Can. J. Math. Vol. **46** (4), 1994 pp. 687–698

EXACT INEQUALITIES FOR THE NORMS OF FACTORS OF POLYNOMIALS

PETER B. BORWEIN

ABSTRACT. This paper addresses a number of questions concerning the size of factors of polynomials. Let p be a monic algebraic polynomial of degree n and suppose $q_1q_2\cdots q_i$ is a monic factor of p of degree m. Then we can, in many cases, exactly determine

$$\max\left\{\frac{\|q_1\|\,\|q_2\|\cdots\|q_i\|}{\|p\|}\right\}.$$

Here $\|\cdot\|$ is the supremum norm either on $[-1,1]$ or on $\{|z|\leq 1\}$. We do this by showing that, in the interval case, for each m and n, the n-th Chebyshev polynomial is extremal. This extends work of Gel'fond, Mahler, Granville, Boyd and others. A number of variants of this problem are also considered.

1. **Introduction.** How large can the norms of factors of a polynomial be? Variations of this problem have attracted considerable attention over the years (see [2]–[14]). We exactly solve this problem in the following forms:

Suppose $p = q\cdot r$ where p, q and r are polynomials of degree n, m and $n-m$ respectively. Then for all m and n

$$\|q\|_{[-1,1]}\|r\|_{[-1,1]}\leq K_{m,n}\|p\|_{[-1,1]} \tag{1}$$

where

$$K_{m,n}:=2^{n-1}\prod_{k=1}^{m}\left(1+\cos\frac{2k-1}{2n}\pi\right)\prod_{k=1}^{n-m}\left(1+\cos\frac{2k-1}{2n}\pi\right) \tag{2}$$

and this bound is exactly attained by the Chebyshev polynomial of degree n. (We denote by $\|\cdot\|_{[-1,1]}$ the sup norm on the interval $[-1,1]$.) This is the content of Theorem 1. (Theorem 1 is originally due to Kneser [10]; see also Aumann [2]. We offer a new proof of this result that easily modifies to the other cases we wish to consider.)

Suppose now that q is a factor of p of degree m and suppose that p and q are both monic. Then for each n and m,

$$|q(-2)|\leq 2^{m-1}\prod_{k=1}^{m}\left(1+\cos\frac{2k-1}{2n}\pi\right)\|p\|_{[-2,2]}. \tag{3}$$

Research supported in part by NSERC of Canada.
Received by the editors September 29, 1992.
AMS subject classification: 26D05, 30C10, 41A10.
Key words and phrases: polynomials, factors, Chebyshev polynomials, inequalities.

https://doi.org/10.4153/CJM-1994-038-8 Published online by Cambridge University Press

This bound is attained by the Chebyshev polynomial of degree n on $[-2, 2]$ with q being the factor composed of the m roots closest to 2. This is the content of Theorem 2.

We then generalize these results to the many factors case. We prove in, Theorem 4, that

$$\|q_1\|_{[-1,1]}\|q_2\|_{[-1,1]}\cdots\|q_j\|_{[-1,1]} \leq 2^{m-n}(3.20991\cdots)^m\|p\|_{[-1,1]}$$

where $q_1\cdots q_j$ is a monic factor of degree m of a monic polynomial p of degree n. This result is sharp for $j \geq 2$. A version of this on the disc is given in Theorem 8.

If

$$x = \frac{1}{2}(z + \frac{1}{z})$$

then

$$p(x) = t(z)t\left(\frac{1}{z}\right) \tag{4}$$

is a mapping between polynomials of degree n on the interval $[-1, 1]$ and self reciprocal polynomials, of the above form, of degree $2n$ on the boundary of the unit disc. So there is some equivalence between factoring self reciprocal polynomials on the disc and factoring on the interval.

From this and Theorem 1 we deduce that if $p = q \cdot r$ are *real* polynomials of degree n, m and $n - m$ respectively then

$$\|q\|_D\|r\|_D \leq (K_{m,n})^{1/2}\|p\|_D$$

and that this bound is exactly obtained, for even n and m, by $z^n + 1$. (Here $\|\cdot\|_D$ is the sup norm on the complex unit disc D.) This is done in Theorem 6.

Boyd [5], using a Mahler measure argument, gives a very pretty proof that

$$\|q\|_D\|r\|_D \leq c^n\|p\|_D$$

with $c = 1.7916\cdots$ (Here $\|\cdot\|_D$ is the sup norm on the complex unit disc). This is an asymptotically best possible result, and improves on earlier bounds of $c = 4$ due to Gel'fond [7] and $c = 2$ due to Mahler [11]. However, there is no dependence on the degrees of the factors.

This result is extended to the many factor case in Boyd [6], a result we reproduce by different methods as Theorem 7.

Related problems concerning the size of single factors (with some normalization conditions) are considered in [3], [5] and [14] and various other problems on norms of factors and products are treated in [3], [4], [12] and [13].

In particular in Boyd [5] it is shown that if $p = q \cdot r$ with p, q and r all monic then

$$\|q\|_D \leq d^n\|p\|_D \tag{5}$$

where $d = 1.3813\cdots (:= M(1+x+y)$ where M is the Mahler measure of a polynomial). This again is asymptotically best possible and improves on earlier results of Granville [9] and Glesser [8] who derived (5) with $d := \frac{(1+\sqrt{5})}{2}$ and $d := \frac{3}{2}$ respectively.

Mignotte [14] derives the inequality

$$\|q\|_D \le 2^m \|p\|_D$$

(recall that m is the degree of q) and shows that this is asymptotically sharp for some n roughly of size $m^2 \log m$.

A variant of these last results, which is sharper for certain polynomials, such as those with integer coefficients, is also given in Section 5. This is Theorem 5.

2. **The two factor case.** Throughout this note we will use the notation $\|\ \| := \|\ \|_{[-1,1]}$ to denote the supremum norm on $[-1,1]$ and more generally $\|\ \|_\chi$ to denote the supremum norm on χ. In this section and the next section the polynomials in question are algebraic polynomials with possibly complex coefficients. We denote by π_n the set of such polynomials of degree at most n. We begin by offering a new proof of an old result.

THEOREM 1. *Let p be any polynomial of degree n and suppose $p = q \cdot r$ where q is a factor of degree m and r is a factor of degree $n - m$. Then*

$$\|q\|_{[-1,1]}\|r\|_{[-1,1]} \le \frac{1}{2} C_{n,m} C_{n,n-m} \|p\|_{[-1,1]}$$

where

$$C_{n,m} := 2^m \prod_{k=1}^{m} \left(1 + \cos \frac{2k-1}{2n}\pi\right).$$

Furthermore, for any n and $m \le n$ the inequality is sharp in the case that p is the Chebyshev polynomial of degree n. In this case the factor q is chosen to be the factor with m roots closest to -1.

On applying the midpoint rule to $\log(2 + 2\cos x)$, we see that

$$\frac{1}{n}\int_0^{m/n} \log(2 + 2\cos \pi x) = \sum_{k=1}^{m} \log\left(2 + 2\cos \frac{2k-1}{2n}\pi\right) + 0\left(\frac{1}{(n-m)^2}\right).$$

Recall that the error in the midpoint rule is $cf''(\zeta)/n^2$ and in this case $f''(x) = \frac{-\pi^2}{(1+\cos \pi x)^2}$. Thus

$$(6)\qquad \begin{aligned} C_{n,m} &= e^{\log \prod_{k=1}^{m} 2(1+\cos \frac{2k-1}{2n}\pi)} \\ &= \left(e^{1/n \sum_{k=1}^{m} \log(2+2\cos \frac{2k-1}{2n}\pi)}\right)^n \\ &= e^{0(n/(n-m)^2)} (e^{I(n,m)})^n \end{aligned}$$

where

$$I(n,m) := \int_0^{m/n} \log(2 + 2\cos x)\, dx.$$

Note that

$$(7)\qquad C_{n,\lfloor n/2\rfloor}^{1/n} \sim \left(e^{\int_0^{1/2} \log(2+2\cos \pi x)\, dx}\right) = (1.7916\cdots)$$

https://doi.org/10.4153/CJM-1994-038-8 Published online by Cambridge University Press

and that

$$C_{n,\lfloor 2n/3\rfloor}^{1/n} \sim \left(e^{\int_0^{2/3}\log(2+2\cos\pi x)\,dx}\right) = (1.9081\cdots). \tag{8}$$

(The first constant appears in Boyd [5] as does the square root of the second constant. We would expect this from the transformation (4). Similar asymptotics are in [10]). From (7) and Theorem 1 we have

COROLLARY 1. *Let p be any polynomial of degree n and suppose $p = q\cdot r$ with q and r polynomials then*

$$\|q\|_{[-1,1]}\|r\|_{[-1,1]} \le 2^{n-1}\prod_{k=1}^{\lfloor n/2\rfloor}\left(1+\cos\frac{2k-1}{2n}\pi\right)^2\|p\|_{[-1,1]}$$
$$\lesssim (3.20991\cdots)^n\|p\|_{[-1,1]}$$

(with equality for the Chebyshev polynomial of degree n). ∎

Here and throughout we use $a_n \lesssim b_n$ for positive sequences to mean $\limsup \frac{a_n}{b_n} \le 1$. The proof of the theorem proceeds through a number of lemmas. For the remainder of the proof we assume n and $m < n$ fixed. Let p be of degree n and suppose $p = q\cdot r$ where q has degree m and r has degree $n-m$. It is an easy consequence of the finite dimensionality of the set of polynomials of degree n that

$$\sup\left\{\frac{\|q\|\,\|r\|}{\|p\|} : p = qr,\ p\in\pi_n,\ q\in\pi_m,\ r\in\pi_{n-m}\right\} \tag{$*$}$$

is attained for some p, q, r. We proceed to show that such an extremal p^* is in fact the Chebyshev polynomial of degree of n (and that the factors q and r are as advertised).

That is

$$p^*(x) = T_n(x) = \cos n\cos^{-1}(x) = \frac{1}{2}\prod_{k=1}^{n}2\left(x-\cos\frac{2k-1}{2n}\pi\right)$$

and the extremal factors, q^* and r^*, are

$$\frac{1}{\sqrt{2}}\prod_{k=1}^{m}2\left(x-\cos\frac{2k-1}{2n}\pi\right) \quad\text{and}\quad \frac{1}{\sqrt{2}}\prod_{k=m+1}^{n}2\left(x-\cos\frac{2k-1}{2n}\pi\right)$$

respectively. Note that

$$\|q^*\| = |q(-1)| = \frac{1}{\sqrt{2}}C_{n,m}$$

and

$$\|r^*\| = |r(1)| = \frac{1}{\sqrt{2}}C_{n,n-m}.$$

(These elementary properties of T_n are available in [1] or [15].)

Our first observation is that we may assume that $|q|$ has a maximum at -1 and $|r|$ has a maximum at 1. This follows since the supremum in $(*)$ is invariant under a change of

https://doi.org/10.4153/CJM-1994-038-8 Published online by Cambridge University Press

variables $x \longmapsto ax + b$ and a corresponding change on the underlying interval. (That is, the sup in $(*)$ is interval independent.) So if $||q|| = |q(\alpha)|$, $\alpha \in (-1, 1)$ and $||r|| = |r(\beta)|$, $\beta \in (-1, 1)$. Then

$$\sup\left\{\frac{||q||_{[\alpha,\beta]}||r||_{[\alpha,\beta]}}{||p||_{[\alpha,\beta]}}\right\} \geq \sup\left\{\frac{||q||\,||r||}{||p||}\right\}$$

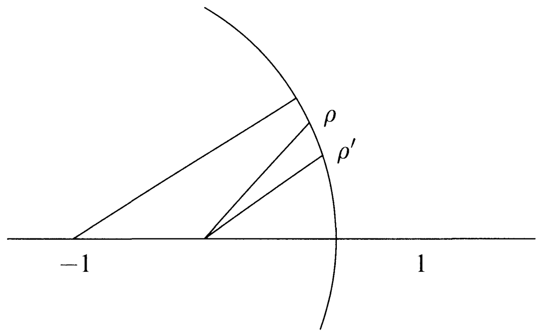
since $||p||_{[\alpha,\beta]} \leq ||p||$. But we can scale this back to the interval $[-1, 1]$ and get a new extremal of the desired type. (Note that $\alpha = \beta$ is a trivial case.)

So we now assume we have an extremal example where q and r attain their maxima at -1 and $+1$ respectively. (From this it follows that the p in the sup in $(*)$ has exact degree n.)

LEMMA 1. *Under the above assumption on p, q and r we may assume that p has all real roots.*

PROOF. This argument (and most of the following arguments) is a perturbation argument. We show that a purely complex root may be perturbed a bit to construct an example that contradicts extremality.

So suppose ρ, with $\mathrm{im}(\rho) > 0$, is a complex root of q (as in the diagram)



If we rotate ρ to ρ' along a circle of radius $|\rho + 1|$ in a clockwise fashion. Then $|q(-1)|$ and $|r(1)|$ are unchanged and these are still maxima. However, for any $x \in (-1, 1]$ the perturbed $p(x)$ has smaller absolute value then the original $p(x)$. (This is equivalent to observing that if $\rho + 1 = re^{i\theta}$ then $|\rho - x|^2 = r^2 - 2(x + 1)r\cos\theta + (x_1)^2$ is an increasing function of θ for $0 \leq \theta \leq \pi$.) This finishes the proof. ∎

So we may add to our assumption on the extremal p that p has all real roots.

LEMMA 2. *Under the above assumptions on p, q and r we may assume that p has all its roots in* $[-1, 1]$.

PROOF. We consider the case where q has a root $\rho > 1$ and observe that a perturbation of ρ to ρ' where $1 \leq \rho' < \rho$ has the effect of leaving $r(1)$ unchanged while

https://doi.org/10.4153/CJM-1994-038-8 Published online by Cambridge University Press

$|q(-1)|/\|p\|$ does not increase. (And $|q(-1)|$ is still a maximum of q on $[-1,1]$). This follows because, for fixed $x_0 \in [-1,1]$.

$$\frac{\rho-(-1)}{\rho-x_0}$$

is a decreasing function of ρ on $[1,\infty)$. (A similar argument holds for $\rho \in [-\infty, 1)$ only now we perturb ρ to $\rho' < p \leq -1$.) ∎

It is now clear that the extremal q and r are chosen by splitting the roots of p so that the m roots closest to 1 are the roots of q and the remaining $n-m$ roots are the roots of r. (Otherwise one would interchange two roots, and increase both $|q(-1)|$ and $|r(1)|$ without altering $\|p\|$).

We have now shown that we may assume our extremal example is of the form

$$\begin{aligned} p(x) &= a\prod_{k=1}^{n-m}(x-\alpha_k)\prod_{k=1}^{m}(x-\beta_k) \\ &= ar(x)q(x) \end{aligned}$$

where

$$-1 \leq \alpha_1 \leq \cdots \leq \alpha_{n-m} \leq \beta_1 \leq \cdots \leq \beta_m \leq 1.$$

Furthermore $|r(x)|$ has a maximum at +1 and $|q(x)|$ has a maximum at -1. We now also, for convenience, assume $\|p\| = 1$. We are ready to prove Theorem 1.

PROOF OF THEOREM 1. We show three things

1] $|p(-1)| = 1$ and $|p(1)| = 1$,

2] $|p(x)| = 1$ for some $x \in (\alpha_i, \alpha_{i+1})$ $i = 1, \ldots, n-m-1$

$|p(x)| = 1$ for some $x \in (\beta_i, \beta_{i+1})$ $i = 1, \ldots, m-1$.

3] $|p(x)| = 1$ for some $x \in (\alpha_{n-m}, \beta_1)$.

These three facts show that p is indeed the Chebyshev polynomial $\pm T_n$ (since T_n is the unique maximally oscillatory polynomial on $[-1,1]$).

To prove 1] we observe that if $|p(-1)| \neq 1$ then we would increase the interval on which we consider the problem to $[-(1+\varepsilon), 1]$ without increasing the norm of p. However since r is monotone on $[-\infty, \alpha_1]$ we have (strictly) increased the norm of r and have violated extremality. (A similar argument applies to $p(1)$.)

To prove 2] we suppose α_i and α_{i+1} are adjacent roots and let

$$s(x) = (x-\alpha_i)(x-\alpha_{i+1}).$$

Let $\varepsilon > 0$, we can find $0 < \delta_1, \delta_2 < \varepsilon$ so that

$$t(x) = \big(x-(\alpha_i-\delta_1)\big)\big(x-(\alpha_{i+1}+\delta_2)\big)$$

satisfies

$$\begin{aligned} \|t(x)-s(x)\| &< \varepsilon \\ |t(-1)| &= |s(-1)| \end{aligned}$$

and

$$|t(x)| < |s(x)| \quad x \in (-1, \alpha_i - \delta_1] \cup [\alpha_{i+1} + \delta_2, 1).$$

Now if $||p||_{[\alpha_i,\alpha_{i+1}]} \neq 1$ we have, for sufficiently small ε that

$$p^*(x) := p(x) \cdot t(x)/s(x)$$

is a new extremal and is extremal on an interval $[-1, 1+\delta]$ (since $|p^*(1)| < 1$). Now if p^* is scaled back to the interval $[-1, 1]$ we violate the extremality of p as in the first part of this proof. (The argument for (β_i, β_{i+1}) is identical.)

To prove 3] we consider the effect of moving β_1 slightly to the right and α_{n-m} an equal amount to the left. If $||p||_{[\alpha_{n-m},\beta_1]} \neq 1$ this increases both $|r(-1)|$ and $|q(1)|$ but does not increase $||p||_{[-1,1]}$. This finishes the proof. ∎

3. **The single factor case.**

THEOREM 2. *Suppose that p is a monic polynomial of degree n and that q is a monic factor of p of degree m. Then*

$$|q(-\beta)| \leq ||p||_{[-\beta,\beta]}(\beta^{m-n}2^{n-1}) \prod_{k=1}^{m}\Big(1 + \cos\frac{2k-1}{2n}\pi\Big).$$

This bound is attained for each n and $m \leq n$ by the Chebyshev polynomial of degree n on $[-\beta, \beta]$ (normalized to be monic). In this case q is chosen to be the factor with m roots closest to β.

COROLLARY 2. *Suppose p is a monic polynomial of degree n and q is a monic factor of p of degree m. Then*

$$|q(-2)| \leq ||p||_{[-2,2]}2^{m-1} \prod_{k=1}^{m}\Big(1 + \cos\frac{2k-1}{2n}\pi\Big) = \frac{1}{2}C_{n,m}||p||_{[-2,2]}$$

and the inequality is sharp for all m and n. Furthermore for all $m \leq n$,

$$C_{n,m}^{1/n} \lesssim C_{n,\lfloor 2n/3\rfloor}^{1/n} \sim (1.9081\cdots).$$

PROOF OF COROLLARY 2. Take $\beta = 2$ in Theorem 2. Note that

$$2^m \prod_{k=1}^{m}\Big(1 + \cos\frac{2k-1}{2n}\pi\Big) \leq 2^{\gamma(n)} \prod_{k=1}^{\gamma(n)}\Big(1 + \cos\frac{2k-1}{2n}\pi\Big)$$

where $\gamma(n) + 1$ ($\sim \frac{2n}{3}$) is the smallest positive integer k with $\cos(\frac{2k-1}{2n}\pi) \leq -1/2$.

So

$$C_{n,m}^{1/n} \lesssim C_{n,\lfloor 2n/3\rfloor}^{1/n}$$

and by (8)

$$C_{n,m}^{1/n} \lesssim (1.9081\cdots).$$ ∎

This is the real analogue of the complex results cited in the introduction. Since if we wish to preserve monicity in the mapping (4) we must use the transformation $x = (z + \frac{1}{z})$ which maps the boundary of the disc onto $[-2, 2]$ (not $[-1, 1]$).

SKETCH OF PROOF OF THEOREM 2. The proof of Theorem 2 is very similar to that of Theorem 1 and we only outline it.

Fix m and n. We consider the maximization problem

$$(**) \qquad \max\{|q(-\beta)|/\|p\|_{[-\beta,\beta]} : q \text{ divides } p,\ q \in \pi_m, p \in \pi_n,\ q \text{ and } p \text{ both monic}\}.$$

We can argue now, exactly as in Lemma 1, that p has all real roots and that these all lie in $[-\beta, \infty)$ since any rotation of roots as in Lemma 1 at least maintains extremality. Arguing as in Lemma 2 gives that p has all roots in $[-1, 1]$. Thus q must be composed of the m roots of p closest to 1.

The argument of the proof of Theorem 1 now applies essentially verbatim (note that all the perturbations preserve monotonicity) and proves that an extremal p can be chosen to be the Chebyshev polynomial on $[-\beta, \beta]$ normalized to be monic. Thus on $[-1, 1]$

$$p(x) = \prod_{k=1}^{n}\left(x - \cos\frac{2k-1}{2n}\pi\right)$$

and

$$q(x) = \prod_{k=1}^{m}\left(x - \cos\frac{2k-1}{2n}\pi\right)$$

from which the result follows (on considering $\beta^n p(x/\beta)$ and $\beta^m q(x/\beta)$ on $[-\beta, \beta]$). ∎

4. **The many factor case.**

THEOREM 3. *Suppose that p is a monic polynomial of degree n and that q and r are monic factors of p of degree m_1 and m_2 respectively. (Suppose further that qr is also a factor of p.) Then*

$$|q(-\beta)|\,|r(\beta)| \le \|p\|_{[-\beta,\beta]}(\beta^{m_1+m_2-n}2^{n-1})\prod_{k=1}^{m_1}\left(1+\cos\frac{2k-1}{2n}\pi\right)\prod_{k=1}^{m_2}\left(1+\cos\frac{2k-1}{2n}\pi\right).$$

This bound is attained for the Chebyshev polynomial of degree n on $[-\beta, \beta]$ normalized to be monic.

The proof of Theorem 3 is analogous to the proofs of Theorem 1 or 2 on considering the maximization problem

$$(***) \qquad \max\{|q(-\beta)|\,|r(\beta)|/\|p\|_{[-\beta,\beta]} : p, q, r \text{ as above}\}.$$

THEOREM 4. *Suppose that p is monic polynomial of degree n and suppose that p has a monic factor of degree m of the form $q_1q_2\cdots q_j$ where q_i is of degree d_i (so $m = d_1 + \cdots + d_j$). Then*

$$\|q_1\|_{[-\beta,\beta]}\|q_2\|_{[-\beta,\beta]}\cdots\|q_j\|_{[-\beta,\beta]}$$
$$\le \|p\|_{[-\beta,\beta]}\beta^{m-n}2^{n-1}\prod_{k=1}^{\lfloor\frac{m}{2}\rfloor}\left(1+\cos\frac{2k-1}{2n}\pi\right)\prod_{k=1}^{\lceil\frac{m}{2}\rceil}\left(1+\cos\frac{2k-1}{2n}\pi\right)$$
$$\lesssim \|p\|(2\beta)^{m-n}(3.20991\cdots)^m \quad (m, n \to \infty)$$

(with equality in the first inequality for all $j \geq 2$ and all n and m for the Chebyshev polynomial of degree n).

PROOF. For each q_i, write

$$q_i = r_i s_i$$

where r_i is the monic factor of q_i composed of the roots of q_i in $\{\mathrm{Re}(z) \leq 0\}$ while s_i is composed of the roots of q_i in $\{\mathrm{Re}(z) > 0\}$.

So

$$\|r_i\|_{[-\beta,\beta]} = |r_i(\beta)|$$

and

$$\|s_i\|_{[-\beta,\beta]} = |s_i(-\beta)|$$

Thus

$$\|q_1\|_{[-\beta,\beta]}\|q_2\|_{[-\beta,\beta]} \cdots \|q_j\|_{[-\beta,\beta]} \leq |r_1(\beta)\cdots r_j(\beta)|\,|s_1(-\beta)\cdots s_j(-\beta)|.$$

We now apply Theorem 3 to the two factors $r_1 \cdots r_j$ and $s_1 \cdots s_j$ to finish the proof. ∎

Note that the $m = n$ case reproduces Theorem 2 of [6] in the real case.

5. **Inequalities on the disc.** We now derive the inequalities on the disc from those on the interval. Suppose $t \in \pi_n$ is monic and suppose $s \in \pi_m$ and $v \in \pi_{n-m}$ are monic factors of t. (So $t = s \cdot v$.) Of course all of t, s and v achieve their maxima on D somewhere on the boundary. Now consider

$$p(x) := t(z)t(1/z)$$

and

$$q(x) := s(z)s(1/z) \quad r(x) := v(z)v(1/z)$$

with

$$x := (z + 1/z).$$

The effect of this transformation on linear factors is as follows

$$(z-\alpha)\Big(\frac{1}{z}-\alpha\Big) = -\alpha\Big(x - \frac{1+\alpha^2}{\alpha}\Big)$$

so p is of degree n, q is of degree m and r is of degree $n-m$. Also

$$\|p\|_{[-2,2]} \leq \|t\|_D^2$$

and for a on the boundary of the unit circle

$$|q(a+a^{-1})| = |s(a)s(a^{-1})|.$$

Furthermore the leading coefficent of p is essentially just the product of the roots of t which is of size $|t(0)|$ while the lead coefficient of q is of size $|s(0)|$ and the lead coefficient of r is of size $|r(0)|$.

From these transformations and the interval inequalities we can prove the following theorems.

THEOREM 5. *Suppose $t = s \cdot v$ is a monic polynomial of degree n and s is a monic factor t of degree m. Then*

$$|v(0)|^{1/2}\|s\|_D \le \left(\frac{1}{2}C_{n,m}\right)^{1/2}\|t\|_D.$$

This is sharp, for even n and m, for $t(z) = z^n + 1$.

PROOF. We can assume, by performing an initial rotation if necessary, that

$$\|s\|_D = |s(-1)|.$$

So from Corollary 2 we deduce that

$$\begin{aligned}\|s\|_D^2 = |s(-1)|^2 = |q(-2)| &\le |s(0)/t(0)|2^{m-1}\prod_{k=1}^{m}\left(1+\cos\frac{2k-1}{2n}\pi\right)\|p\|_{[-2,2]} \\ &\le |s(0)/t(0)|2^{m-1}\prod_{k=1}^{m}\left(1+\cos\frac{2k-1}{2n}\pi\right)\|t\|_D^2.\end{aligned} \tag{9}$$

Now $s(0)/t(0) = 1/v(0)$. ∎

THEOREM 6. *Suppose $t = s \cdot v$ where $s \in \pi_m$ and $v \in \pi_{n-m}$ and where t, s and v are* real *polynomials. Then*

$$\|s\|_D\|v\|_D \le \left(\frac{1}{2}C_{n,m}\cdot C_{n,n-m}\right)^{1/2}\|t\|_D.$$

As before,

$$C_{n,m} := 2^m\prod_{k=1}^{m}\left(1+\cos\frac{2k-1}{2n}\pi\right)$$

and

$$(C_{n,m}\cdot C_{n,n-m})^{1/(2n)} \lesssim (1.79162\cdots).$$

This is sharp, for even n and m, for $t(z) = z^n + 1$.

PROOF. From Theorem 1 we deduce, for a and b on the boundary of the unit disc and for arbitrary s and v, that

$$\begin{aligned}|s(a)\cdot s(a^{-1})|\,|v(b)\cdot v(b^{-1})| &= |q(a+a^{-1})|\,|r(b+b^{-1})| \\ &\le \frac{1}{2}C_{n,m}\cdot C_{n,n-m}\|p\|_{[a+1/a,b+1/b]} \\ &\le \frac{1}{2}C_{n,m}\cdot C_{n,n-m}\|p\|_{[-2,2]} \\ &\le \frac{1}{2}C_{n,m}\cdot C_{n,n-m}\|t\|_D^2.\end{aligned}$$

The result now follows on choosing a and b to be points on $\{|z| = 1\}$ where s and v achieve their norms. ∎

The next result reproduces Theorem 2 of Boyd [6].

https://doi.org/10.4153/CJM-1994-038-8 Published online by Cambridge University Press

THEOREM 7. *Suppose $t = s_1 s_2 \cdots s_j$ are real polynomials, with t of degree n, then*

$$\|s_1\|_D \|s_2\|_D \cdots \|s_j\|_D \le \frac{1}{\sqrt{2}} \Big(\prod_{k=1}^{\lceil \frac{n}{2} \rceil} 2\Big(1 + \cos \frac{2k-1}{2n}\pi\Big)\Big) \|t\|_D$$
$$\lesssim (1.7916\cdots)^n \|t\|_D$$

This is sharp for even n for $t(z) = z^n + 1$.

PROOF. Use Theorem 4 and proceed as in the previous theorem. ∎

THEOREM 8. *Suppose $t = vs_1s_2 \cdots s_j$ with all polynomials real. Suppose t is monic of degree n and $s_1s_2 \cdots s_j$ is monic of degree m. Then*

$$|v(0)|^{\frac{1}{2}} \|s_1\|_D \|s_2\|_D \cdots \|s_j\|_D \le \frac{1}{\sqrt{2}} \Big(\prod_{k=1}^{\lceil \frac{m}{2} \rceil} 2\Big(1 + \cos \frac{2k-1}{2n}\pi\Big)\Big) \|t\|_D.$$

This is sharp for even n and m for $t(z) = z^n + 1$ for $j \ge 2$.

PROOF. This follows from Theorem 4, as in Theorem 3, with the following additional detail (as in Theorem 4). We first factor each s_j as

$$s_j = u_j v_j$$

where u_j has all roots in $\{\mathrm{Re}(z) \le 0\}$ and v_j has all roots in $\{\mathrm{Re}(z) > 0\}$. It now follows that

$$\|s_1\|_D \cdots \|s_j\|_D \le |u_1(1) \cdots u_j(1)| \, |v_i(-1) \cdots v_j(-1)|$$

and we proceed as before. ∎

If we don't assume that the polynomials in question are real we can get the following.

COROLLARY 3. *Suppose $t = s \cdot v$ where $s \in \pi_m$ and $v \in \pi_{n-m}$. Then*

$$|s(-1)| \, |v(1)| \le \Big(\frac{1}{2} C_{n,m} \cdot C_{n,n-m}\Big)^{1/2} \|t\|_D$$

and if t is monic

$$|t(0)|^{1/2} \|s\|_D \|v\|_D \le \frac{1}{2} (C_{n,m} \cdot C_{n,n\text{-}m})^{1/2} (\|t\|_D)^2.$$

These are sharp, for even n and m, for $t(z) = z^{n+1}$.

PROOF. The first inequality follows as in the proof of Theorem 6 with $a := -1$ and $b := 1$. The second part is immediate from (9) of the proof of Theorem 5. ∎

REFERENCES

1. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1965.

2. G. Aumann, *Satz, über das Verhalten von Polynomen auf Kontinuen*, Sitz. Preuss. Akad. Wiss. Phys.-Math. Kl., (1933), 926–931.

3. B. Beauzamy and P. Enflo, *Estimations de produits de polynômes*, J. Number Theory **21**(1985), 390–412.

4. B. Beauzamy, E. Bombieri, P. Enflo and H. L. Montgomery, *Products of polynomials in many variables*, J. Number Theory **36**(1990), 219–245.

5. D. W. Boyd, *Two sharp inequalities for the norm of a factor of a polynomial*, Mathematika, to appear.

6. ———, *Sharp inequalities for the product of polynomials*, to appear.

7. A. O. Gel'fond, *Transcendental and Algebraic Numbers*, Dover, New York, 1960; translation by L. F. Boron, Russion edition, 1952.

8. P. Glesser, *Nouvelle majoration de la norme des facteurs d'un polynôme*, C. R. Math. Rep. Acad. Sci. Canada **12**(1990), 224–228.

9. A. Granville, *Bounding the coefficients of a divisor of a given polynomial*, Monatsh. Math. **109**(1990), 271–277.

10. H. Kneser, *Das Maximum des Produkts zweies Polynome*, Sitz. Preuss. Akad. Wiss. Phys.-Math. Kl., (1934), 429–431.

11. K. Mahler, *An application of Jensen's formula to polynomials*, Mathematika **7**(1960), 98–100.

12. ———, *On some inequalities for polynomials in several variables*, J. London Math. Soc. **37**(1962), 341–344.

13. ———, *A remark on a paper of mine on polynomials*, Illinois J. Math. **8**(1964), 1–4.

14. M. Mignotte, *Some useful bounds*. In: Computer Algebra, Symbolic and Algebraic Computation, (eds. B. Burchberger, *et al.*), Springer, New York, 1982, 259–263.

15. T. Rivlin, *Chebyshev Polynomials*, 2nd Edition, Wiley, New York, 1990.

Department of Mathematics
Statistics and Computing Science
Dalhousie University
Halifax, Nova Scotia
B3H 3J5

https://doi.org/10.4153/CJM-1994-038-8 Published online by Cambridge University Press