COMPOSITIONS OF SET OPERATIONS

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1. Introduction. The set operations under consideration are Borel operations and Souslin's operation ($\mathscr{A}$). With respect to a given family of sets and in a setting free of any topological structure there are defined three Borel families (Definitions 3.1) and the family of Souslin sets (Definition 4.1). Conditions on an initial family are determined under which iteration of the Borel operations with Souslin's operation ($\mathscr{A}$) on the initial family and the families successively produced results in a non-decreasing sequence of families of analytic sets (Theorem 5.2.1 and Definition 3.5). A classification of families of analytic sets with respect to an initial family of sets is indicated in a manner analogous to the familiar classification of Borel sets (Definition 5.3).

2. Notation.

2.1. σH is the union of the members of H; i.e.,

$$\sigma H = \{x\colon x \in A \text{ for some } A \in H\}.$$

2.2. $A \sim B$ is the set of points in A which are not in B.

2.3. ω is the set of finite ordinals; i.e., ω is the least denumerable ordinal.

2.4. 0 is the least ordinal and the empty set.

2.5. For n in ω, $\bar{n} = \{m\colon m \in \omega \text{ and } 0 \leqq m \leqq n\}$.

2.6. Z is the set of all sequences of natural numbers; i.e., Z is the set of all functions on ω to ω.

2.7. Z_n' is the set of all functions on $\bar{n}$ to ω.

2.8. If f is a function and A is a set, then $f|A$ is the restriction of f to A; i.e., $f|A$ is the function g on $A \cap$ domain f such that for each $x \in A \cap$ domain f, $g(x) = f(x)$.

3. The Borel families. In Definitions 3.1 it is helpful to think of H as a covering family of subsets of some space. However, the definitions apply to a family H consisting of any sets whatever.

3.1. *Definitions.*

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3.1.1. bH is the smallest family which contains H and is closed to countable non-vacuous union and countable non-vacuous intersection.

3.1.2. βH is the smallest family which contains H and is closed to countable union and set difference.

3.1.3. BH is the smallest family which contains H and is closed to countable (including vacuous) union and complementation with respect to σH.

Thus bH, βH, and BH are the Borel families long associated with H, and the following relations hold for any family H.

3.2. THEOREMS.

3.2.1. $H \subset bH \subset \beta H \subset BH$.

3.2.2. $\sigma H = \sigma bH = \sigma\beta H = \sigma BH$.

3.2.3. $0 \in \beta H$ *and* $\sigma H \in BH$.

The inclusions in 3.2.1 are generally proper. However, the compositions of b, β, and B yield no new set operations. These well-known compositions are most easily presented in a tabular form. (In 3.3 the fact that $\beta bH = \beta H$ for any family H is recorded by the entry in the first column of the second row within the table.)

3.3. THEOREM. *For any family* H,

	bH	βH	BH
b	bH	βH	BH
β	βH	βH	BH
B	BH	BH	BH

In 3.4 and 3.6 below are assembled some closely related conditions on a family H under which some or all of the Borel families over H coincide.

3.4. THEOREMS.

3.4.1 (Sierpiński [**7**]). $bH = \beta H$ *if and only if* $A \sim B \in bH$ *whenever* A *and* B *belong to* H.

3.4.2. $\beta H = BH$ *if and only if* $\sigma H \sim A \in \beta H$ *whenever* $A \in H$.

3.4.3. *If* $\sigma H \in H$, *then* $bH = BH$ *if and only if* $\sigma H \sim A \in bH$ *whenever* $A \in H$.

Proof of 3.4.3. If $bH = BH$, then $\sigma H \sim A \in bH$ whenever $A \in H$.

Conversely, if $\sigma H \in H$ and $\sigma H \sim A \in bH$ whenever $A \in H$, then the empty set belongs to bH, and bH is closed to countable (including vacuous) union. Let $G = \{A: A \in bH \text{ and } \sigma H \sim A \in bH\}$. Then $H \subset G \subset bH$. Now for each $n \in \omega$ let $A_n \in G$. Then $\bigcup_{n\in\omega} A_n \in bH$ and for each $n \in \omega$, $\sigma H \sim A_n \in bH$. Moreover,

$$\sigma H \sim \bigcup_{n\in\omega} A_n = \bigcap_{n\in\omega} (\sigma H \sim A_n) \in bH$$

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and

$$\sigma H \sim (\sigma H \sim A_0) = A_0 \in bH.$$

Thus G contains H and is closed to countable (including vacuous) union and complementation with respect to σH. Consequently, $BH \subset G \subset bH$ and the desired conclusion is at hand.

The well-known case in which H is the family of closed sets in a metric space is a case for which the Borel families coincide. More generally, whenever H is an internal family (Definition 3.5), $bH = \beta H = BH$.

3.5. *Definition* (Morse [**6**]). H is internal if and only if H is closed to finite non-vacuous union and the complements (with respect to σH) of members of H are in bH.

Thus the family of closed sets in a metric space is internal.

3.6. THEOREMS.

3.6.1. *BH is internal.*

Proof. BH is closed to countable union, countable intersection, and complementation with respect to $\sigma BH (= \sigma H)$. Moreover $BH = bBH$.

3.6.2. *βH is internal if and only if $\beta H = BH$.*

Proof. If $\beta H = BH$, then βH is internal, by 3.6.1.
Now let

$$G = \{A : A \in \beta H \text{ and } \sigma H \sim A \in \beta H\}.$$

Then G is a subset of βH. As in the proof of 3.4.3, it is easily seen that G is closed to countable union and complementation with respect to σH. If βH is internal, then the complements of members of βH are in $b\beta H = \beta H$, and so the complements of members of H are in βH. Thus if βH is internal, $H \subset G$. Consequently, $BH \subset G \subset \beta H$, and the desired conclusion is at hand.

3.6.3 (Morse [**6**]). *If $\sigma H \in H$, then bH is internal if and only if $bH = BH$.*

3.6.4. *If $\sigma H \in H$ and H is internal, then $bH = \beta H = BH$.*

Proof. Suppose H is internal and let $A \in H$. Then $\sigma H \sim A \in bH \subset \beta H$ and $bH = \beta H = BH$ by 3.4.3.

4. Souslin sets. Another operation applicable to an arbitrary family of sets is Souslin's operation $(\mathscr{A})$ [**8**]. An operation $(\mathscr{A})$ applied to sets selected from a given family H can be defined as follows: For each finite sequence $\langle x_1, x_2, \ldots, x_n \rangle$ of natural numbers let $E_{x_1, x_2, \ldots, x_n} \in H$. Then A is the result of an operation $(\mathscr{A})$ applied to the family H if

$$A = \bigcup (E_{x_1} \cap E_{x_1, x_2} \cap E_{x_1, x_2, x_3} \cap \ldots), \tag{1}$$

where the union extends over all infinite sequences of natural numbers, and for each such sequence the intersection extends over all the truncations of that sequence.

In 4.1 below is defined the family SH of all Souslin sets which result from applications of operations ($\mathscr{A}$) to the family H. Souslin sets have been studied extensively but usually in a topological setting. Some indication of the extent of the theory can be found in [**1**]. The purpose of this section is to show relations between Souslin sets and Borel sets which are not dependent upon a topological structure in the underlying space.

In the following definition we employ the Notations 2.5–2.8 to express the equation (1) above.

4.1. *Definition.* For a family H consisting of any sets whatever, SH is the family of all sets A such that

$$A = \bigcup_{x \in Z} \bigcap_{n \in \omega} E(x|\bar{n})$$

for some function E on $\bigcup_{n\in\omega} Z_n{}'$ to H.

Well known and due to Lusin, Sierpiński, and Souslin [**3**; **5**; **8**] are the following relations which hold for any family H.

4.2. THEOREM. $bH \subset SH = SSH = SbH = bSH$.

We shall have need of the following.

4.3. THEOREMS.

4.3.1. $\sigma SH = \sigma H$.

4.3.2. $\sigma SH \in BH$.

We should like to enlarge the table in 3.3 to include the compositions of Souslin's operation with the Borel operations. In view of 4.2 the compositions of S with β and B must be determined.

4.4. THEOREM. *If $\sigma H \in H$, then $S\beta H = SBH$ and $\beta SH = BSH$.*

Proof. Let $\sigma H \in H$; then $\sigma H \sim \alpha \in \beta H$ whenever $\alpha \in H$, and by 3.4.2, $\beta H = BH$. Thus $S\beta H = SBH$. Again if $\sigma H \in H$, then $\sigma H \in SH$ by 3.2.1 and 4.2. So that by 4.3.1 and 3.1.2, $\sigma SH \sim \alpha = \sigma H \sim \alpha \in \beta SH$ whenever $\alpha \in SH$. Thus $\beta SH = BSH$ by 3.4.2.

4.5. THEOREM. *If H is internal and $\sigma H \in H$ (or equivalently, if $\sigma H \in H$ and $\sigma H \sim \alpha \in bH$ whenever $\alpha \in H$), then $SBH = S\beta H = SbH = SH$ and $\beta SH = BSH$.*

Proof. Apply 3.4.3, 4.2, and 4.4.

An enlargement of the table in 3.3 is given in 4.6. Of course, in view of 3.6.4, the first two columns within the table below are redundant.

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4.6. THEOREM. *If H is internal and $\sigma H \in H$, then*

	bH	βH	BH	SH
b	bH	βH	BH	SH
β	βH	βH	BH	BSH
B	BH	BH	BH	BSH
S	SH	SH	SH	SH

In general $BSH \neq BH$ and $BSH \neq SH$, for amongst Souslin's fundamental results in [**8**] is the following.

THEOREM. *If H is the family of closed linear sets, then SH contains sets which are not in BH, and a set and its complement are both in SH if and only if they are both in BH.*

Here $BSH \neq BH$ because $BH \subset SH \subset BSH$ and $BSH \neq SH$ since BSH is closed to complementation and SH is not.

That a set and its complement are both Souslin if and only if they are both Borel is a corollary of the fact that disjoint Souslin sets can be separated by Borel sets. This separability property holds for any family H which is closed to finite intersection. The following theorem and its proof are due to Sion. See [**1**], where the theorem and its proof are apparently cast in a topological setting but are actually free of that context.

4.7. THEOREM. *If H is closed to finite intersection, A and B belong to SH, and $A \cap B = 0$, then there exist A' and B' in bH such that $A \subset A'$, $B \subset B'$, and $A' \cap B' = 0$.*

Proof. Let $A = \bigcup_{x\in Z} \bigcap_{m\in\omega} E(x|\bar{m})$ and $B = \bigcup_{x\in Z} \bigcap_{m\in\omega} F(x|\bar{m})$, where E and F are on $\bigcup_{m\in\omega} Z_m'$ to H. Since H is closed to finite intersection, it can be assumed without loss of generality that

$$E(x|\overline{m+1}) \subset E(x|\bar{m}) \quad \text{and} \quad F(x|\overline{m+1}) \subset F(x|\bar{m}) \quad \text{for each } x \in Z \text{ and } m \in \omega.$$

For each $n \in \omega$ and $z \in Z_n'$, let

$$T_n(z) = \{x : x \in Z \text{ and } x|\bar{n} = z\}, \qquad g_n(z) = \bigcup_{x\in T_n(z)} \bigcap_{m\in\omega} E(x|\bar{m}),$$

and

$$h_n(z) = \bigcup_{x\in T_n(z)} \bigcap_{m\in\omega} F(x|\bar{m}).$$

For each $x \in Z$ let

$$E'(x) = \bigcap_{m\in\omega} E(x|\bar{m}) \quad \text{and} \quad F'(x) = \bigcap_{m\in\omega} F(x|\bar{m}).$$

Then for each $x \in Z$, $E'(x) \subset A$ and $F'(x) \subset B$. Consequently, if $A \cap B = 0$, then $E'(x) \cap F'(y) = 0$ for each x and y in Z. Moreover, for each x and y in Z there is $n \in \omega$ such that

$$E(x|\bar{n}) \cap F(y|\bar{n}) = 0,$$

$$g_n(x|\bar{n}) = \bigcup_{z\in T_n(x|\bar{n})} \bigcap_{k\in\omega} E(z|\bar{k}) \subset E(x|\bar{n}),$$

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and

$$h_n(y|\bar{n}) = \bigcup_{z\in T_n(y|\bar{n})} \bigcap_{k\in\omega} F(z|\bar{k}) \subset F(y|\bar{n}).$$

Thus for each x and y in Z there is $n \in \omega$ such that $g_n(x|\bar{n})$ and $h_n(y|\bar{n})$ can be separated by members of H. It follows that A and B can be separated by members of bH, since

$$\begin{aligned} A &= \bigcup_{x\in Z} \bigcap_{m\in\omega} E(x|\bar{m}) \\ &= \bigcup_{n\in\omega} \bigcup_{z\in Z_n'} \bigcup_{x\in T_n(z)} \bigcap_{m\in\omega} E(x|\bar{m}) \\ &= \bigcup_{n\in\omega} \bigcup_{z\in Z_n'} g_n(z) \end{aligned}$$

and

$$\begin{aligned} B &= \bigcup_{x\in Z} \bigcap_{m\in\omega} F(x|\bar{m}) \\ &= \bigcup_{n\in\omega} \bigcup_{z\in Z_n'} \bigcup_{x\in T_n(z)} \bigcap_{m\in\omega} F(x|\bar{m}) \\ &= \bigcup_{n\in\omega} \bigcup_{z\in Z_n'} h_n(z), \end{aligned}$$

which completes the proof because A and B are represented as countable unions of sets which can be separated by members of H.

5. Iteration of the Borel and Souslin operations.

In view of the remarks concluding § 4 we inquire into the compositions of BS with the Borel operations and with S. Specifically, there are the compositions $\alpha(BS)$ and $(BS)\alpha$, where α is any one of b, β, B, S or BS. Evidently, in the present context we must agree that $(BS)\alpha = B(S\alpha)$ when α is b, β, B or S and that $(BS)(BS) = B(S(BS))$.

Of the nine compositions that can be formed, all save SBS (*i.e.*, $S(BS)$) and $BSBS$ (or $B(S(BS))$) are easily seen to be equivalent to BS when applied to an internal family H for which $\sigma H \in H$ (or equivalently to a family H such that $\sigma H \in H$ and $\sigma H \sim \alpha \in bH$ whenever $\alpha \in H$). In order to appraise the extent of the class of sets obtained by continued iteration of the operations B and S we introduce Definitions 5.1 and 5.3 below.

5.1. *Definition.* For an internal family H such that $\sigma H \in H$ (or equivalently, for a family H such that $\sigma H \in H$ and $\sigma H \sim \alpha \in bH$ whenever $\alpha \in H$) let $A_{-1}H = H$, and for each $n \in \omega$ let $A_{2n}H = BA_{2n-1}H$ and $A_{2n+1}H = SA_{2n}H$. Thus

$$\begin{gathered} A_{-1}H = H, \qquad A_0H = BH, \\ A_1H = SBH = SH, \text{ in view of 4.5,} \\ A_2H = BSH, \qquad A_3H = SBSH, \text{ etc.} \end{gathered}$$

5.2. THEOREMS.

5.2.1. *If H is internal and $\sigma H \in H$, then $H \subset A_nH \subset A_{n+1}H$ whenever $n \in \omega$.*

5.2.2. *If H is internal and $\sigma H \in H$, then $BA_{2n}H = A_{2n}H$ and $SA_{2n+1}H = A_{2n+1}H$ whenever $n \in \omega$.*

5.3. *Definition.* For an internal family H such that $\sigma H \in H$, the Analytic H sets of class (-1) are the members of H, and for each $n \in \omega$ the Analytic H sets of class n are the members of $A_nH \sim A_{n-1}H$.

Thus the members of $BH \sim H$ are the Analytic H sets of class 0, the members of $SH \sim BH$ are the Analytic H sets of class 1, etc. If H is a finite field of sets, then $H = BH = SH$ and there are no Analytic H sets of class higher than (-1). If H is the family of closed linear sets, then there are Analytic H sets of classes 0 and 1, and in view of the remarks following 4.6 there are Analytic H sets of class 2. We conjecture but are unable to prove that if H is the family of closed linear sets, then there are Analytic H sets of class n for each $n \in \omega$. We continue with a few observations which have bearing on this conjecture.

5.4 *Definition.* A family H is complemental if and only if it is closed to complementation with respect to σH.

Thus whatever the family K, BK is complemental; for if $\alpha \in BK$, then $\sigma BK \sim \alpha$ $(= \sigma K \sim \alpha) \in BK$.

5.5. THEOREM. *If H is internal, $\sigma H \in H$, $n \in \omega$, and there are* Analytic H *sets of class $n - 1$ but none of class n, then $A_{n-1}H$ is complemental.*

Proof. If for $n \in \omega$, $A_{n-1}H \neq 0$ but $A_nH \sim A_{n-1}H = 0$, then $A_nH = A_{n-1}H$. Thus either n is even and $A_nH = BA_{n-1}H$ or n is odd (hence $n \geqq 1$) and $n - 1$ is even and $A_{n-1}H = BA_{n-2}H$. Since BK is complemental whatever the family K, the conclusion is at hand.

5.6. LEMMA. *If H is internal and $\sigma H \in H$, then for each $n \in \omega$, A_nH is internal if and only if A_nH is complemental.*

Proof. Let $\sigma H \in H$ and $n \in \omega$. It follows from 3.2 and 4.2 that

$$\sigma A_nH = \sigma H \in H \subset A_nH.$$

In view of 3.3 and 4.2, $bA_nH = A_nH$ and A_nH is closed to finite union. Finally, let $\alpha \in A_nH$. If A_nH is complemental, then $\sigma A_nH \sim \alpha \in A_nH = bA_nH$ and A_nH is internal. If A_nH is internal, then $\sigma A_nH \sim \alpha \in bA_nH = A_nH$ and A_nH is complemental.

5.7. THEOREM. *If H is internal, $\sigma H \in H$, and $A_{2n+1}H$ is complemental for some $n \in \omega$, then there are no* Analytic H *sets of class k for $k \geqq 2n + 2$.*

Proof. As in the preceding lemma, we have $\sigma A_nH = \sigma H \in H \subset A_nH$ for each $n \in \omega$. Let n be such a member of ω that $A_{2n+1}H$ is complemental (and so, internal by 5.6). In view of 5.1, 4.5, and 5.2.2 we have:

$$A_{2n+3}H = SBA_{2n+1}H = SA_{2n+1}H = A_{2n+1}H.$$

Hence $A_{2n+3}H = A_{2n+2}H = A_{2n+1}H$ and there are no Analytic H sets of class k

if $k \geqq 2n + 2$ for the continued iteration of the Borel and Souslin operations yields no new sets.

Thus if $\sigma H \in H$ and H and SH are complemental (internal), then there are no Analytic H sets of class k for $k \geqq 2$. In this case $BSH = SH$ and the table in 4.6 is closed.

We conclude with a different sort of appraisal of the extent of the family of Analytic H sets of all classes. Let $\mathscr{L}H$ be the smallest family which contains H and is closed to the application of Souslin's operation $(\mathscr{A})$ and complementation with respect to σH. Evidently, each of the families $A_n H$ is contained in $\mathscr{L}H$, if H is internal and $\sigma H \in H$. It is known (Kuratowski [**2**]) that when H is the family of closed linear sets, $\mathscr{L}H$ is contained in the projective classes P_2 and C_2. (The projective class P_2 consists of those linear sets which are the projections of complements of plane analytic sets, and the projective class C_2 consists of the linear complements of sets in P_2.)

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