–Mills theory in chapter 6. There is a difference, however, due to the fact that the connection in the general relativity case is complex. In principle, its complex conjugate is a complicated expression given by the reality conditions. Therefore we cannot quite write for the transform as we did in chapter 6,

$$\Psi(\gamma) = \int dAW_\gamma[A]^*\Psi[A] \tag{8.18}$$

since the expression for $W_\gamma[A]^*$ would, in principle, be a complicated non-polynomial expression in terms of A. Moreover, as we argued before, it is not clear that one wants to implement the reality conditions at this level. One may want to impose them later as relations among observables of the theory.

In order to be able to proceed we will assume in the following manipulations that A is real. This is not unjustified, since the manipulations in terms of real As yield operator expressions in the loop representation that have exactly the same commutation relations as their counterparts in the connection representation. In this sense the loop transform is a very useful heuristic device for finding appropriate loop counterparts to operators in the connection representation. The reader should be aware that the following calculations are heuristic and not meant to be precise derivations. It is remarkable that through this procedure one can recover exactly the same expression for the constraints as we did in the previous section. This suggests that a measure may exist such that the manipulations can be made rigorous taking into account the complex nature of the connections.

We therefore define

$$\hat{O}\Psi(\gamma) \equiv \int dAW_\gamma[A]\hat{O}\Psi[A] = \int dA\hat{O}^\dagger W_\gamma[A]\Psi[A], \tag{8.19}$$

where by $\hat{O}^\dagger$ we mean the operator $\hat{O}$ but with a reverse factor ordering. Therefore the practical calculation of transforming an operator consists in evaluating its action on a Wilson loop as if it were a calculation in the

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connection representation and rearranging the result as a manipulation purely in terms of loops. One should remember that when considering the action on the Wilson loop one should choose for the operator one wishes to transform the *opposite* factor ordering to the one chosen for its action on wavefunctions $\Psi[A]$.

We start with the vector constraint. Its action on a Wilson loop is given by

$$F^i_{ab}(x)\frac{\delta}{\delta A^i_a(x)}W_\gamma[A] = F^i_{ab}(x)\oint_\gamma dy^a\delta(y-x)\mathrm{Tr}(\mathbf{H}(\gamma_o^y)\tau^i\mathbf{H}(\gamma_y^o))$$
$$= \oint_\gamma dy^a\delta(y-x)\mathrm{Tr}(\mathbf{H}(\gamma_o^x)\mathbf{F}_{ab}\mathbf{H}(\gamma_x^o)). \quad (8.20)$$

Recalling the action of a loop derivative on a Wilson loop introduced in chapter 1 we get

$$F^i_{ab}(x)\frac{\delta}{\delta A^i_a(x)}W_\gamma[A] = \oint_\gamma dy^a\delta(y-x)\Delta_{ab}(\gamma_o^y)W_\gamma[A], \quad (8.21)$$

and therefore we can write for the diffeomorphism constraint in the loop representation

$$\hat{C}(\vec{N}) = \int d^3xN^b(x)\oint_\gamma dy^a\delta(x-y)\Delta_{ab}(\gamma_o^y). \quad (8.22)$$

This is exactly the expression we introduced in the first chapter as the generator of diffeomorphisms on functions of the group of loops and we checked in that chapter that it satisfied the correct algebra of diffeomorphisms,

$$[\hat{C}(\vec{N}),\hat{C}(\vec{M})] = \hat{C}(\mathcal{L}_{\vec{N}}\vec{M}). \quad (8.23)$$

Sometimes one may use the shorthand notation

$$\hat{C}(\vec{N}) = \int d^3xN^b(x)X^{ax}(\gamma)\Delta_{ab}(\gamma_o^x), \quad (8.24)$$

where $X^{ax}(\gamma)$ is the first order multitangent to the loop, but care should be exercised if the loop has multiple points (intersections).

The reader may appreciate the remarkable fact that a formalism so heuristic in nature manages to yield the expected result. We started with the action of the diffeomorphism constraint in the connection representation and by the most direct and obvious manipulation we end up with an expression with the desired geometric action in terms of loops. Encouraged by this result we will follow the same procedure for the Hamiltonian constraint.

The calculations for the Hamiltonian constraint are of the same nature, the only care to be taken is the presence of a second functional derivative, which requires a regularization. We will perform here only a formal calculation in order to simplify the presentation, we postpone the discussion of regularization issues to the next section. In fact, at the formal level we have already performed the required calculation in the previous chapter,

$$\begin{aligned}\hat{\mathcal{H}}(x)W_\gamma[A] &= \epsilon^{ijk}F^i_{ab}(x)\frac{\delta}{\delta A^j_a}\frac{\delta}{\delta A^k_b}\\ &= F^k_{ab}(x)\epsilon_{ijk}\oint_\gamma dy^b\oint_{\gamma_o^y} dz^a\delta(x-y)\delta(x-z)\mathrm{Tr}(\tau^i\mathbf{H}(\gamma_z^y)\tau^j\mathbf{H}(\gamma_{y\,o}^{z}))\\ &\quad+F^k_{ab}(x)\epsilon_{ijk}\oint_\gamma dy^b\oint_{\gamma_y^{o\prime}} dz^a\delta(x-y)\delta(x-z)\mathrm{Tr}(\tau^j\mathbf{H}(\gamma_y^z)\tau^i\mathbf{H}(\gamma_{z\,o}^{y})).\end{aligned} \tag{8.25}$$

We now rearrange this expression using the identity,

$$i\epsilon^{lmn}\mathrm{Tr}(\tau^m\mathbf{A}\tau^n\mathbf{B}) = \mathrm{Tr}(\tau^l\mathbf{A})\mathrm{Tr}(\mathbf{B}) - \mathrm{Tr}(\mathbf{A})\mathrm{Tr}(\tau^l\mathbf{B}), \tag{8.26}$$

where $\mathbf{A},\mathbf{B}$ are $SU(2)$ matrices. The integrands can then be rewritten as

$$\begin{aligned}\epsilon^{ijk}\mathrm{Tr}(\tau^i\mathbf{H}(\gamma_z^y)\tau^j\mathbf{H}(\gamma_{y\,o}^{z})) &= \mathrm{Tr}(\tau^k\mathbf{H}(\gamma_z^y))\mathrm{Tr}(\mathbf{H}(\gamma_{y\,o}^{z}))\\ &\quad-\mathrm{Tr}(\mathbf{H}(\gamma_z^y))\mathrm{Tr}(\tau^k\mathbf{H}(\gamma_{y\,o}^{z})),\end{aligned} \tag{8.27}$$

$$\begin{aligned}\epsilon^{ijk}\mathrm{Tr}(\tau^j\mathbf{H}(\gamma_y^z)\tau^i\mathbf{H}(\gamma_{z\,o}^{y})) &= \mathrm{Tr}(\tau^k\mathbf{H}(\gamma_{z\,o}^{y}))\mathrm{Tr}(\mathbf{H}(\gamma_y^z))\\ &\quad-\mathrm{Tr}(\mathbf{H}(\gamma_{z\,o}^{y}))\mathrm{Tr}(\tau^k\mathbf{H}(\gamma_y^z)),\end{aligned} \tag{8.28}$$

and noticing that

$$\mathrm{Tr}(\mathbf{H}(\gamma_{y\,o}^{z})) = \mathrm{Tr}(\mathbf{H}(\gamma_{z\,o}^{y})), \tag{8.29}$$

$$\mathrm{Tr}(\tau^k\mathbf{H}(\gamma_z^y)) = -\mathrm{Tr}(\tau^k\mathbf{H}(\gamma_y^z)), \tag{8.30}$$

we get for the action of the Hamiltonian,

$$\begin{aligned}\hat{\mathcal{H}}(x)W_\gamma[A] &= F^k_{ab}(x)(\oint_\gamma dy^b\oint_{\gamma_o^y} dz^a + \oint_\gamma dy^b\oint_{\gamma_y^o} dz^a)\delta(x-y)\delta(x-z)\\ &\quad\times\mathrm{Tr}(\tau^k\mathbf{H}(\gamma_{z\,o}^{y}))\mathrm{Tr}(\mathbf{H}_y^z) - \mathrm{Tr}(\mathbf{H}_{z\,o}^{y})\mathrm{Tr}(\tau^k\mathbf{H}(\gamma_y^z)).\end{aligned} \tag{8.31}$$

The two sets of integrals can be combined into a single one, and inserting the F^i_{ab} in the holonomies we get,

$$\begin{aligned}\hat{\mathcal{H}}(x)W_\gamma[A] &= \oint_\gamma dy^b\oint_\gamma dz^a\delta(x-y)\delta(x-z)\\ &\quad\times\mathrm{Tr}(\mathbf{F}_{ab}(x)\mathbf{H}(\gamma_{z\,o}^{y}))\mathrm{Tr}(\mathbf{H}(\gamma_y^z) - \mathrm{Tr}(\mathbf{H}_{z\,o}^{y})\mathrm{Tr}(\mathbf{F}_{ab}(x)\mathbf{H}(\gamma_y^z)).\end{aligned} \tag{8.32}$$

We now rearrange the products of holonomies into a single one using the generalization of the Mandelstam identities when elements of the algebra are involved. One could have left the expression as it was and then the

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action of the Hamiltonian constraint on a wavefunction of a single loop would be a function of a multiloop. This has been the approach taken in some papers [138]. Here, as we said before, we reexpress everything in terms of single loops. The identity needed is

$$\mathrm{Tr}(\tau^i \mathbf{A})\mathrm{Tr}(\mathbf{B}) = \mathrm{Tr}(\tau^i(\mathbf{AB} + \mathbf{AB}^{-1})) = \mathrm{Tr}(\tau^i(\mathbf{BA} + \mathbf{B}^{-1}\mathbf{A})), \quad (8.33)$$

where $\mathbf{A}, \mathbf{B}$ are again elements of the $SU(2)$ group. Rearranging terms with this identity, we get

$$\begin{aligned}\hat{\mathcal{H}}(x)W_\gamma[A] = &\oint_\gamma dy^{[b} \oint_\gamma dz^{a]} \delta(x-y)\delta(x-z) \\ &\times \mathrm{Tr}(\mathbf{F}_{ab}(x)[\mathbf{H}(\gamma_y^z)\mathbf{H}(\gamma_{y\,o}^{\,z}) + \mathbf{H}(\gamma_{y\,o}^{\,z})\mathbf{H}(\gamma_y^z)]), \quad (8.34)\end{aligned}$$

We can rearrange this expression in terms of loop derivatives,

$$\begin{aligned}\hat{\mathcal{H}}(x)W_\gamma[A] = &\oint_\gamma dy^{[b} \oint_\gamma dz^{a]} \delta(x-y)\delta(x-z) \\ &\times \Delta_{ab}(\gamma_o^x)\mathrm{Tr}([\mathbf{H}(\gamma_y^z \circ \gamma_{y\,o}^{\,z}) + \mathbf{H}(\gamma_{y\,o}^{\,z} \circ \gamma_y^z)]), \quad (8.35)\end{aligned}$$

from which we can read off the expression of the Hamiltonian constraint in the loop representation,

$$\begin{aligned}\hat{\mathcal{H}}(\underset{\sim}{N})\Psi(\gamma) = &\int d^3x \underset{\sim}{N}(x) \oint_\gamma dy^{[b} \oint_\gamma dz^{a]} \delta(x-y)\delta(x-z) \\ &\times \Delta_{ab}(\gamma_o^x)[\Psi(\gamma_y^z \circ \gamma_{y\,o}^{\,z}) + \Psi(\gamma_{y\,o}^{\,z} \circ \gamma_y^z)]. \quad (8.36)\end{aligned}$$

It should be pointed out that the notation in the above two expressions for the loop derivative precisely means

$$\Delta_{ab}(\gamma_o^x)\Psi(\gamma_y^z \circ \gamma_{y\,o}^{\,z}) \equiv \Delta_{ab}(\beta_o^x)\Psi(\beta)|_{\beta=\gamma_y^z \circ \gamma_{y\,o}^{\,z}} \quad (8.37)$$

and similarly for the action of the loop derivative on the holonomy. From now on we will use this notation whenever the Hamiltonian constraint is involved. Again, this expression coincides with the one introduced in the previous section directly obtained as a limit of the T operators. We see that the two approaches yield the same constraints.

One can perform another rearrangement that simplifies the expression of the Hamiltonian constraint even further. Going back to the expression in terms of F_{ab}^i (8.34), there are two terms in the expression of the Hamiltonian. Each of them is a trace of an element of the algebra times elements of the group. Such traces are equal to minus the trace of the inverse argument. If one replaces the argument of the second trace by its inverse, one obtains exactly the same expression as the argument of the first trace, with y and z exchanged. One can relabel y and z in the second term (one gains an additional minus sign from the antisymmetrization in $dy^{[a}dz^{b]}$) and one gets back (in the limit in which the regulator is removed) the same term as the first one. Continuing with the derivation as presented above one gets for the final action of the Hamiltonian

$$H(\underset{\sim}{N})\Psi(\gamma) = 2\int d^3x \underset{\sim}{N}(x) \oint_\gamma dy^{[b} \oint_\gamma dz^{a]} \delta(x-y)\delta(x-z) \times \Delta_{ab}(\gamma_o^x)\Psi(\gamma_y^z \circ \gamma_{y\,o}^z). \tag{8.38}$$

Because the equality presented only holds in the limit in which the regulator is removed, the above expression can be thought of as a different regularization of the Hamiltonian constraint introduced before.

It is remarkable that such a compact expression embodies all the information of the time evolution of the Einstein equations in the language of loops.

The constraint algebra involving the Hamiltonian constraint that we derived above has been computed at the formal level in reference [141] and it reproduces the classical algebra. Care should be exercised when computing the constraint algebra, since the problem necessarily requires a regularization, as has been emphasized in the papers by Tsamis and Woodard[142] and Friedman and Jack [143]. The formal computation of the constraints is useful, however, to illustrate a series of computational techniques in loop space and to clarify the meaning of the expressions of the constraints in the loop representation.

8.4 Physical states and regularization

In the previous section we found expressions for the Hamiltonian and diffeomorphism constraints of quantum gravity in the loop representation. In this section we will discuss the construction of solutions to these constraints. We will start with the diffeomorphism constraint and then we will analyze the Hamiltonian. We will elaborate further on the Hamiltonian constraint in chapters 9, 10 and 11. In order to operate properly with the quantum constraints on wavefunctions we will be required to study the regularization of the constraints.

8.4.1 Diffeomorphism constraint

Let us start with the diffeomorphism constraint. In section 1.3.4 we showed that the diffeomorphism constraint acts on functions of loops by infinitesimally deforming the loop argument along a vector $\vec{N}$. The deformation is the same that the loop would suffer if it existed in a spatial manifold on which a diffeomorphism is performed along a vector $\vec{N}$. Therefore if a wavefunction $\Psi(\gamma)$ in the loop representation is to be annihilated by the diffeomorphism constraint it should be invariant under deformations of the loop argument. Such functions are known as knot invariants. Another way of putting this is to say that the function only depends on the knot class of the loop. The knot class of a loop is given by the orbit of the diffeomorphism group in loop space that contains the given loop.

Therefore by considering such functions of loops one immediately solves the diffeomorphism constraint. The diffeomorphism invariance of general relativity therefore is very elegantly coded into knot invariance in the loop representation. There is an abundant literature on the study of knot invariants, and we will return in more detail to issues of knot theory in the next chapter. Notice that the situation is qualitatively different from that in the traditional variables for quantum gravity. There one considered functionals of a spatial metric $\Psi[q]$. The invariance under diffeomorphisms implied that one was dealing with functionals of the "geometry" (or more precisely its diffeomorphism invariant properties) rather than functionals of a metric. The situation is also qualitatively different from that in the connection representation that we discussed in the previous chapter. Again, there one had to consider functions of a connection that were invariant under diffeomorphisms $\Psi[A]$. Although some isolated examples of these can be given, it is quite evident that one can construct many more examples of functions of loops invariant under diffeomorphisms. For instance, functions that depend on the number of intersections of a loop or the number of corners or kinks in the loops are examples of functions that are invariant under diffeomorphisms. So are the "characteristic functions" in loop space: functions that give 1 if the argument is in a certain knot class and zero otherwise. Although we have seen that the use of loops played a role in the connection representation, we see that the shift in point of view offered by the loop representation is very important in the task of finding the physical states that are annihilated by the constraints. We will find many solutions to the constraints in the loop representation of which the counterpart in terms of connections is either not known and is expected to be quite complicated or ill defined. Knot theory captures in a natural way the non-local, topological properties of a theory invariant under diffeomorphisms. The connection between knot theory and quantum gravity was first noticed by Rovelli and Smolin [38].

8.4.2 Hamiltonian constraint: formal calculations

In order to discuss the solutions to the Hamiltonian constraint one needs to introduce a regularization. The issue of the regularization of the Hamiltonian constraint is the subject of intense investigations at present. Basically the problem is that all known regularization procedures are difficult to make compatible with diffeomorphism invariance and typically intro-

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duce conflicts or ambiguities in the resulting regularized theory. We will first introduce a point-splitting regularization in loop space and discuss the action of the Hamiltonian constraint on a generic function of loop $\Psi(\gamma)$. We will *not* at the moment assume that the function is invariant under deformations of the loops, i.e., the state will not, in general, be annihilated by the diffeomorphism constraint. This is the most natural thing to do, since the Hamiltonian constraint is an operator that is not invariant under diffeomorphisms and therefore its action is not well defined on the space of knot invariants. In general the action of the Hamiltonian on a knot invariant will produce as a result a function of a loop that is not invariant under diffeomorphisms.

There is a second motivation for considering the action of the Hamiltonian on all function of loops, related to the details of the definitions we give for the constraints. This is due to the fact that the loop derivative that we defined in chapter 1 is not, in general, well defined on functions that are invariant under diffeomorphisms. This can be readily seen. The notion of a loop derivative involves, in general, a change of topology in the loop. Therefore in its definition,

$$\Psi(\pi_o^x \circ \delta\gamma \circ \pi_x^o \circ \gamma) = (1 + \tfrac{1}{2}\sigma^{ab}(x)\Delta_{ab}(\pi_o^x))\Psi(\gamma), \tag{8.39}$$

it could happen that the loop argument of Ψ in the left-hand side is in a different knot class that that of the right-hand side. The addition of the infinitesimal loop would therefore not amount to a small change in the loop function and the limit involved in the derivative is not well defined. The situation is the loop analog of the derivative of the Heaviside theta function at the origin in elementary calculus. The usual way to deal with this problem (that leads to the calculus of distributions) is to consider the Heaviside function as a limit of a set of differentiable functions. Similarly here we would like to regard the functions invariant under diffeomorphisms as suitable limits of non-invariant functions that are loop differentiable. The action of the Hamiltonian constraint on a diffeomorphism invariant function will also be defined in a limiting process.

There have been several proposals for the Hamiltonian constraint in the loop representation [39, 138, 139, 16, 140]. Some of them do not involve the use of loop derivatives or use derivatives that are different from the one we introduce in this book. All of them, however, are based on the idea of appending an infinitesimal loop to the knot and therefore do not have a clear and unambiguous topological action in terms of knots.

We consider the Hamiltonian introduced in the last section

$$\begin{aligned}\hat{\mathcal{H}}(\underset{\sim}{N})\Psi(\gamma) = \int d^3x \underset{\sim}{N}(x) \oint_\gamma dy^{[b} \oint_\gamma dz^{a]} \delta(x-y)\delta(x-z) \\ \times \Delta_{ab}(\gamma_o^x)\Psi(\gamma_y^z \circ \gamma_{y\,o}^z).\end{aligned} \tag{8.40}$$

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As we pointed out before, the above expression is formal and a regularization is needed for its proper definition. Before discussing the regularization let us qualitatively study the action of the formal constraint on a function of a loop. Taking the results from the connection representation as a guide, we know that the action of the Hamiltonian constraint is different if loops with and without intersections are involved. In the loop representation wavefunctions must take values for *all* piecewise differentiable loops. We will therefore study separately the action of the Hamiltonian constraint on a generic loop function Ψ assuming that the argument is a smooth loop, a loop with a kink or a loop with intersections.

The action of the formal Hamiltonian on a function of a loop $\Psi(\gamma)$ is very simple in the case in which the argument is a smooth non-intersecting loop at the point where the Hamiltonian acts. In that case, in the formal expression of the Hamiltonian there is a single contribution per point x belonging to the loop γ. The contribution is proportional (through a divergent factor) to the double contraction of the tangent to the loop at the point with the loop derivative $\dot{\gamma}^a\dot{\gamma}^b\Delta_{ab}\Psi(\gamma)$ (where $\dot{\gamma}^a$ is the tangent vector to the loop in a certain parametrization). Since one is contracting a symmetric tensor with an antisymmetric one the result vanishes. This is the counterpart in the loop representation of the same result that we found in the connection representation at the formal level: non-intersecting smooth loops yield solutions of the Hamiltonian constraint. In general, the action of the Hamiltonian involves a splitting and rerouting of the argument of the wavefunction. For the case of non-intersecting loops or kinks, the contribution gives back the same loop as the original one since $\gamma_y^z \to \gamma$ and $\gamma_{y\,o}^{\,z} \to \iota$ in the limit (or vice-versa depending on the order of y and z along the loop). The rerouting is non-trivial only at intersections. At the formal level of this discussion, the Hamiltonian has a non-vanishing contribution at intersections and kinks but not at points where the loops are smooth.

The fact that the Hamiltonian constraint has a (formally) vanishing action at points where loops are smooth and non-intersecting led [38, 39] to the construction of a historically very important set of "physical states" of quantum gravity by simply considering wavefunctions $\Psi(\gamma)$ with support only on smooth non-intersecting loops, i.e.,

$$\Psi(\gamma) = \begin{cases} \Psi_0(\gamma) & \text{if } \gamma \text{ is smooth and non-intersecting,} \\ 0 & \text{otherwise,} \end{cases} \tag{8.41}$$

where $\Psi_0(\gamma)$ is any knot invariant. Formally the Hamiltonian has vanishing action on this state since it gives no contribution if the loop γ is either smooth (for the reasons explained above) or intersecting (since the state vanishes for such loops). This state has the appearance of a "step function" in loop space. The reader may question the applicability of a

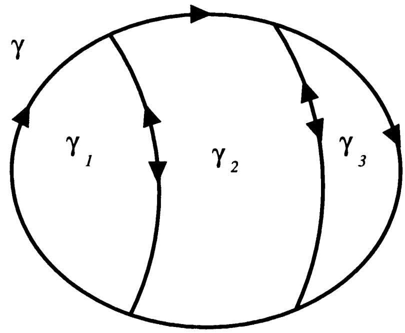

Fig. 8.1. The loop used in the Mandelstam identity that is not satisfied by the naive states

differential operator in loop space to such a state. In principle, the action could be well defined since the Hamiltonian in this case does not change the number of intersections of the loop and therefore has a separate action in the two regions into which the definition of the state partitions the loop space.

Unfortunately, there is a serious objection to these kinds of naive states. This was noticed by Rovelli and Smolin ([39] page 135). The problem is that, as we emphasized at the beginning of this chapter and throughout this book, a state in the loop representation is not any function of a loop, but has to satisfy several properties, among them the Mandelstam identities. The Mandelstam identities imply relations among the values that a wavefunction takes when evaluated on loops with and without intersection. It is easy to check that the above proposed wavefunctions do *not* satisfy the appropriate relations. For instance, consider a non-intersecting loop γ obtained by the composition of loops γ_1, γ_2 and γ_3 as shown in the figure 8.1, and apply the Mandelstam identity

$$\Psi(\gamma_1\circ\gamma_2\circ\gamma_3)+\Psi(\gamma_1\circ\gamma_2\circ\gamma_3^{-1}) = \Psi(\gamma_2\circ\gamma_1\circ\gamma_3)+\Psi(\gamma_2\circ\gamma_1\circ\gamma_3^{-1}). \quad (8.42)$$

The first term in the left-hand side is $\Psi(\gamma)$ and all the other terms involve intersections (and multiple lines) between the different components. Therefore the state has vanishing value on all the terms in the expression except on the first where it is $\Psi_0(\gamma)$ and one is led to the contradiction: $\Psi_0(\gamma) = 0$.

One could think of constructing a set of states motivated by the non-intersecting ones by assigning proper values to loops with intersections via the Mandelstam identity. This was suggested in reference[39]. Very recently, the introduction of the spin-network [146] ideas gave a concrete meaning to this construction. There is rapid development at present in trying to exploit these states for physical purposes [144].

There is another way in which states based on non-intersecting loops can be thought of as generating genuine solutions to the Hamiltonian constraint, using the notions of bras and kets. Consider the space of kets $|\Psi>$ and let us assume that we know an inner product in loop space such that the Hamiltonian is a self-adjoint operator (notice that the inner product is not on the physical space but on all states). We define the bra $<\alpha|$ by

$$\Psi(\alpha) \equiv <\alpha|\Psi> . \tag{8.43}$$

Notice that the bras, from their definition, satisfy the Mandelstam identities, for instance $<\alpha| = <\alpha^{-1}|$, etc.

By definition, the action of the Hamiltonian on $\Psi(\alpha)$ is

$$\hat{H}\Psi(\alpha) \equiv <\alpha|H|\Psi>, \tag{8.44}$$

from which one can immediately read off the action of the Hamiltonian on a bra $<\alpha|$, being given by the usual expression in the loop representation. If one now considers a bra $<\alpha|$ with α a smooth loop then $<\alpha|\hat{H}| = 0$. Making use of the assumption that the Hamiltonian is a self-adjoint operator one has that $\hat{H}|\alpha>= 0$ and therefore

$$<\gamma|\hat{H}|\alpha>= \hat{H}\Psi_\alpha(\gamma) = 0. \tag{8.45}$$

That is, if one knows the inner product in the space of loops under which the Hamiltonian is a self-adjoint operator, one can construct a family of functions of loops $\Psi_\alpha(\gamma)$ (where the smooth non-intersecting loop α plays the role of a parameter) that are annihilated by the Hamiltonian constraint simply by taking the inner product $<\gamma|\alpha>$. These states satisfy the Mandelstam constraint. Notice that the wavefunctions depend on a loop γ that can have arbitrary intersections and kinks. Though this construction constitutes an interesting observation, the fact that it relies on the introduction of an inner product in loop space under which the Hamiltonian is self-adjoint makes it of little use in practice.

There is a chance that one could modify the definition of the naive states in order make them compatible with the Mandelstam constraints. In particular, Smolin[145] has a proposal based on the use of an area operator; however, it is not clear whether under the proposed modification one still manages to solve the Hamiltonian constraint.

Let us now discuss the regularized action of the Hamiltonian constraint.

8.4.3 Hamiltonian constraint: regularized calculations

We again consider the Hamiltonian introduced in the last section,

$$\hat{\mathcal{H}}(\underset{\sim}{N})\Psi(\gamma) = \int d^3x \underset{\sim}{N}(x) \oint_\gamma dy^{[b} \oint_\gamma dz^{a]} \delta(x-y)\delta(x-z)$$

$$\times\Delta_{ab}(\gamma_o^x)\Psi(\gamma_y^z \circ \gamma_{y\,o}^z), \tag{8.46}$$

but we point split one of the Dirac delta functions,

$$\hat{\mathcal{H}}_\epsilon(\underset{\sim}{N})\Psi(\gamma) = \int d^3x \underset{\sim}{N}(x) \oint_\gamma dy^{[b} \oint_\gamma dz^{a]}\delta(x-y) f_\epsilon(y-z)$$
$$\times\Delta_{ab}(\gamma_o^y)\Psi(\gamma_y^z \circ \gamma_{y\,o}^z), \tag{8.47}$$

where $f_\epsilon(y-z)$ is a usual symmetric regulator. For the sake of concreteness, we can consider a family of Gaussians,

$$f_\epsilon(x-y) = (\pi\epsilon)^{-3/2} \exp\left(\frac{-|x-y|^2}{\epsilon}\right). \tag{8.48}$$

One can consider other families of regulators, like families of Heaviside functions $f_\epsilon(x,y) = \Theta_\epsilon(x,y)/\epsilon^3$ where $\Theta_\epsilon(x,y) = 3/4\pi$ if $|x-y| < \epsilon$ and zero otherwise. The background metric enters in all cases since one has to compute the distance between x and y.

Notice that there are several possibilities to regularize and the regularized expressions will, in general, be different and coincide only in the limit. For instance, we could have split the other delta function that appears in the definition of the Hamiltonian.

The introduction of the point-splitting implies that the paths that appear in the expression of the regularized constraint do not close a loop. This is equivalent to the introduction of a non-gauge invariant point-splitting in the connection representation, the breaking of gauge invariance being manifest in the loop representation in the appearance of open paths. When the regulators are removed, the open ends of paths coincide and one recovers closed loops and gauge invariance. One could simply choose to work in a regularized framework with open loops and recover gauge invariance only as a limit after regularization. Another procedure is to close the loops by adding arbitrary small paths and restore gauge invariance in the regularized expressions. In the limit, the contributions from the added paths drop out. In the connection representation one does not have any privileged paths to restore gauge invariance in the point-splitting. In the loop representation one can always choose to close the loops through their original trajectory before reroutings and splittings, as was done in references [138, 139], or through other prescribed paths [16, 39]. Notice that these constructions hide implicit assumptions about the behavior of the wavefunctions of loops $\Psi(\gamma)$. It is not true that for all functions the contributions of the infinitesimal added paths drop out in the limit. These kinds of statements imply a certain notion of continuity of the functions in loop space that at the moment is not well understood.

Let us now redo the calculation of the action of the Hamiltonian constraint acting on a function of loops in the case in which it acts on a point of the loop that has no kinks nor intersections. To make the calculation as explicit as possible we introduce a parametrization for the loop $\gamma(s)^a$ with $s \in [0,1]$ and we rewrite the Hamiltonian (8.47),

$$\begin{aligned}\hat{\mathcal{H}}_\epsilon(\underset{\sim}{N})\Psi(\gamma) = \int_0^1 ds \int_0^1 dt \dot{\gamma}^{[b}(s)\dot{\gamma}^{a]}(t)\underset{\sim}{N}(\gamma(s))\\ \times f_\epsilon(\gamma(s)-\gamma(t))\Delta_{ab}(\gamma_o^s)\Psi(\gamma_s^t \circ \gamma_{s\,o}^{\,t}).\end{aligned} \tag{8.49}$$

We now split the integral in t,

$$\begin{aligned}\hat{\mathcal{H}}_\epsilon(\underset{\sim}{N})\Psi(\gamma) = \left(\int_0^1 ds \int_s^1 dt + \int_0^1 ds \int_0^s dt\right) \dot{\gamma}^{[b}(s)\dot{\gamma}^{a]}(t)\\ \times \underset{\sim}{N}(\gamma(s)) f_\epsilon(\gamma(s)-\gamma(t))\Delta_{ab}(\gamma_0^s)\Psi(\gamma_s^t \circ \gamma_{s\,o}^{\,t}).\end{aligned} \tag{8.50}$$

The above expression involves open loops, as we discussed. One needs to close them appending infinitesimal loops going from s to t in one of the terms and from t to s in the other. Since we assume the point of action is smooth, there is no ambiguity in the closing process and one gets $\gamma_s^t \circ \gamma_{s\,o}^{\,t} \to \gamma^{-1}$ when $t > s$ and $\gamma_s^t \circ \gamma_{s\,o}^{\,t} \to \gamma$ when $t < s$.

If we now replace, in the limit $\epsilon \to 0$, $\dot{\gamma}^a(t) \to \dot{\gamma}^a(s) + \ddot{\gamma}^a(s)(t-s)$ and $\gamma^a(s) - \gamma^a(t) \to \dot{\gamma}^a(s)(s-t)$, the terms involving two tangent vectors cancel out, exactly as they did in the formal calculation. Introducing the variable u, defined as $t-s$ for the first integral and $s-t$ for the second, one is left with

$$\hat{\mathcal{H}}_\epsilon(\underset{\sim}{N})\Psi(\gamma) = 2\int_0^1 \int_0^1 dsdu\, u\dot{\gamma}^{[b}(s)\ddot{\gamma}^{a]}(s)\underset{\sim}{N}(\gamma(s)) f_\epsilon(u\,\dot{\gamma}(s))\Delta_{ab}(\gamma_0^s)\Psi(\gamma) \tag{8.51}$$

and noticing that with the Gaussian regulator

$$\frac{\epsilon}{2|\dot{\gamma}|^2}\partial_u f_\epsilon(u\,\dot{\gamma}(s)) = -u f_\epsilon(u\,\dot{\gamma}(s)), \tag{8.52}$$

we get for the leading action of the Hamiltonian,

$$\hat{\mathcal{H}}_\epsilon(\underset{\sim}{N})\Psi(\gamma) = -\frac{1}{\pi^{3/2}\epsilon^{1/2}}\int_0^1 ds \frac{\dot{\gamma}^{[b}(s)\ddot{\gamma}^{a]}(s)}{|\dot{\gamma}(s)|^2}\underset{\sim}{N}(\gamma(s))\Delta_{ab}(\gamma_0^s)\Psi(\gamma). \tag{8.53}$$

We see that the action of the Hamiltonian is divergent. This will be the case for all kinds of loops and points in the loop and we will be forced to define a renormalized Hamiltonian as the regulated operator that has a finite limit for $\epsilon \to 0$, i.e.,

$$\hat{H} = \lim_{\epsilon \to 0} \sqrt{\epsilon}\hat{H}^\epsilon. \tag{8.54}$$

We see that the action of the Hamiltonian constraint on a smooth point of a loop, after the constraint is appropriately regularized and renormalized, is non-vanishing, contrary to what the naive calculation suggested. The resulting terms depend on higher derivatives of the loop and are usually referred to as "acceleration terms" [134]. The result (8.53) is invariant under reparametrization of the loops but depends explicitly on a background metric through $|\dot{\gamma}(s)|^2$, reflecting the fact that the regulator we took is not invariant under diffeomorphisms.

Notice that the expression (8.53) can be reinterpreted as the action on a loop state of a diffeomorphism along the vector field $\dot{\gamma}^{[b}(s)\ddot{\gamma}^{a]}(s)/|\dot{\gamma}(s)|^2$. This is not a standard diffeomorphism along a fixed external vector field, but the vector field is defined by the loop. If the loop has intersections, then this vector field is not well defined. If the loop is smooth, however, one could construct smooth vector fields $\vec{N}$ on the manifold such that on the loop take the same value as $\dot{\gamma}^{[b}(s)\ddot{\gamma}^{a]}(s)/|\dot{\gamma}(s)|^2$ and the wavefunction should be annihilated by them (if it is invariant under diffeomorphisms). Therefore we see that the contribution from the acceleration terms vanishes if one considers wavefunctions of smooth loops that are invariant under diffeomorphisms and one can solve the Hamiltonian constraint. This is an improvement on the situation in the connection representation. As we pointed out in the previous chapter, there one also finds acceleration terms when one regulates using point-splitting and that means that the Wilson loops do not satisfy the Hamiltonian constraint. In the loop representation, since we can deal with diffeomorphism invariant states, one can make the contributions from the acceleration terms vanish. Therefore we see that — ignoring the objections already stated concerning the Mandelstam constraints — the naive states based on loops without intersections also solve the constraints when a proper regularization is taken into account.

Let us now consider the action of the Hamiltonian at a point where the loop has a kink [138], i.e., a discontinuity in the tangent vector to the curve, but there is only one line going in and out of the point, i.e., there are no intersections. Such a situation is illustrated in the figure 8.2. In the expression of the Hamiltonian there is now a contribution of lower order than in the previous case, stemming from the fact that at the point of the kink s_0 there are two possible values for the tangent to the loop which we denote $\dot{\gamma}^a_+$ and $\dot{\gamma}^a_-$. Therefore, in the formal computation one gains a term $\dot{\gamma}^a_+\dot{\gamma}^a_-\Delta_{ab}$ that does not vanish. The regularized calculation gives as result

$$\hat{\mathcal{H}}_\epsilon(\underset{\sim}{N})\Psi(\gamma) = 2\frac{\dot{\gamma}_+^{[b}\dot{\gamma}_-^{a]}\underset{\sim}{N}(x_i)}{(\pi\epsilon)^{3/2}}\int_0^1\int_0^1 ds\,dt \times\exp\left(-\frac{s^2+t^2+2st\vec{\dot{\gamma}}_+\cdot\vec{\dot{\gamma}}_-}{\epsilon}\right)\Delta_{ab}(\gamma_o^{x_i})\Psi(\gamma), \quad (8.55)$$

where x_i is the point at which the kink lies. If there were more than one kink in the loop, the expression would be the same for each of them and a

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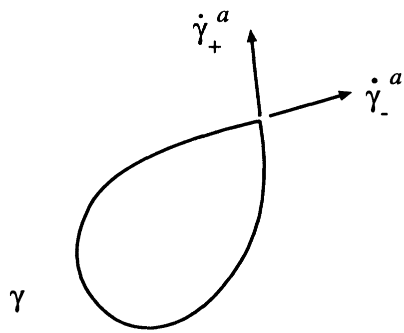

Fig. 8.2. A loop with a kink. Notice the convention for the tangent vectors $\dot\gamma^a_\pm$.

discrete sum along all the kinks should be introduced. In this expression we have assumed that a parametrization was chosen such that $|\dot\gamma_\pm|^2 = 1$.

The integral can be explicitly computed, giving

$$\hat{\mathcal{H}}_\epsilon(\underset{\sim}{N})\Psi(\gamma) = 2\frac{\dot\gamma_+^{[b}\dot\gamma_-^{a]}}{\sqrt{1-(\vec{\dot\gamma}_+\cdot\vec{\dot\gamma}_-)^2}}\frac{\underset{\sim}{N}(x_i)}{(\pi\epsilon)^{1/2}} \times\left(\frac{1}{4}-\frac{\arcsin(\vec{\dot\gamma}_+\cdot\vec{\dot\gamma}_-)}{2\pi}\right)\Delta_{ab}(\gamma_o^{x_i})\Psi(\gamma). \qquad (8.56)$$

Again, we see this contribution from the Hamiltonian has to be renormalized with a factor $\sqrt{\epsilon}$ to obtain a finite contribution. We also see that the expression is background dependent through the angle that the two tangents to the loop at the kink form measured with the background metric. The expression of the action of the Hamiltonian on a kink can be rewritten in terms of a quantity called the normalized area element,

$$\sigma_N^{ab}(\gamma) = \frac{\dot\gamma_+^{[b}\dot\gamma_-^{a]}}{\sqrt{1-(\vec{\dot\gamma}_+\cdot\vec{\dot\gamma}_-)^2}}. \qquad (8.57)$$

The word normalized is used in the sense that the norm of the vector dual to the area element is independent of the angle of the tangent vectors of the loop and therefore is independent of the background metric introduced for the regularization. The normalized area element is ill defined when the two tangent vectors coincide. However the product that appears in the action of the Hamiltonian on a kink,

$$\sigma_N^{ab}(\gamma)\left(\frac{1}{4}-\frac{\arcsin(\vec{\dot\gamma}_+\cdot\vec{\dot\gamma}_-)}{2\pi}\right), \qquad (8.58)$$

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is well defined. It vanishes in the limit in which the two tangent vectors are the same and therefore the loop is smooth. This agrees with the result that we derived before in which the tangent–tangent contribution to the Hamiltonian at smooth points vanished, the leading order being given by the acceleration terms. We will notice a different behavior in the case of intersections.

It is remarkable that much like in the case of the acceleration terms, the action of the Hamiltonian on a kink can be reduced to a diffeomorphism. Consider the usual expression for the diffeomorphism constraint,

$$\hat{C}(\vec{N})\Psi(\gamma) = \int d^3x N^a(x) \oint_\gamma dy^b \delta(x-y)\Delta_{ab}(\gamma_o^y)\Psi(\gamma) \tag{8.59}$$

and consider the particular vector field

$$N^a_\epsilon(x) = \underset{\sim}{M}(x) \oint_\gamma dz^a \frac{1}{\sqrt{\pi\epsilon}} \exp\left(-\frac{|z-x|^2}{\epsilon}\right). \tag{8.60}$$

It is immediate to see that,

$$\hat{\mathcal{H}}(\underset{\sim}{M})\Psi(\gamma) = \lim_{\epsilon\to 0} \frac{1}{\pi\sqrt{\epsilon}} \hat{C}(\vec{N}_\epsilon)\Psi(\gamma). \tag{8.61}$$

Therefore we see that the action of this particular diffeomorphism on the loop state is exactly the same as that of the Hamiltonian in the regularized limit if the loop is smooth with at most a finite number of kinks and no intersections. We therefore see another difference with the connection representation, where Wilson loops with kinks simply failed to solve the Hamiltonian constraint. In the loop representation, if one considers states that have support on loops with kinks and are diffeomorphism invariant, they automatically solve the Hamiltonian constraint (again there can be a conflict with the Mandelstam identities that prevents us from considering such functions as true states of the gravitational field).

We finally discuss the case of a loop with intersections. We will focus our attention on double intersections but higher order ones are a straightforward generalization. The calculation is very similar to the one we performed for the case of kinks, except that now there are four possible contributions coming from taking the four lines adjacent to the intersection in groups of two. The contribution per pair is exactly the same as that of a single kink (8.56) with the difference that the argument of the wavefunction is not the loop γ in the regularized limit but a rerouting of the loop at the intersection takes place. The vectors $\dot{\gamma}^a_\pm$ in this case correspond to the two tangent vectors in the particular pair of lines considered. An orientation convention has to be determined *a priori* as was done in the case of the kinks in figure 8.2.

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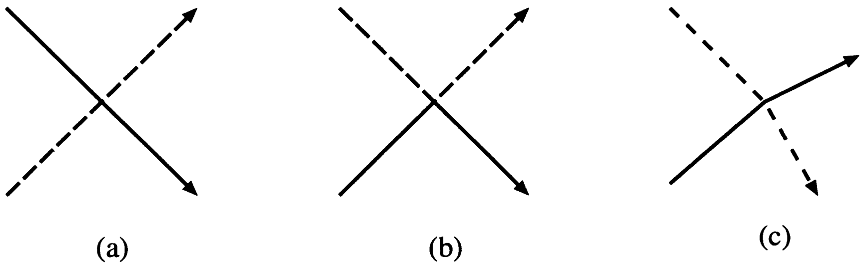

Fig. 8.3. Three different possibilities at a double intersection: (a) a straight-through intersection; (b) intersection with a kink; (c) intersection with more than two tangent vectors. Cases (b) and (c) are usually referred to as cases with "kinks at the intersection"

At a double intersection there are several different possibilities, illustrated in figre 8.3. The case of a straight-through intersection gives a qualitatively different result than the cases with kinks at the intersection. In the former case, the four contributions coming from taking the lines in pairs add up in such a way that the $\arcsin(\vec{\dot{\gamma}}_+ \cdot \vec{\dot{\gamma}}_-)$ terms in (8.56) all drop out and we get

$$\hat{\mathcal{H}}_\epsilon(\underset{\sim}{N})\Psi(\gamma) = 2\sigma^{ab}_N(\gamma)\frac{\underset{\sim}{N}(x_i)}{(\pi\epsilon)^{1/2}}\Delta_{ab}(\gamma_o^{x_i})\Psi(\gamma_{x_i}^{x_i} \circ \gamma_{x_i\,o}^{x_i}). \tag{8.62}$$

It is remarkable that the expression depends on the tangent vectors only through the normalized area element and therefore it is independent of the background metric used for the regularization. This result was first noticed by Rovelli and Smolin [140]. Unfortunately, the resulting expression is ill defined in the limit in which the tangent vectors coincide, as opposed to the case of a single kink.

If there are kinks at the intersection, the above cancellation of the $\arcsin(\vec{\dot{\gamma}}_+ \cdot \vec{\dot{\gamma}}_-)$ terms does not happen and one is left with a background dependent result. Several terms appear, some having the same rerouting effect as in the straight-through intersection but others having as the argument of the wavefunction the loop γ, as happened at a kink.

The action of the Hamiltonian on an intersection cannot be rewritten as a genuine diffeomorphism as was the case of the action on a kink or the acceleration terms. Attempts have been made to interpret the Hamiltonian at intersections in this way ("shift operator") [39, 139] but they all amount to a reinterpretation of the terms we have derived, without a genuine connection with diffeomorphisms. These reinterpretations may help to visualize the action of the Hamiltonian at intersections. At a smooth point in the loop the action of the Hamiltonian can be viewed as a diffeomorphism along the tangent to the loop.

As can be concluded from this section, the action of the regularized Hamiltonian in loop space is only non-trivial at points where the loops have intersections. The resulting action of the Hamiltonian at such points is relatively simple, it amounts to the sum of terms consisting of a straightforward rerouting of the argument of the wavefunction acted upon by a loop derivative contracted with the normalized area element of the loop at the intersection point.

At this point it is worthwhile pondering whether the point-splitting procedure introduced has been enough to produce well defined expressions for the constraints in the loop representation. The answer is positive if one makes certain assumptions about the wavefunctions considered. A strong assumption is the existence of a loop derivative of the wavefunctions. As was mentioned above, the loop derivative is ill defined for wavefunctions that are diffeomorphism invariant. In general, the action of appending an infinitesimal loop does not preserve the knot class of a given loop. Moreover, the particular way in which the infinitesimal loop is added can influence the final result. The way in which this conflict may be resolved is through the use of suitable limiting procedures for the definition of the wavefunctions, such that they are diffeomorphism invariant in the limit. Outside the limit, the loop derivative is well defined. A practical implementation of this proposal is the use of extended loops, to which we will return in chapter 11. Another proposal is to take the limits involved in a different way such that loop derivatives do not explicitly appear. We refer the reader to reference [140] for more details.

8.5 Conclusions

We have applied the loop representation ideas to the quantization of general relativity based on the Ashtekar new variables formulation. We introduced explicit expressions for the constraint equations at a formal and regularized level. We discussed some general issues concerning the space of states of the theory. In the following chapters we will discuss applications of these ideas. In the next chapter we will discuss the inclusion of matter and the use of approximations. In chapter 10 we will elaborate on the connections with knot theory. In chapter 11 we will discuss a regularization that gives rise to a new representation in terms of extended loops.