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DIVISIBILITY OF DIRECT SUMS IN TORSION THEORIES

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Introduction. Given a hereditary torsion theory $(\mathscr{A}, \mathscr{B})$ on the category Mod R of right R-modules we obtain in this paper necessary and sufficient conditions for the direct sum $\oplus_{\alpha \in J} M_\alpha$ of a given family $\{M_\alpha\}_{\alpha \in J}$ of R-modules to be divisible for the torsion theory $(\mathscr{A}, \mathscr{B})$. Using this criterion we show that if $\{M_\alpha\}_{\alpha \in J}$ is a family of R-modules having the property that $\oplus_{\alpha \in K} M_\alpha$ is divisible for every countable subset K of J then $\oplus_{\alpha \in J} M_\alpha$ is itself divisible. If $\mathscr{F} = \{I | I$ a right ideal in R such that $R|I \in \mathscr{A}\}$ is the filter of "dense" right ideals of R associated with the torsion theory $(\mathscr{A}, \mathscr{B})$, we show that if arbitrary direct sums of divisible modules for the torsion theory $(\mathscr{A}, \mathscr{B})$ are divisible then the family $\mathscr{F}$ satisfies the ascending chain condition. The results obtained in this paper are generalisations of our earlier results concerning direct sums of injective modules [**4**]. If $(\mathscr{A}, \mathscr{B})$ is the hereditary torsion theory for which $\mathscr{A}$ is the class of all R-modules and $\mathscr{B}$ consists only of the module 0, then the notion of a divisible module is the same as that of an injective module and we recover the results obtain in [**4**].

1. Preliminaries. Throughout this paper R will denote a ring with $1 \neq 0$ and our attention will be confined to the category Mod R of unital right R-modules. In [**2**] S. E. Dickson defined a torsion theory on Mod R to be a pair $(\mathscr{A}, \mathscr{B})$ of non-empty classes of modules satisfying the following conditions:

(1) $\mathscr{A} \cap \mathscr{B} = \{0\}$, the set consisting only of the module 0

(2) If $A \to A'' \to 0$ is exact with $A \in \mathscr{A}$ then $A'' \in \mathscr{A}$

(3) If $0 \to B' \to B$ is exact with $B \in \mathscr{B}$ then $B' \in \mathscr{B}$ and

(4) For every module M there is an exact sequence $0 \to A \to M \to B \to 0$ with $A \in \mathscr{A}$, $B \in \mathscr{B}$.

The modules in $\mathscr{A}$ are called the *torsion modules* for the torsion theory and those in $\mathscr{B}$ are called *torsion free*. If $(\mathscr{A}, \mathscr{B})$ is a torsion theory, then $\mathscr{A}$ is closed under isomorphic images, factor modules, extensions and arbitrary direct sums, whereas $\mathscr{B}$ is closed under isomorphic images, submodules, extensions and arbitrary direct products [**2**, Theorem 2.3]. A torsion theory $\mathscr{A}$ is called *hereditary* if $\mathscr{A}$ is closed under submodules or equivalently $\mathscr{B}$ is closed under injective hulls [**2**, Theorem 2.9]. In [**3**] J. Lambek refers to a torsion theory in the sense of S. E. Dickson as a pretorsion theory and the term torsion

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theory is reserved only to a hereditary torsion theory in the sense of S. E. Dickson. We will follow the terminology of Dickson. We now recall the notion of a divisible module introduced by J. Lambek [3] and the dual notion of a codivisible module introduced by P. E. Bland [1].

Definition 1.1. A module M is called *divisible* if given any exact sequence $0 \to L \xrightarrow{f} N$ with coker $f \in \mathscr{A}$ and any $g : L \to M$, there exists $h : N \to M$ such that $h \circ f = g$.

Dually, M is called *codivisible* if given any exact sequence $N \xrightarrow{f} L \to 0$ with ker $f \in \mathscr{B}$ and any $g : M \to L$, there exists $h : M \to N$ such that $f \circ h = g$.

It is clear that the direct product of any family of divisible modules is divisible and the direct sum of any family of codivisible modules is codivisible. Any direct summand of a divisible (respectively, codivisible) module is divisible (respectively, codivisible). Associated to any hereditary torsion theory $(\mathscr{A}, \mathscr{B})$ on Mod R there is a unique filter of right ideals

$$\mathscr{F} = \{I|I \text{ right ideal in } R \text{ such that } R|I \in \mathscr{A}\}.$$

Each one of the ideals I in $\mathscr{F}$ is dense in R in the sense of [**3**, page 3]. We will be using the following characterisation of divisible modules, valid for hereditary torsion theories.

LEMMA 1.2. *A module M is divisible if and only if given any $f : I \to M$ with $I \in \mathscr{F}$ there exists an extension $h : R \to M$ of f or equivalently there exists an element $m \in M$ satisfying $f(\lambda) = m\lambda$ for all $\lambda \in I$.*

Let $\{M_\alpha\}_{\alpha \in J}$ be a family of modules. We write $\bar{x}$ for the element $(x_\alpha)_{\alpha \in J}$ in the direct product $\prod_{\alpha \in J} M_\alpha$. For any $\bar{x} \in \prod_{\alpha \in J} M_\alpha$ we set $I_{\bar{x}} = \{\lambda \in R|\bar{x}\lambda \in \oplus_{\alpha \in J} M_\alpha\}$.

Definition 1.3. An element $\bar{x} \in \prod_{\alpha \in J} M_\alpha$ will be called *special* if there exists a finite subset F of J such that $x_\alpha\lambda = 0$ for all $\alpha \in J - F$ and $\lambda \in I_{\bar{x}}$.

The right annihilator $\{\lambda \in R|x\lambda = 0\}$ of an element x of a module M will be denoted by $r(x)$. Throughout this paper $(\mathscr{A}, \mathscr{B})$ denotes a hereditary torsion theory and $\mathscr{F}$ the filter $\{I|I \text{ right ideal of } R \text{ with } R|I \in \mathscr{A}\}$ of dense ideals in R. The sum of all torsion submodules of M is itself in $\mathscr{A}$ and thus for any module M, there exists a largest torsion submodule $T(M)$ of M. $T(M)$ will be referred to as the torsion submodule of M.

Definition 1.4. An element $x \in M$ will be referred to as a *torsion element* of M if and only if $x \in T(M)$.

It is well-known that $x \in T(M)$ if and only if $r(x) \in \mathscr{F}$ (Lemma 2.1 of [**1**]).

2. Divisibility of direct sums. Let $\{M_\alpha\}_{\alpha \in J}$ be a given family of modules, N the factor module $(\prod_{\alpha \in J} M_\alpha)/(\oplus_{\alpha \in J} M_\alpha)$ and $p : \prod_{\alpha \in J} M_\alpha \to N$ the canonical quotient map.

THEOREM 2.1. *$\oplus_{\alpha \in J} M_\alpha$ is divisible if and only if each M_α is divisible and every element $x \in p^{-1}(T(N))$ is a special element.*

Proof. Assume each M_α divisible and every $\bar{x} \in p^{-1}(T(N))$ special. Let $f : I \rightarrow \oplus_{\alpha \in J} M_\alpha$ be any map with $I \in \mathscr{F}$. Since $\prod_{\alpha \in J} M_\alpha$ is divisible there exists an element $\bar{u} \in \prod_{\alpha \in J} M_\alpha$ such that $f(\lambda) = \bar{u}\lambda$ for all $\lambda \in I$. In particular $\bar{u}I \subset \oplus_{\alpha \in J} M_\alpha$. Hence $I \subset I_{\bar{u}}$. The map $\varphi : R \rightarrow N$ defined by $\varphi(\lambda) = p(\bar{u})\lambda = p(\bar{u}\lambda)$ for all $\lambda \in R$ satisfies $\varphi(I) = 0$. If $\eta : R \rightarrow R/I$ denotes the canonical quotient map, it follows that φ factors through η. If $\bar{\varphi} : R/I \rightarrow N$ denotes the induced map, we have $\bar{\varphi}(R/I) = p(\bar{u})R$. Since $R/I \in \mathscr{A}$ by condition (2) in the definition of a torsion theory we get $p(\bar{u}) R \in \mathscr{A}$. Thus $p(\bar{u}) \in T(N)$ or $\bar{u} \in p^{-1}(T(N))$. Hence by assumption, $\bar{u}$ is a special element. Hence $u_\alpha I_{\bar{u}} = 0$ for almost all α. Since $I \subset I_{\bar{u}}$ we get $u_\alpha I = 0$ for almost all α. Let $\bar{v} \in \oplus_{\alpha \in J} M_\alpha$ be the element whose α-component is u_α or 0 according as $u_\alpha I \neq 0$ or $u_\alpha I = 0$. Then it is clear that $\bar{u}\lambda = \bar{v}\lambda$ for all $\lambda \in I$. The map $h : R \rightarrow \oplus_{\alpha \in J} M_\alpha$ defined by $h(\lambda) = \bar{v}\lambda (\lambda \in R)$ extends f. From Lemma 1.2 it follows that $\oplus_{\alpha \in J} M_\alpha$ is divisible. This proves sufficiency.

Conversely, assume $\oplus_{\alpha \in J} M_\alpha$ is divisible. Being a direct summand of $\oplus_{\alpha \in J} M_\alpha$ it follows that each M_α is divisible. Let $\bar{x} \in p^{-1}(T(N))$. Clearly $I_x = r(p(\bar{x}))$. Since $p(\bar{x}) \in T(N)$ it follows that $I_x \in \mathscr{F}$. Consider the map $f : I_x \rightarrow \oplus_{\alpha \in J} M_\alpha$ given by $f(\lambda) = \bar{x}\lambda$. The divisibility of $\oplus_{\alpha \in J} M_\alpha$ now implies that there exists an extension $h : R \rightarrow \oplus_{\alpha \in J} M_\alpha$ of f. If $\bar{y} = h(1) \in \oplus_{\alpha \in J} M_\alpha$ we have $y_\alpha = 0$ for almost all α and $x_\alpha I_{\bar{x}} = y_\alpha I_{\bar{x}} = 0$ for almost all α. Thus $\bar{x}$ is a special element. This proves necessity.

THEOREM 2.2. *Suppose $\{M_\alpha\}_{\alpha \in J}$ is a family of R-modules such that for every countable subset K of J, $\oplus_{\alpha \in K} M_\alpha$ is divisible. Then $\oplus_{\alpha \in J} M_\alpha$ is itself divisible.*

Proof. Assume if possible that $\oplus_{\alpha \in J} M_\alpha$ is not divisible. Write N_J for the factor module $(\prod_{\alpha \in J} M_\alpha)/(\oplus_{\alpha \in J} M_\alpha)$ and $p_J : \prod_{\alpha \in J} M_\alpha \rightarrow N_J$ for the canonical quotient map. From Theorem 2.1, there exists an element $\bar{x} \in P_J^{-1}(T(N_J))$ which is not special. Then $x_\alpha I_{\bar{x}} \neq 0$ for infinitely many $\alpha \in J$. Let K be an infinite countable subset of $\{\alpha \in J | x_\alpha I_{\bar{x}} \neq 0\}$. Let $N_K = (\prod_{\alpha \in K} M_\alpha)/(\oplus_{\alpha \in K} M_\alpha)$ and $p_K : \prod_{\alpha \in K} M_\alpha \rightarrow N_K$ the quotient map. Let $\bar{y} \in \prod_{\alpha \in K} M_\alpha$ be defined by $y_\alpha = x_\alpha$ for all $\alpha \in K$. Then clearly $I_{\bar{x}} \subset I_{\bar{y}}$ and $I_{\bar{y}} = r(p_K(y))$. From $I_{\bar{x}} = r(p_J(x)) \in \mathscr{F}$ we see that $I_{\bar{y}} \in \mathscr{F}$. Hence $r(p_K(y)) \in \mathscr{F}$. This implies that $p_K(\bar{y}) \in T(N_K)$. Clearly $y_\alpha I_{\bar{y}} \supseteq x_\alpha I_{\bar{x}} \neq 0$ for all $\alpha \in K$. By Theorem 2.1, this implies that $\oplus_{\alpha \in K} M_\alpha$ is not divisible, a contradiction. This completes the proof of Theorem 2.2.

THEOREM 2.3. *If arbitrary direct sums of divisible modules are divisible, then $\mathscr{F}$ satisfies the ascending chain condition.*

Proof. Suppose $\mathscr{F}$ does not satisfy the ascending chain condition. Then there exists an infinite sequence $I_1 \subset I_2 \subset I_3 \subset \ldots$ of right ideals $I_j \in \mathscr{F}$ with $I_j \neq I_{j+1}$ for every $j \geqq 1$. Let $A_j = R/I_j$, $\eta_j : R \rightarrow A_j$ the canonical

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projection and $x_j = \eta_j(1) \in A_j$. Let M_j be the divisible hull of A_j [**3**, Proposition 0.7, page 10]. Consider the element $\bar{x} = (x_j)_{j \geqq 1}$ of $\prod_{j \geqq 1} M_j$. Let $N = (\prod_{j \geqq 1} M_j)/(\oplus_{j \geqq 1} M_j)$ and $p : \prod_{j \geqq 1} M_j \to N$ the canonical quotient map. For any $\lambda \in I_j$ we have $x_k \lambda = 0$ whenever $k \geqq j$. Hence $I_j \subset I_x$ for all $j \geqq 1$.In particular we get $r(p(\bar{x})) = I_{\bar{x}} \in \mathscr{F}$. Hence $p(\bar{x}) \in T(N)$. Let λ_j be any element of I_{j+1} which is not in I_j. Then $x_j \lambda_j \neq 0$ and $\lambda_j \in I_{\bar{x}}$. Hence $x_j I_{\bar{x}} \neq 0$ for each $j \geqq 1$. It follows that $\bar{x}$ is not a special element in $\prod_{j \geqq 1} M_j$ and hence by Theorem 2.1, $\oplus_{j \geqq 1} M_j$ is not divisible. This proves Theorem 2.3.

Remarks 2.4. (1) In case $(\mathscr{A}, \mathscr{B})$ is the hereditary torsion theory with $\mathscr{A} =$ the class of all R-modules and $\mathscr{B} = \{0\}$, the notion of a divisible module agrees with that of an injective module. Then Theorem 2.1 gives necessary and sufficient conditions for a direct sum $\oplus_{\alpha \in J} M_\alpha$ to be injective. It then asserts that $\oplus_{\alpha \in J} M_\alpha$ is injective if and only if each M_α is injective and every element $\bar{x} \in \prod_{\alpha \in J} M_\alpha$ is special. This is equivalent to Theorem 3 of [**4**]. Also Theorem 2.2 in this case reduces to Theorem 4 of [**4**].

(2) We do not know whether the converse of Theorem 2.3 is true.

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