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A VALUATION FORMULA FOR MULTI-ASSET, MULTI-PERIOD BINARIES IN A BLACK–SCHOLES ECONOMY

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Abstract

We present a new valuation formula for a generic, multi-period binary option in a multi-asset Black–Scholes economy. The payoff of this so-called M-binary is the most general possible, subject to the condition that a simple analytic expression exists for the present value. Portfolios of M-binaries can be used to statically replicate many European exotics for which there exist closed-form Black–Scholes prices.

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1. Introduction

Many European style exotic options have payoffs which can be represented as portfolios of simpler contracts. These simpler contracts are often other exotic options called *binary* or *digital* options. A binary option is generally one whose payoff is determined by an exercise condition. If the exercise condition is satisfied at one or possibly more future dates, the binary pays out a predetermined amount dependent on the terms of the contract. If the exercise condition is not satisfied, the binary expires worthless.

As a simple example, consider a standard European call option of strike price k and expiry date T. The payoff $C_T(x) = (x - k)^+ = (x - k)\mathbb{I}(x > k)$ can be viewed as a portfolio of: long one asset binary with T payoff $A_T(x) = x\mathbb{I}(x > k)$, and short k bond binaries each with T payoff $B_T(x) = \mathbb{I}(x > k)$. The indicator $\mathbb{I}(x > k)$ is the exercise condition for both these binaries. This example obviously involves just a single asset and a single expiry date.

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We present in this paper a valuation formula for the most general multi-asset, multi-period binary option, which has an analytic expression in the Black–Scholes framework. This binary option has a payoff which is any product of arbitrary powers of the asset prices at any of multiple monitoring dates. This payoff is conditional on a wide set of exercise scenarios which may involve similar products. We refer to this generic binary option as an M-binary.

Examples of European style exotic options which are M-binary decomposable include: multi-asset, single-period asset and bond (or cash) digitals; single-asset, multi-period asset and bond (or cash) digitals; power (or turbo) options; compound options; chooser options; reset options with multiple strike and date resets; fixed period shout options; fixed strike, discrete geometric mean Asian options; floating strike, discrete geometric mean Asian options; multi-asset turbo options (options with power-law payoffs); discretely monitored barrier options; discretely monitored lookback options; best and worst rainbow options; options on the maximum/minimum of multiple assets; exchange and sequential exchange options; outperformance rainbow options; compound options on the M-best (or worst) of N-risky assets; ratchets and cliquets; balloon options; gaps and supershares; cylinder options; capped options; range options (for example, Everest and Kilimanjaro options); Napoleon options.

As can be seen, the M-binary is the key building block for a very large selection of exotic options. To price a given exotic option, the main effort required is to decompose the exotic option payoff into the replicating portfolio of M-binaries and then it only remains to identify the associated M-binary parameters.

If we utilize the reflection principle for Brownian motion, described as the "method of images" in [1] and recently extended to multivariate settings in [9], this list can be enlarged even further to include a whole family of barrier, ladder and lookback options and other exotic extensions. We are not the first to decompose exotic options into portfolios of binary options. Rubinstein and Reiner [8], Heynen and Kat [4] and Ingersoll [7] explore various aspects of the same concept. What we present here is a new formula for a generic binary option which is the most general possible, subject to the condition that it has a tractable, analytic representation in the Black–Scholes framework. Consequently, the formula is of little help when pricing options for which no closed-form valuation exists, such as additive basket options, or options whose payoff does not involve powers of assets or products of such powers.

2. The M-binary payoff

Due to the multivariate nature of the M-binary we shall price in this paper, it is helpful to adopt a vector and matrix notation. Additionally, we find it convenient to use the following (nonstandard) convention.

DEFINITION 2.1. For any vector $\mathbf{v}$ and matrix M such that $M\mathbf{v}$ is defined, make use of the shorthand

$$\mathbf{v}^M \equiv \exp(M \log \mathbf{v}),$$

where $\log \mathbf{v} = (\log v_1, \log v_2, \ldots)'$ denotes componentwise function evaluation, as does $\exp \mathbf{v}$, and $\mathbf{v}'$ denotes the transpose of $\mathbf{v}$.

We are now in a position to introduce a formal definition of an M-binary. The option involves a somewhat general payoff and, as a consequence, a number of parameters that require explanation. Rather than appear to define these parameters "out of the blue", we prefer to state the M-binary's expiry payoff and *then* discuss the meaning of the parameters.

DEFINITION 2.2. An N-asset, M-period multi-binary with parameters $\{\alpha, \mathbf{a}, S, A\}$ is a binary option with expiry T payoff of the form

$$V_T(\mathbf{X}) = \mathbf{X}^{\alpha}\, \mathbb{I}_m(S\mathbf{X}^A > S\mathbf{a}). \tag{2.1}$$

The meanings of the parameters $\{\alpha, \mathbf{a}, S, A\}$, the price vector $\mathbf{X}$ and other notational issues are discussed in the following remarks.

REMARK 1.

(1) The payoff depends on at least one of the asset prices X_i $(i = 1, 2, \ldots, N)$ at each of the monitoring times T_k $(k = 1, 2, \ldots, M)$, assumed to satisfy $T_1 < T_2 < \cdots < T_M \le T$. Note that T is the payout date of the option and need not coincide with the last monitoring date T_M. We shall be interested in pricing the M-binary for all times t before T_1, the first monitoring date.
$\mathbf{X}$ is the smallest vector of asset prices at different monitoring times required to completely determine the payoff of the option. In particular, $\mathbf{X} = X_{(ik)} = X_i(T_k)$ refers to asset price X_i at monitoring time T_k. We emphasize that it is not required that every asset X_i at every time T_k be present in $\mathbf{X}$, only those $X_i(T_k)$ that directly determine the option's payoff.
If we let n be the dimension of $\mathbf{X}$, then it should be clear that $N \le n \le NM$, where the minimum value obtains if each asset price occurs at only one monitoring time, and the maximum value obtains if each asset occurs at all M monitoring times. We refer to $\mathbf{X}$ as the *payoff vector* for the M-binary and n as the *payoff dimension*.

(2) The actual payoff at time T (if nonzero) is the scalar $\mathbf{X}^{\alpha}$, which is our shorthand notation for the product of powers:

$$\mathbf{X}^{\alpha} = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}.$$

Thus, α is a row vector of indices and has the same length as $\mathbf{X}$. If a component $X_{(ik)}$ does not appear in the payoff, then we simply set the corresponding index $\alpha_{(ik)} = 0$. In the extreme case when the payoff is 1 dollar (that is, a bond binary), then $\alpha_{(ik)} = 0$ for all components. If the payoff is the price $X_{(ik)}$ of a single asset i at time T_k, then we set $\alpha_{(ik)} = 1$ and all other components of α equal to zero. In this way, we can construct very general payoffs for our M-binary exotic option.

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(3) The symbol $\mathbb{I}_m$ in (2.1) is our notation for an m-dimensional indicator function, and gives the *exercise condition* for the M-binary. It is defined for any m-dimensional vectors, $\mathbf{Y}$ and $\mathbf{a}$, by

$$\mathbb{I}_m(\mathbf{Y} > \mathbf{a}) = \mathbb{I}(Y_1 > a_1)\mathbb{I}(Y_2 > a_2) \cdots \mathbb{I}(Y_m > a_m).$$

That is, $\mathbb{I}_m$ is a product of m one-dimensional indicator functions. The matrix S in (2.1) is an m-dimensional diagonal matrix whose diagonal entries are either $+1$ or -1. If $S_{ii} = +1$, then the corresponding indicator is $\mathbb{I}(Y_i > a_i)$, while if $S_{ii} = -1$, then it is $\mathbb{I}(-Y_i > -a_i)$, or equivalently, $\mathbb{I}(Y_i < a_i)$. The matrix S therefore determines the direction of the inequalities in the multiple exercise conditions. We call $\mathbf{a}$ the *exercise price* vector for the M-binary.

(4) The most interesting part of the M-binary payoff function is the term $\mathbf{X}^A$. Here, A is an $(m \times n)$ *exercise condition* matrix. Each row of A contains the indices to be applied to $\mathbf{X}$ for each of the m indicator functions implied by $\mathbb{I}_m$. Hence row j of A gives rise to the indicator function $\mathbb{I}(X_1^{A_{j1}} X_2^{A_{j2}} \cdots X_n^{A_{jn}} > a_j)$. Different choices of the matrix A allow a considerable degree of flexibility in the kinds of exercise conditions that can be applied. For example, if $A = I_n$, the n-dimensional identity matrix, then the exercise condition reduces to the simple expression

$$\mathbb{I}_n(\mathbf{X} > \mathbf{a}) = \mathbb{I}(X_1 > a_1)\mathbb{I}(X_2 > a_2) \cdots \mathbb{I}(X_n > a_n).$$

On the other hand, if $A = (1/n)\mathbf{1}_n$ is a row vector with all components equal to $1/n$, then $m = 1$ and

$$\mathbb{I}_m(\mathbf{X}^A > a) = \mathbb{I}(G_n > a),$$

where $G_n = (X_1 X_2 \cdots X_n)^{1/n}$ is the geometric mean of the n asset prices $X_1, X_2, \ldots, X_n$.
Since it is always possible to express an exercise condition of dimension $m > n$ as a combination of exercise conditions each with dimension not greater than n, we can, without loss of generality, assume $m \leq n$.

As an example of the utility of the M-binary, we show how to choose the matrix A so that we can price exotic options whose payoffs depend on the maximum or minimum of an observed set of option prices and a given amount of cash k.

PROPOSITION 2.3. *Let A be an $(n \times n)$ matrix and $\mathbf{a}$ an n-dimensional vector with elements defined by*

$$A_{ij} = \begin{cases} 1 & \text{if } j = p \\ -1 & \text{if } i = j \neq p \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad a_i = \begin{cases} k & \text{if } i = p \\ 1 & \text{if } i \neq p \end{cases}, \tag{2.2}$$

where p is some integer in the range $1 \leq p \leq n$ and k is a positive constant. The exercise condition $\mathbb{I}_n(s\mathbf{X}^A > s\mathbf{a})$ is satisfied for $s = 1$ if $\max(\mathbf{X}) = X_p > k$ *or for $s = -1$ if* $\min(\mathbf{X}) = X_p < k$.

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Now consider the M-binary portfolio with expiry payoff

$$Q^s(\mathbf{X}, T; k) = \sum_{p=1}^{n} X_p \mathbb{I}_n(s\mathbf{X}^A > s\mathbf{a}) + k\mathbb{I}_n(s\mathbf{X} < sk\mathbf{1}_n),$$

where $(A, \mathbf{a})$ are defined as in (2.2) and their implicit dependence on p is understood. This portfolio pays the best (if $s = 1$) or worst (if $s = -1$) of n given assets and k units of cash; that is, $\max(\mathbf{X}, k)$ and $\min(\mathbf{X}, k)$, respectively. Calls and puts on the maximum and minimum of several assets can then be obtained using the identities $[\max(\mathbf{X}) - k]^+ = \max(\mathbf{X}, k) - k$ and $[k - \min(\mathbf{X})]^+ = k - \min(\mathbf{X}, k)$.

If the option payoff depends on only the maximum or minimum asset price and not on the given cash amount k, then (2.2) should be modified by omitting the pth row of A and the pth component of $\mathbf{a}$. We omit the proof of Proposition 2.3, which is relatively straightforward.

3. The M-binary present value

We adopt the Black–Scholes framework extended to multiple assets, that is, the N assets are assumed to follow correlated geometric Brownian motions. In particular, let r denote the risk-free interest rate, q_i the dividend yield of asset i and σ_i its volatility. Then the risk-neutral price process of the ith asset, X_i, satisfies the stochastic differential equation

$$\frac{dX_i}{X_i} = (r - q_i)\, ds + \sigma_i\, dB_i, \qquad X_i(t) = x_i, \tag{3.1}$$

where B_i and B_j are correlated Brownian motions with instantaneous correlation coefficient ρ_{ij}. With these assumptions, we are now ready to state the main result of the paper.

THEOREM 3.1. *Let $V_t(\mathbf{x})$ denote the time t present value of an M-binary with expiry payoff (2.1). In the Black–Scholes economy described above, the arbitrage-free present value of the M-binary is given by the expression*

$$V_t(\mathbf{x}) = \mathbf{x}^\alpha e^\theta \mathcal{N}_m(S\mathbf{d}, SCS), \tag{3.2}$$

where

$$\begin{gathered}
\theta = -r\tau + \alpha'\mu + \frac{1}{2}\alpha'\Gamma\alpha, \qquad \mathbf{d} = D^{-1}\big[\log(\mathbf{x}^A/\mathbf{a}) + A(\mu + \Gamma\alpha)\big], \\
C = D^{-1}(A\Gamma A')D^{-1}, \qquad \Gamma = \Sigma R \Sigma', \\
R = \rho_{ij}\frac{\min(\tau_k, \tau_l)}{\sqrt{\tau_k \tau_l}}, \qquad D = \sqrt{\operatorname{diag}(A\Gamma A')}, \\
\mu = \left(r - q_i - \frac{1}{2}\sigma_i^2\right)\tau_k, \qquad \Sigma = \operatorname{diag}(\sigma_i\sqrt{\tau_k}), \qquad \tau_k = T_k - t, \qquad \tau = T - t.
\end{gathered} \tag{3.3}$$

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REMARK 2.

(1) The present value $V_t(\mathbf{x})$ is valid for all times satisfying $t < T_1$, the minimum monitoring time. The term $\mathbf{x}$ on the right-hand side of (3.2) denotes the present value of the payoff vector $\mathbf{X} = X_{(ik)}$. Clearly, all the components $x_{(ik)}$ are equal to x_i regardless of the time index k.
(2) For a zero-mean, unit-variance Gaussian random vector $\mathbf{Z}$ with correlation matrix R, the function $\mathcal{N}_m(\mathbf{d}, R) \equiv \mathbb{E}\{\mathbb{I}_m(\mathbf{Z} < \mathbf{d})\}$ denotes the multivariate normal cumulative distribution function.
(3) The parameters θ, $\mathbf{d}$, C, D, Γ, μ and Σ all depend on the monitoring times T_k. C is an $(m \times m)$ correlation matrix, Γ is an $(n \times n)$ covariance matrix, D is a positive definite $(m \times m)$ diagonal matrix which normalizes the covariance matrix $A\Gamma A'$ to a correlation matrix.
(4) While the formula is presented for constant parameters $\{q_i, \sigma_i, \rho_{ij}\}$, there is no difficulty in extending it to the deterministic time-varying parameter case.

4. Proof of main theorem

Theorem 3.1 is a consequence of four fundamental results. These are straightforward to prove (or at least well known), hence they are stated without proof. Then follows the proof of our main result, *sans* integrals and any explicit change of numeraire.

In a complete, arbitrage-free, multi-asset market the discounted price of any derivative contract is a martingale with respect to the risk-neutral measure. This leads to the following well-known pricing formula, also called the first fundamental theorem on asset pricing (see [3]).

LEMMA 4.1. *For any derivative contract with expiry value $V_T(\mathbf{X})$, the present, time t, value, $V_t(\mathbf{x})$, is given by*

$$V_t(\mathbf{x}) = e^{-r(T-t)}\mathbb{E}\{V_T(\mathbf{X}) \mid \mathbf{X}(t) = \mathbf{x}\},$$

where the expectation is taken with respect to the risk-neutral measure.

LEMMA 4.2. *Given the price dynamics (3.1) for the N assets, the distribution of the log of the payoff vector $\mathbf{X} = X_{(ik)}$ can be written as*

$$\log \mathbf{X} \stackrel{\mathrm{d}}{=} \log \mathbf{x} + \mu + \Sigma\mathbf{Z},$$

where $\mathbf{Z} = Z_{(ik)}$ is a zero-mean, unit-variance Gaussian random vector with correlation matrix R, and μ, Σ and R are given in (3.3).

LEMMA 4.3. *Let $\mathbf{c}$ be a constant vector, $F(\cdot)$ an arbitrary measurable function of n variables, and Z a zero-mean, unit-variance Gaussian random vector with correlation matrix R. Then*

$$\mathbb{E}\{e^{\mathbf{c}'\mathbf{Z}}\, F(\mathbf{Z})\} = e^{(1/2)\mathbf{c}'R\mathbf{c}}\, \mathbb{E}\{F(\mathbf{Z} + R\mathbf{c})\}.$$

We like to call the above result the multivariate Gaussian shift theorem.

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LEMMA 4.4. *Let B be an $(m \times n)$ matrix of rank $m \leq n$, $\mathbf{b}$ a vector of length m and $\mathbf{Z}$ a zero-mean, unit-variance Gaussian random vector with correlation matrix R. Then*

$$\mathbb{E}\{\mathbb{I}_m(B\mathbf{Z} < \mathbf{b})\} = \mathcal{N}_m\big(D^{-1}\mathbf{b};\ D^{-1}(BRB')D^{-1}\big),$$

where $D = \sqrt{\operatorname{diag}(BRB')}$.

We now proceed to tie the above lemmas together in a proof of Theorem 3.1. Firstly, let $\mathbf{Z} = Z_{(ik)}$ be a zero-mean, unit-variance Gaussian random vector with correlation matrix R given by (3.3). Then, by Lemma 4.2,

$$\mathbf{X}^\alpha \overset{\mathrm{d}}{=} \mathbf{x}^\alpha \exp\{\alpha'\mu + \mathbf{c}'\mathbf{Z}\}, \qquad \mathbf{c} = \Sigma\alpha,$$

and the exercise condition $S\mathbf{X}^A > S\mathbf{a}$ is equivalent to

$$SA(\log \mathbf{x} + \mu + \Sigma\mathbf{Z}) > S \log \mathbf{a}.$$

Rearranging this last equation, we arrive at $\mathbb{I}_m(S\mathbf{X}^A > S\mathbf{a}) = \mathbb{I}_m(B\mathbf{Z} > -S\mathbf{u})$, where $B = SA\Sigma$ and $\mathbf{u} = \log(\mathbf{x}^A/\mathbf{a}) + A\mu$. Hence from Lemmas 4.1 and 4.3 we obtain

$$\begin{aligned}
V_t(\mathbf{x}) &= e^{-r\tau}\,\mathbb{E}\{\mathbf{X}^\alpha \mathbb{I}_m(S\mathbf{X}^A > S\mathbf{a})\} \\
&= \mathbf{x}^\alpha e^{-r\tau+\alpha'\mu}\,\mathbb{E}\{e^{\mathbf{c}'\mathbf{Z}}\mathbb{I}_m(B\mathbf{Z} > -S\mathbf{u})\} \\
&= \mathbf{x}^\alpha e^{\theta}\,\mathbb{E}\{\mathbb{I}_m(B(\mathbf{Z} + R\Sigma\alpha) > -S\mathbf{u})\} \\
&= \mathbf{x}^\alpha e^{\theta}\,\mathbb{E}\{\mathbb{I}_m(B\mathbf{Z} > -SD\mathbf{d})\} = \mathbf{x}^\alpha e^{\theta}\,\mathbb{E}\{\mathbb{I}_m(B\mathbf{Z} < SD\mathbf{d})\}.
\end{aligned}$$

Observe that $S\mathbf{u} + BR\Sigma\alpha = SD\mathbf{d}$ where D and $\mathbf{d}$ are defined by (3.3). The last line follows from the symmetry of zero-mean Gaussian vectors, that is, $\mathbb{E}\{F(\mathbf{Z})\} = \mathbb{E}\{F(-\mathbf{Z})\}$ for any measurable function F. Theorem 3.1 now follows from Lemma 4.4 with $B = SA\Sigma$. □

REMARK 3. It is clear from the above proof that a formula of the form (3.2) exists for any price dynamics so long as the log-prices are normally distributed. In particular, this includes the case of deterministic, time-varying parameters $\{r, q_i, \sigma_i, \rho_{ij}\}$. While this extra generality adds no burden to the proof of the result, it adds significantly to the already intimidating expanse of notation. It is for this reason the details are omitted.

5. An ESO example

The main result of this paper is captured by (3.2) and (3.3). While these equations have wide applicability across many areas of financial asset pricing, we present here one example as an illustration of the method.

Our example is a rather simple model of an executive stock option (ESO) which has been used in practice to value corporate remuneration packages. At time $t = 0$, an executive is granted at-the-money (ATM) stock (call) options in the company, with a strike price set at the current share price $X_0 = k$. These options vest at time T_1 subject to a performance test, and if successfully passed must be exercised on their expiry date at time T_2, where $0 < T_1 < T_2$. The performance criterion is that the share

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price $X_1 = X(T_1)$ must exceed a benchmark index $Y_1 = Y(T_1)$ at time T_1. The index can be a general market index such as the ASX 200 index on the Australian stock exchange, or the S&P 500 index on the New York stock exchange. More commonly, it will be a customized index related to the company business. If the options vest, that is $X_1 > Y_1$, then the executive receives the call options and is required to hold them to their expiration date at time T_2. In reality, executives may exercise the options at any time *post* the vesting date T_1, which technically makes these options American (or Bermudan) rather than European. However, it is also well known (for example, see Hull and White [6]) that executives do not exercise their options optimally, and often exercise when the options are sufficiently deep in-the-money.

It is assumed that the share price X_t and the performance index Y_t follow geometrical Brownian motions with volatilities σ_1 and σ_2 respectively and with correlation coefficient ρ. For simplicity, we shall also assume that no dividends are paid either for the company stock or the index. We denote by x, y the current spot prices of the stock and index.

Under the above conditions, the European version of the ESO has payoff

$$V(\mathbf{X}, T) = (X_2 - k)^+ \mathbb{I}(X_1 > Y_1),$$

which is easily replicated as a portfolio of two-asset, two-period M-binaries as follows:

$$V(\mathbf{X}, T) = V_1(\mathbf{X}, T) - kV_2(\mathbf{X}, T),$$

where

$$\begin{aligned} V_1(\mathbf{X}, T) &= X_2 \mathbb{I}(X_1 > Y_1)\mathbb{I}(X_2 > k) = X_2 \mathbb{I}(X_1 Y_1^{-1} > 1)\mathbb{I}(X_2 > k), \\ V_2(\mathbf{X}, T) &= \mathbb{I}(X_1 > Y_1)\mathbb{I}(X_2 > k) = \mathbb{I}(X_1 Y_1^{-1} > 1)\mathbb{I}(X_2 > k). \end{aligned}$$

The parameters for the M-binary $V_1(\mathbf{X}, T)$ are:

$$\mathbf{X} = \begin{pmatrix} X_1 \\ Y_1 \\ X_2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ x \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} 1 \\ k \end{pmatrix},$$

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Furthermore, with $\tau_1 = T_1 - t$ and $\tau_2 = T_2 - t$, we obtain

$$\Sigma = \begin{pmatrix} \sigma_1\sqrt{\tau_1} & 0 & 0 \\ 0 & \sigma_2\sqrt{\tau_1} & 0 \\ 0 & 0 & \sigma_1\sqrt{\tau_2} \end{pmatrix}, \quad R = \begin{pmatrix} 1 & \rho & \tau_{12} \\ \rho & 1 & \rho\tau_{12} \\ \tau_{12} & \rho\tau_{12} & 1 \end{pmatrix}; \quad \tau_{12} = \sqrt{\frac{\tau_1}{\tau_2}},$$

$$\Gamma = \Sigma R \Sigma' = \begin{pmatrix} \sigma_1^2\tau_1 & \rho\sigma_1\sigma_2\tau_1 & \sigma_1^2\tau_1 \\ \rho\sigma_1\sigma_2\tau_1 & \sigma_2^2\tau_1 & \rho\sigma_1\sigma_2\tau_1 \\ \sigma_1^2\tau_1 & \rho\sigma_1\sigma_2\tau_1 & \sigma_1^2\tau_2 \end{pmatrix}, \quad \mu = \begin{pmatrix} \left(r - \frac{1}{2}\sigma_1^2\right)\tau_1 \\ \left(r - \frac{1}{2}\sigma_2^2\right)\tau_1 \\ \left(r - \frac{1}{2}\sigma_1^2\right)\tau_2 \end{pmatrix},$$

$$\theta = -r\tau_2 + \left(r - \frac{1}{2}\sigma_1^2\right)\tau_2 + \frac{1}{2}\sigma_1^2\tau_2 = 0.$$

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Next we compute

$$A\Gamma A' = \begin{pmatrix} \sigma^2\tau_1 & (\sigma_1^2 - \rho\sigma_1\sigma_2)\tau_1 \\ (\sigma_1^2 - \rho\sigma_1\sigma_2)\tau_1 & \sigma_1^2\tau_2 \end{pmatrix} \quad \text{where } \sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2.$$

Then

$$D = \begin{pmatrix} \sigma\sqrt{\tau_1} & 0 \\ 0 & \sigma_1\sqrt{\tau_2} \end{pmatrix}, \quad C = \begin{pmatrix} 1 & \rho^* \\ \rho^* & 1 \end{pmatrix}, \quad \rho^* = \left(\frac{\sigma_1 - \rho\sigma_2}{\sigma}\right)\sqrt{\frac{\tau_1}{\tau_2}},$$

$$\mathbf{x}^\alpha = x^0 y^0 x^1 = x, \quad A\mu = \begin{pmatrix} \frac{1}{2}(\sigma_2^2 - \sigma_1^2)\tau_1 \\ (r - \frac{1}{2}\sigma_1^2)\tau_2 \end{pmatrix}, \quad A\Gamma\alpha = \begin{pmatrix} (\sigma_1^2 - \rho\sigma_1\sigma_2)\tau_1 \\ \sigma_1^2\tau_2 \end{pmatrix}.$$

Finally, we compute

$$\begin{aligned} \mathbf{d} &= \begin{pmatrix} \sigma\sqrt{\tau_1} & 0 \\ 0 & \sigma_1\sqrt{\tau_2} \end{pmatrix}^{-1} \left[\log\begin{pmatrix} x/y \\ x/k \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(\sigma_2^2 - \sigma_1^2)\tau_1 \\ (r - \frac{1}{2}\sigma_1^2)\tau_2 \end{pmatrix} + \begin{pmatrix} (\sigma_1^2 - \rho\sigma_1\sigma_2)\tau_1 \\ \sigma_1^2\tau_2 \end{pmatrix} \right] \\ &= \begin{pmatrix} \left\{\log(x/y) + \frac{1}{2}\sigma^2\tau_1\right\} \Big/ \sigma\sqrt{\tau_1} \\ \left\{\log(x/k) + (r + \frac{1}{2}\sigma_1^2)\tau_2\right\} \Big/ \sigma_1\sqrt{\tau_2} \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \end{aligned}$$

say. With this notation, we obtain the result

$$V_1(x, y, t) = x\mathcal{N}_2\left\{\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \begin{pmatrix} 1 & \rho^* \\ \rho^* & 1 \end{pmatrix}\right\} = x\,\mathcal{N}(d_1, d_2;\ \rho^*). \tag{5.1}$$

For $V_2(\mathbf{X}, t)$, the only change compared to $V_1(\mathbf{X}, T)$ is $\alpha = [0, 0, 0]'$. This leads to

$$\theta = -r\tau_2, \quad \mathbf{x}^\alpha = x^0 y^0 x^0 = 1,$$

$$\mathbf{d}' = \begin{pmatrix} \left\{\log(x/y) + \frac{1}{2}(\sigma_2^2 - \sigma_1^2)\tau_1\right\} \Big/ \sigma\sqrt{\tau_1} \\ \left\{\log(x/k) + (r - \frac{1}{2}\sigma_1^2)\tau_2\right\} \Big/ \sigma_1\sqrt{\tau_2} \end{pmatrix} = \begin{pmatrix} d_1' \\ d_2' \end{pmatrix},$$

say. Note that $d_1' = d_1 - \rho^*\sigma_1\sqrt{\tau_2}$ and $d_2' = d_2 - \sigma_1\sqrt{\tau_2}$. Hence, we obtain

$$V_2(x, y, t) = e^{-r\tau_2}\,\mathcal{N}(d_1', d_2';\ \rho^*). \tag{5.2}$$

Putting the two expressions (5.1) and (5.2) together, we get the ESO's present value as

$$V(x, y, t) = x\,\mathcal{N}(d_1, d_2;\ \rho^*) - ke^{-r\tau_2}\,\mathcal{N}(d_1', d_2';\ \rho^*). \tag{5.3}$$

Expression (5.3) has the same Black–Scholes structure as for a standard European call option, with the usual univariate normals being replaced by bivariate normals. Furthermore, the correlation coefficient of the bivariate normals is not ρ, as might be expected, but rather the more complicated ρ^*. It is clear from this rather simple example that multi-asset, multi-period exotics can be very complicated indeed. Our formula however, allows one to price such exotics in a systematic way, using only elementary matrix operations.

6. Conclusion

The present value of the M-binary is the main result of the paper. The underlying asset price dynamics are assumed to be multivariate geometric Brownian motion which extends the traditional univariate Black–Scholes model. The price, for all times prior to the first monitoring date, is guaranteed to be arbitrage-free. The corresponding hedging parameters can be determined from the vector $\mathbf{h} = \operatorname{grad} V_t(\mathbf{x})$, while other "greeks" can also be obtained by partial differentiation of (3.2) with respect to the various parameters.

The formula only applies to European style derivatives and underlying asset dynamics with constant parameters, although it is a straightforward matter to extend it to include deterministic time-varying interest rates, dividend yields, volatilities and correlation coefficients. If the underlying asset model requires stochastic volatilities, then there is no escaping the use of numerical methods such as multivariate Monte Carlo schemes. The analytic formula presented in this paper could then be used as an ideal control variate to stabilize the variance of the estimate.

It is interesting to observe that the price of the M-binary is expressed in terms of a multivariate Gaussian distribution function whose order m is determined solely by the dimension of the exercise condition and is independent of the number of assets and the number of monitoring periods. Thus, even if the payoff is multivariate, but the exercise condition is only one-dimensional, the price will only depend on univariate Gaussians.

It is well known that multivariate Gaussians can present numerical problems if the order is too high. The univariate case ($m = 1$) is linearly related to the error function and presents no difficulty. An efficient algorithm for the bivariate case ($m = 2$) due to Drezner was popularized in the mathematical finance literature by Hull [5]. Algorithms for higher-order Gaussians have been published by Genz [2].

The formulae in (3.2) reduce in their complexity for some classes of the input parameters $\{\alpha, \mathbf{a}, S, A\}$. We discuss two of these here.

(1) We have already noted that the exercise condition simplifies if the matrix A is an identity matrix. Other simplifications for the case $A = I_n$ are: matrix D reduces $D = \Sigma = \sigma_i \sqrt{\tau_k}$ so that the correlation matrix C is just R, the joint asset–time correlation matrix of the Gaussian vector $\mathbf{Z}$.

(2) If the payoff is one unit of cash, then all the elements of α are zero and the term e^{θ} reduces to the usual discount factor $e^{-r\tau}$. On the other hand, if the payoff is the value of asset i at time T, then this factor reduces to $e^{\theta} = e^{-q_i\tau}$, as might be expected. Both these cases are illustrated in the ESO example of Section 5.

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