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SUMS OF DEFICIENCIES OF ALGEBROID FUNCTIONS

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Let $f(z)$ be an n-valued algebroid function of finite lower order. In the present paper, we give a spread relation of $f(z)$ and some applications of the spread relation.

1. INTRODUCTION

Let $f(z)$ be an n-valued algebroid function of finite lower order μ, defined by an irreducible equation

$$A_0 f^n + A_1 f^{n-1} + \cdots + A_{n-1} f + A_n = 0 \tag{1.1}$$

where $A_0, A_1, \cdots, A_n$ are entire functions without common zeros.

Fix a sequence (r_j) of Pólya peaks of order μ of $f(z)$ (or $T(r, f)$). Let $f_j(z)$ be the jth determination of $f(z)$ and $\Lambda(r)$ a positive function with

$$\Lambda(r) = o(T(r, f)), \quad r \to \infty. \tag{1.2}$$

Define the sets of arguments $E'_\Lambda(r, a) \subset (-\pi, \pi]$ by

$$E'_\Lambda(r, a) = \{\theta\colon \min_{1 \leqslant j \leqslant n} \left|f_j(re^{i\theta}) - a\right| < e^{\Lambda(r)}, a \neq \infty\},$$

$$E'_\Lambda(r, \infty) = \{\theta\colon \max_{1 \leqslant j \leqslant n} \left|f_j(re^{i\theta})\right| > e^{\Lambda(r)}\},$$

and let

$$\sigma'_\Lambda(a) = \liminf_{j \to \infty} \operatorname{meas} E'_\Lambda(r_j, a)$$

$$\sigma'(a) = \inf_\Lambda \sigma'_\Lambda(a)$$

where the infimum is taken over all functions $\Lambda(r)$ satisfying (1.2). Niino ([5]) proved the following spread relation

$$\sigma'(a) \geqslant \min\left\{2\pi, \frac{4}{\mu} \arcsin \sqrt{\frac{\delta(a, f)}{2}}\right\}. \tag{1.3}$$

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Now we assume that

$$\|A(z)\| = \left(|A_0|^2 + |A_1|^2 + \cdots + |A_n|^2\right)^{1/2},$$

$$\|a\| = \begin{cases} \left(|a|^{2n} + |a|^{2n-2} + \cdots + |a|^2 + 1\right)^{1/2}, & a \neq \infty \\ 1, & a = \infty, \end{cases}$$

$$F(z, a) = \begin{cases} A_0 a^n + A_1 a^{n-1} + \cdots + A_{n-1} a + A_n, & a \neq \infty \\ A_0, & a = \infty, \end{cases}$$

$$m(r, a, A) = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{\|A\| \cdot \|a\|}{F(z, a)} \right| d\theta, \ z = re^{i\theta},$$

$$\mu(r, A) = \frac{1}{2n\pi} \int_0^{2\pi} \log \max_{0 \leqslant j \leqslant n} \left| A_j(re^{i\theta}) \right| d\theta.$$

Set

$$T(r, a, A) = m(r, a, A) + N(r, 0, F(z, a));$$

by Jensen's formula, we have

$$T(r, a, A) = \frac{1}{2\pi} \int_0^{2\pi} \log(\|A\| \cdot \|a\|) d\theta + O(1).$$

Since
$$\max_{0 \leqslant j \leqslant n} |A_j(z)| \leqslant \|A(z)\| \leqslant (n+1)^{1/2} \max_{0 \leqslant j \leqslant n} |A_j(z)|,$$

we have
$$|T(r, a, A) - n\mu(r, A)| = O(1).$$

By using Valiron's result ([6]), we get

$$|T(r, a, A) - nT(r, f)| = O(1),$$

so that
$$\delta(a, f) = 1 - \limsup_{r \to \infty} \frac{N(r, 0, F(z, a))}{T(r, a, A)}.$$

With these notations, we define the sets of arguments $E_\Lambda(r, a) \subset (-\pi, \pi]$ by

$$E_\Lambda(r, a) = \{\theta : \frac{\|A\| \cdot \|a\|}{|F(z, a)|} > e^{\Lambda(r)}, \quad z = re^{i\theta}\}$$

and let
$$\sigma_\Lambda(a) = \liminf_{j \to \infty} \operatorname{meas} E_\Lambda(r_j, a)$$

$$\sigma(a) = \inf_\Lambda \sigma_\Lambda(a)$$

where the infimum is taken over all functions satisfying (1.2).

In the present paper, we prove a spread relation analogous to (1.3) with the spread $\sigma'(a)$ replaced by $\sigma(a)$ and give some applications of the spread relation.

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2. Spread relations

In the following statements the notations of the introduction are taken for granted. For a complex number a, we set

$$m^*(z, a) = \sup_E \frac{1}{2\pi} \int_E \log \frac{\|A\| \cdot \|a\|}{|F(\xi, a)|} d\omega, \quad \xi = re^{i\omega},$$
$$(z = re^{i\theta}, 0 < r < \infty, 0 \leqslant \theta \leqslant \pi)$$

where the supremum is taken over all measurable sets $E \subset (-\pi, \pi]$ of Lebesgue measure 2θ, and

$$T^*(z) = T^*(z, a) = m^*(z, a) + N(r, 0, F(z, a)).$$

The function $T^*(z)$ is defined on the set

$$H_1 = \{z\colon \operatorname{Im} z \geqslant 0, \quad z \neq 0\}.$$

It follows from the definition of this function that for arbitrary r such that $0 < r < \infty$ and a complex number a we have

$$\sup T^*\left(re^{i\theta}\right) = T(r, a, A), \tag{2.1}$$
$$T^*(r) = N(r, 0, F(z, a)). \tag{2.2}$$

Lemma 2.1. *$T^*(z)$ is subharmonic in the half plane $\operatorname{Im} z > 0$ and is continuous in H_1.*

Proof: By a result of Goldberg ([**3**]), we know that $\log \|A\|$ is subharmonic so that $\log(\|A\| \cdot \|a\|)$ is subharmonic. Since $F(z, a)$ is an entire function, we have $\log|F(z, a)|$ is a subharmonic function. By the Theorem A′ in [**2**], Lemma 2.1 follows. □

Theorem 2.1. *Let $f(z)$ be an n-valued algebroid function of lower order μ $(0 < \mu < \infty)$, defined by the equation (1.1); then*

$$\sigma(a) \geqslant \min\left\{2\pi, \frac{4}{\mu} \arcsin \sqrt{\frac{\delta(a, f)}{2}}\right\}, \tag{2.3}$$

where a is a deficient value of $f(z)$.

Proof: We consider the following two cases.

(1) $4 \arcsin \sqrt{(\delta(a, f)/2)}/\mu < 2\pi$.

To deduce inequality (2.3) we should use Lemma 2.1 and the proof of (1.4) in [**1**]; let us outline the method of the proof of inequality (2.3) (for details see the proof of relation (1.4) in [**1**, p.429–434].

We set

$$\gamma = \frac{2}{\pi\mu} \arcsin \sqrt{\frac{\delta(a, f)}{2}}. \tag{2.4}$$

The following inequality is fulfilled for the function

$$v(z) = \begin{cases} 0, & z = 0 \\ T^*(z^\gamma), & z = re^{i\theta}, 0 < r < \infty, 0 \leqslant \theta \leqslant \pi \end{cases}$$

which is subharmonic in the half plane $\operatorname{Im} z > 0$ (see Lemma 2.1):

$$v\left(re^{i\theta}\right) \leqslant \int_{-R}^{R} v(t) A(t, r, \theta, R) dt + \int_0^\pi v\left(Re^{i\varphi}\right) B(\varphi, r, \theta, R) d\varphi$$

where A and B are kernels (see [1, p.430]).

We use the estimates $B(\varphi, r, \theta, R) < 32(r/R)$, $(0 < \theta < \pi, 0 < \varphi < \pi, 0 < r < R/2)$ and $A(t, r, \theta, R) \leqslant P(t, r, \pi - \theta)$, $A(-t, r, \theta, R) \leqslant P(t, r, \theta)$, where

$$P(t, r, \theta) = \frac{1}{\pi} \frac{r \sin\theta}{t^2 + 2rt\cos\theta + r^2}.$$

Taking into account properties (2.1) and (2.2) of the function $T^*(z)$, we get

$$\begin{aligned} v\left(re^{i\theta}\right) \leqslant & \int_0^R N(t^\gamma, 0, F(z, a)) P(t, r, \pi - \theta) dt \\ & + \int_0^R T(t^\gamma, a, A) P(t, r, \theta) dt + 32(r/R) T(R^\gamma, a, A) \\ & (0 < \theta < \pi, \quad 0 < r < R/2). \end{aligned} \tag{2.5}$$

Let (r_j) be a sequence of Pólya peaks of order μ of $T(r, a, A)$ (or $T(r, f)$) and $\left(r'_{j'}\right)$ be the sequence occuring in the definition of Pólya peaks (see [1, p.418]) such that $r'_{j'}/r_j \to \infty$ $(j \to \infty)$.

Let us set

$$s_j = (r_j)^{1/\gamma} \text{ and } s'_{j'} = \left(r'_{j'}\right)^{1/\gamma}.$$

The following relations are valid:

$$\begin{aligned} \int_0^{s'_{j'}} N(t^\gamma, 0, F) P(t, s_j, \pi - \theta) dt \quad \leqslant \quad & (1 - \delta(a, f)) T(r_j, a, A) \\ & \times \left\{ \frac{\sin(\pi - \theta)\gamma\mu}{\sin \pi\gamma\mu} + o(1) \right\}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} \int_0^{s'_{j'}} T(t^\gamma, a, A) P(t, s_j, \theta) dt \quad \leqslant \quad & T(r_j, a, A) \left\{ \frac{\sin\theta\mu\gamma}{\sin\pi\mu\gamma} + o(1) \right\}, \\ & (j \to \infty, \quad 0 < \theta < \pi), \end{aligned} \tag{2.7}$$

where $o(1)$ does not depend on θ,

$$\frac{s_j}{s'_{j'}}T\left(\left(s'_{j'}\right)^{\gamma}, a, A\right) = o(T(r_j, a, A)), \ j \to \infty. \tag{2.8}$$

Setting $r = s_j$ and $R = s'_{j'}$ in (2.5) and taking the relations (2.6), (2.7), (2.8) into account, we get ($j \to \infty$, $0 < \theta < \pi$)

$$v\left(s_j e^{i\theta}\right) \quad \leqslant \quad T(r_j, a, A)\left\{\frac{\sin\theta\gamma\mu + (1 - \delta(a, f))\sin(\pi - \theta)\gamma\mu}{\sin\pi\gamma\mu} + o(1)\right\}. \tag{2.9}$$

From the definition of γ we have

$$1 - \delta(a, f) = \cos\pi\gamma\mu.$$

We write the inequality (2.9) in the form

$$\begin{aligned} v\left(s_j e^{i\theta}\right) \quad &\leqslant \quad T(r_j, a, A)\{\cos(\pi - \theta)\gamma\mu + \alpha_j\}, \\ &(j = 1, 2, \cdots, \ 0 < \theta < \pi) \end{aligned}$$

where $\alpha_j \to 0$ as $j \to \infty$. Further, following [1, p.433–434], we arrive at the relation (2.3).

(2) $4/\mu \arcsin\sqrt{(\delta(a, f)/2)} \geqslant 2\pi$.

In this case, we choose a number d such that

$$0 < d < \delta(a, f)$$

and

$$\frac{4}{\mu}\arcsin\sqrt{\frac{d}{2}} < 2\pi.$$

Set

$$\gamma = \frac{2}{\pi\mu}\arcsin\sqrt{\frac{d}{2}};$$

by similar reasoning, we arrive at

$$\sigma(a) \geqslant \frac{4}{\mu}\arcsin\sqrt{\frac{d}{2}}.$$

Letting $d \uparrow d_0 = 2\sin^2(\mu\pi/2)$, we obtain the desired result

$$\sigma(a) = 2\pi.$$

Theorem 2.1 is proved. ∎

THEOREM 2.2. *Let $f(z)$ be an n-valued algebroid function of lower order μ $(0 < \mu < \infty)$, defined by the equation (1.1), and $q \geqslant 2\mu$ be an integer. If*

$$\delta(a, f) \geqslant 1 - \cos\frac{\mu\pi}{q}, \tag{2.10}$$

then

$$\sigma(a) \geqslant \frac{2\pi}{q}. \tag{2.11}$$

PROOF: The proof of this theorem is similar to the proof of case (1) in Theorem 2.1. Let us only observe that we must choose $\gamma = 1/q$ and apply inequality (2.10) to relation (2.9). Relation (4.16) from [1, p.433] reduces to the desired inequality (2.11). □

3. APPLICATIONS

LEMMA 3.1. *Let $f(z)$ be an n-valued algebroid function of lower order μ $(0 < \mu < \infty)$ and let a_i $(i = 0, 1, \cdots, n)$ be any $n+1$ distinct complex numbers. Choose $\Lambda(r) = (T(r, f))^{1/2}$ and define the sets $E_\Lambda(r, a_j)$ in $(-\pi, \pi]$ by*

$$E_\Lambda(r, a_j) = \left\{\theta\colon \frac{\|A(z)\| \cdot \|a_j\|}{|F(z, a_j)|} > e^{\Lambda(\gamma)},\ z = e^{i\theta}r\right\} (j = 0, 1, \cdots, n), \tag{3.1}$$

Then there exists a positive number $r_0 > 0$ such that $r \geqslant r_0$

$$\bigcap_{j=0}^{n} E_\Lambda(r, a_j) = \emptyset.$$

PROOF: We assume that $a_j \neq \infty$ $(j = 0, 1, \cdots, n)$ without loss of generality. Suppose that

$$E(r) = \bigcap_{j=0}^{n} E_\Lambda(r, a_j) \neq \emptyset.$$

We choose $\theta_0 \in E(r)$ and consider the following system of $n+1$ equations.

$$F(re^{i\theta_0}, a_j) = \sum_{k=0}^{n} A_k(re^{i\theta_0})a_j^{n-k} \quad (j = 0, 1, \cdots, n).$$

Since the determinant of the coefficients

$$\det\left(a_j^n, a_j^{n-1}, \cdots, a_j, 1\right) \neq 0,$$

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we can solve this system for the unknowns $A_j(re^{i\theta_0})$ $(0 \leqslant j \leqslant n)$ and obtain (for some constants b_{jk}):

$$A_k(re^{i\theta_0}) = \sum_{j=0}^{n} b_{jk} F(re^{i\theta_0}, a_j), \quad (k = 0, 1, \cdots, n)$$

so that

$$\begin{aligned} |A_q(re^{i\theta_0})| &= \max_{0 \leqslant k \leqslant n} |A_k(re^{i\theta_0})| \\ &\leqslant \max_{0 \leqslant k \leqslant n} \sum_{j=0}^{n} |b_{jk}| \cdot |F(re^{i\theta_0}, a_j)| \\ &\leqslant C |F(re^{i\theta_0}, a_s)|, \ (0 \leqslant s \leqslant n) \end{aligned}$$

where C is a constant and

$$|F(re^{i\theta_0}, a_s)| = \max_{0 \leqslant j \leqslant n} |F(re^{i\theta_0}, a_j)|.$$

This means that for fixed r

$$\begin{aligned} \frac{\|A(re^{i\theta_0})\| \cdot \|a_s\|}{|F(re^{i\theta_0}, a_s)|} &\leqslant \frac{(n+1)^{1/2} |A_q(re^{i\theta_0})| \cdot \|a_s\|}{|F(re^{i\theta_0}, a_s)|} \\ &\leqslant (n+1)^{1/2} C \|a_s\| = \text{ constant}, \end{aligned}$$

which for sufficiently large r contradicts the assumption that θ_0 belongs to $E_\Lambda(r, a_s)$. Lemma 3.1 is thus proved. ∎

LEMMA 3.2. *Let $f(z)$ be an n-valued algebroid function of lower order μ $(0 < \mu < \infty)$, defined by the equation (1.1) and*

$$\Lambda(r) = (T(r, f))^{1/2}.$$

Then, on summing all the deficient values a of $f(z)$, we have

$$\sum_a \sigma(a) \leqslant \sum_a \sigma_\Lambda(a) \leqslant 2n\pi.$$

PROOF: Let a_j $(j = 1, 2, \cdots, N)$ be any N deficient values of $f(z)$. The sets $E_\Lambda(r, a_j)$ $(1 \leqslant j \leqslant N)$ are defined by (3.1). Since for each $\theta_0 \in (-\pi, \pi]$, θ_0 can belong to at most n of the sets $E_\Lambda(r, a_j)$ $(1 \leqslant j \leqslant N)$ for sufficiently large r,

$$\sum_{k=1}^{N} \sigma(a_k) \leqslant \sum_{k=1}^{N} \sigma_\Lambda(a_k) = \sum_{k=1}^{N} \lim_{j \to \infty} \operatorname{meas} E_\Lambda(r_j, a_k) \leqslant 2n\pi.$$

Since N can be arbitrarily large, Lemma 3.2 is thus proved. $\square$

THEOREM 3.1. *Let $f(z)$ be an n-valued algebroid function of lower order μ $(0 < \mu < \infty)$, defined by the equation (1.1) and q $(\geqslant 2\mu)$ be an integer. If $f(z)$ has more than nq deficient values, then there are at most $nq - 1$ deficient values a_k $(k = 1, 2, \cdots, nq - 1)$ such that*

$$\delta_k = \delta(a_k, f) \geqslant 1 - \cos\frac{\mu\pi}{q}, \ (k = 1, 2, \cdots, nq - 1).$$

PROOF: Assume that the assertion is false; we choose $nq + 1$ distinct deficient values a_k $(k = 1, 2, \cdots, nq + 1)$ of $f(z)$ such that

$$\delta_1 \geqslant \delta_2 \geqslant \cdots \delta_{nq} \geqslant 1 - \cos\frac{\mu\pi}{q}, \tag{3.2}$$

$$\delta_{nq+1} > 0, \ (q \geqslant 2\mu, \ \delta_k = \delta(a_k, f), \ 1 \leqslant k \leqslant nq + 1). \tag{3.3}$$

Choosing the integer s $(\geqslant nq)$ large enough, (3.3) implies

$$\delta_{nq+1} \geqslant 1 - \cos\frac{\mu\pi}{s}. \tag{3.4}$$

Now let (r_j) be a sequence of Pólya peaks of $T(r, f)$ and let

$$\Lambda(r) = (T(r, f))^{1/2};$$

by Theorem 2.2, (3.2) and (3.4) imply

$$\sigma(a_k) \geqslant \frac{2\pi}{q}, \quad \sigma(a_{nq+1}) \geqslant \frac{2\pi}{s}, \quad k = 1, 2, \cdots, nq.$$

So
$$\sum_{k=1}^{nq+1} \sigma_\Lambda(a_k) \geqslant \sum_{k=1}^{nq+1} \sigma(a_k) \geqslant 2n\pi + \frac{2\pi}{s}.$$

This contradicts Lemma 3.2, so Theorem 3.1 is proved. $\square$

THEOREM 3.2. *Let $f(z)$ be an n-valued algebroid function of lower order μ $(0 \leqslant \mu \leqslant \infty)$, defined by the equation (1.1). Then on summing over all the deficient values a of $f(z)$, we have*

$$\sum_a \sqrt{\delta(a, f)} \leqslant n\left(\sqrt{2}\mu\pi + 2\mu + 1\right).$$

PROOF: We consider the following two cases.

(1) If $\mu = 0$, by a result of Gu ([4]), $f(z)$ has at most n deficient values, so that Theorem 3.2 holds.

(2) If $0 < \mu < \infty$, we assume that

$$a_1, a_2, \cdots, a_n, \cdots \tag{3.5}$$

are all the deficient values of $f(z)$ and assume that (3.5) has been ordered so that

$$\delta_1 \geqslant \delta_2 \geqslant \cdots \geqslant \delta_n \geqslant \cdots,$$

where $\delta_k = \delta(a_k, f)$, $k = 1, 2, \cdots, n, \cdots$.

Let $q = [2\mu] + 1$ and m be an integer. If $m \leqslant nq$, it is trivial that

$$\sum_{i=1}^{m} \sqrt{\delta(a_i, f)} \leqslant nq \leqslant n(2\mu + 1).$$

If $m > nq$, by Theorem 3.1 we have

$$\delta_{nq} < 1 - \cos\frac{\mu\pi}{q}.$$

Hence, with each $\delta_{nq+i} > 0$ $(1 \leqslant i \leqslant m - nq)$, we may associate a positive integer q_i such that

$$1 - \cos\frac{\mu\pi}{q_i + 1} \leqslant \delta_{nq+i} < 1 - \cos\frac{\mu\pi}{q_i}, \quad i \geqslant 1. \tag{3.6}$$

By Lemma 3.2 and Theorem 2.2, we get

$$\sum_{i=1}^{m-nq} \frac{2\pi}{q_i + 1} \leqslant \sum_{i=nq+1}^{m} \sigma(a_i) \leqslant 2n\pi,$$

so that

$$\sum_{i=1}^{m-nq} \frac{1}{q_i} \leqslant \sum_{i=1}^{m-nq} \frac{2}{q_i + 1} \leqslant 2n. \tag{3.7}$$

From the second inequality in (3.6), we deduce

$$\delta_i^{1/2} < \sqrt{2}\sin\frac{\mu\pi}{2q_i} < \frac{\pi\mu}{q_i\sqrt{2}}$$

and hence

$$\sum_{i=nq+1}^{m} \sqrt{\delta(a_i, f)} \leqslant \sum_{i=1}^{m-nq} \frac{\mu\pi}{\sqrt{2}q_i} \leqslant \sqrt{2}n\mu\pi.$$

Therefore

$$\sum_{i=1}^{m} \sqrt{\delta(a_i, f)} \leqslant \sqrt{2}n\mu\pi + nq \leqslant n\left(\sqrt{2}\mu\pi + 2\mu + 1\right). \tag{3.8}$$

Since m can be arbitrarily large, Theorem 3.2 follows from (3.8). □

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