POUNTS OF SMALL HEIGHT ON AFFINE VARIETIES
DEFINED OVER FUNCTION FIELDS
OF FINITE TRANSCENDENCE DEGREE

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Abstract

We provide a direct proof of a Bogomolov-type statement for affine varieties $V$ defined over function fields $K$ of finite transcendence degree over an arbitrary field $k$, generalising a previous result (obtained through a different approach) of the first author in the special case when $K$ is a function field of transcendence degree 1. Furthermore, we obtain sharp lower bounds for the Weil height of the points in $V(K)$, which are not contained in the largest subvariety $W \subseteq V$ defined over the constant field $k$.

Keywords and phrases: function fields, Weil height, Bogomolov conjecture.

1. Introduction

1.1. Notation. Given an arbitrary field $k$ and a function field $K$ of transcendence degree $m \geq 1$ over $k$, we let $h : A^N(K) \to \mathbb{Q}_{\geq 0}$ represent the Weil height for the points of the corresponding affine space (for any given $N \geq 1$); we refer the reader to the classical geometric construction of the Weil height for points on affine spaces defined over function fields as presented in Serre’s book [7], but we will also sketch briefly in Section 2 an algebraic construction of the Weil height.

1.2. Statement of our main results. Some of the most important theorems in arithmetic geometry in the past 30 years have been the proofs of the Bogomolov conjecture both for powers of the multiplicative group (see [1, 9]) and for abelian varieties (see [8, 10]) defined over $\overline{\mathbb{Q}}$. In both cases, the fundamental principle has been that when $G$ is either $\mathbb{G}_m^N$ or an abelian variety defined over $\overline{\mathbb{Q}}$, the accumulating subvarieties of $G$ for points of small canonical height are the torsion translates of algebraic subgroups of $G$; in other words, if $V \subseteq G$ is an irreducible subvariety with the property that, for any $\epsilon > 0$,

$$\{ P \in V(\overline{\mathbb{Q}}) : \tilde{h}(P) \leq \epsilon \}$$
is Zariski dense in $V$, then $V = Q + W$, where $Q \in G_{\text{tor}}$ and $W$ is a connected algebraic subgroup of $G$.

Motivated by these classical results, the first author studied in [4, 5] a variant of the Bogomolov conjecture for points on affine varieties defined over function fields (alternatively, this could be interpreted as a variant of the Bogomolov conjecture for $G_m^N$ defined over a function field). So, given a function field $K/k$, the accumulating subvarieties of $\mathbb{A}^N$ for points of small Weil height with respect to the places of the function field $K/k$ (see also Section 2 for the definition of the Weil height in this context) should be the subvarieties defined over $\bar{k}$ (the algebraic closure of the constant field) since they are the subvarieties containing a Zariski dense set of points from $\mathbb{A}^N(\bar{k})$ (which all have Weil height 0).

In other words, given an affine subvariety $V \subseteq \mathbb{A}^N$, let $W \subseteq V$ be its largest subvariety defined over $k$, which is simply the Zariski closure of the subset $V(\bar{k})$. Then one expects to find some positive real number $\epsilon$ with the property that for each point $P \in V(\bar{k})$, if the Weil height of the point $P$ satisfies the inequality $h(P) < \epsilon$, then $P \in W(\bar{k})$. The first author proved in [4] that this expectation is indeed met in the case where $k$ is a finite field and then, later in [5], generalised his result to all function fields $K/k$ of transcendence degree 1. In this paper, we prove the following result, which covers all function fields (of arbitrary finite transcendence degree).

**Theorem 1.1.** Let $k$ be a field and let $K$ be a function field of transcendence degree $m \geq 1$ over $k$. We let $h : \mathbb{A}^N(\bar{k}) \to \mathbb{Q}_{\geq 0}$ be the Weil height associated to the function field $K/k$. Let $V \subseteq \mathbb{A}^N$ be an affine subvariety defined over $K$ and let $W \subseteq V$ be the Zariski closure of all points of $V$ with coordinates in $\bar{k}$. Then there exists a positive real number $c_0$ with the property that $h(P) \geq c_0$ for each $P \in (V \setminus W)(\bar{k})$.

**Remark 1.2.** The constant $c_0$ from the conclusion of Theorem 1.1 depends in an explicit way on the data defining the variety $V$. Indeed, as shown in the proof of Theorem 1.1, $c_0$ depends only on the total degrees of the polynomials from a finite set of generators for the vanishing ideal of $V$ and on the degree of $K$ over a rational function field $K_0$ (of transcendence degree $m$ over $k$) used in the definition of the Weil height $h(\cdot)$. (For more details, see Section 2.)

The strategy of proof from [5] (inspired by a clever trick the first author learned from the beautiful paper [2] of Bombieri and Zannier) presented some natural obstructions to a generalisation covering any function field, as explained in [5, Remark 2.7]. So, our proof of Theorem 1.1 (which stemmed from the second author’s attempt at generalising the results of [5] to arbitrary function fields) follows a different strategy than the one employed in [5, 4]. Indeed, we are able to argue in a more direct way to prove the conclusion from Theorem 1.1 and, as a by-product of our method, we obtain also the following sharp lower bound for the Weil height of a point not contained on the largest subvariety of $V$ defined over the constant field.

**Theorem 1.3.** Let $k$ be a field, let $m \in \mathbb{N}$ and let $K := k(t_1, \ldots, t_m)$ be a function field of transcendence degree $m$ over $k$. Let $V \subseteq \mathbb{A}^N$ be the zero locus of finitely many...
polynomials \( g_i \) in \( N \) variables with coefficients in \( K \). Let \( D := \max_i \deg(g_i) \) (where the degree \( \deg(g) \) of a polynomial \( g \in K[x_1, \ldots, x_N] \) is defined to be its total degree in the variables \( x_1, \ldots, x_N \)). We let \( W \subseteq V \) be the Zariski closure of all points of \( V \) with coordinates in \( \bar{k} \). Then for any point \( P \in V(\bar{k}) \), either \( P \in W(\bar{k}) \) or \( h(P) \geq 1/D \).

The key result employed in the proof of Theorem 1.3 is our Proposition 3.1 (proven in Section 3), which is of independent interest and could potentially be useful for other applications. The lower bound of \( 1/D \) for the Weil height of a point \( P \in (V \setminus W)(\bar{k}) \) is the best possible, as shown by the following examples.

### 1.3. Examples

**Example 1.4.** Let \( D \in \mathbb{N} \) and let \( y = tx^D \) be a plane curve \( V \) defined over the function field \( K = k(t) \) (for any given field \( k \)). Then each point \((a, b) \in V(\bar{k})\) where \( b \in \bar{k} \) has Weil height (see Section 2) equal to \( 1/D \), which is precisely the lower bound from Theorem 1.3. Furthermore, the only point on \( V \) with both coordinates in \( \bar{k} \) is \((0, 0)\), that is, in the notation of Theorem 1.3, \( W = \{(0, 0)\} \).

We also note (see our next example) that in the conclusion of either Theorem 1.1 or 1.3, one does indeed have to exclude the subvariety \( W \subseteq V \), which is the largest subvariety defined over \( \bar{k} \), in order to obtain a positive lower bound for the Weil height of the remaining points in \( V(\bar{k}) \).

**Example 1.5.** Consider the plane \( V \subset \mathbb{A}^3 \) given by the equation \( z = tx + y \), defined over the function field \( k(t) \). The largest subvariety of \( V \) defined over \( \bar{k} \) is the line given by the equations \( x = 0 \) and \( z = y \). Clearly, \( W \) contains infinitely many points of arbitrarily small Weil height (not only the ones defined over \( \bar{k} \)); so, in order to obtain a uniform positive lower bound for the Weil height of the points on the plane \( V \), one would have to exclude the entire line \( W \). Furthermore, there exists a Zariski dense set of points in \( V \setminus W \) of Weil height 1 (which is the smallest height predicted by the conclusion of Theorem 1.3); indeed, any point on \( V \) of the form

\[(a, b, ta + b) \quad \text{with} \ a, b \in \bar{k} \text{ and } a \neq 0\]

would have Weil height precisely equal to 1.

### 2. Heights in function fields

In this section we construct the places for arbitrary function fields and define the corresponding Weil height; we refer the reader also to [7] for additional details (including a more geometric approach).

Let \( K \) be a function field of transcendence degree \( m \geq 1 \) over a field \( k \). We let \( t_1, \ldots, t_m \) be algebraically independent (over \( k \)) functions in \( K \) and consider the function field \( K_0 := k(t_1, \ldots, t_m) \). Then the places \( \Omega_{K_0} \) correspond to irreducible hypersurfaces of the projective space \( \mathbb{P}^m_k \) (whose function field equals \( K_0 \)). More explicitly, the places of the function field \( K_0/k \) correspond:
either to the irreducible polynomials \( Q \in k[t_1, \ldots, t_m] \), in which case we associate an (exponential) valuation, denoted \( v_Q \), which is defined to be

\[ v_Q \left( \frac{P_1}{P_2} \right) = \exp_Q(P_1) - \exp_Q(P_2), \]

for any nonzero \( P_1/P_2 \in k(t_1, \ldots, t_m) \) (where \( \exp_Q(P_i) \) simply refers to the exponent of the irreducible polynomial \( Q \) appearing in the factorisation of \( P_i \) in prime factors) and we let \( n_{v_Q} := \deg(Q) \);

• or to the (negative) total degree function (which itself corresponds geometrically to the hyperplane at infinity from \( \mathbb{P}_k^m \)), in which case we associate a valuation, denoted \( v_\infty \), which is defined to be

\[ v_\infty \left( \frac{P_1}{P_2} \right) = \deg(P_2) - \deg(P_1), \]

for any nonzero \( P_1/P_2 \in k(t_1, \ldots, t_m) \) and we let \( n_{v_\infty} := 1. \)

For each nonzero rational function \( R \in k(t_1, \ldots, t_m) \), we have the product formula (or rather, the sum formula since we work with exponential valuations)

\[ \sum_{v \in \Omega_{K_0}} n_v \cdot v(R) = 0. \tag{2.1} \]

Given a finite extension \( L \) of \( K_0 \), we let \( \Omega_L \) be the set of places of \( L \) lying above the places from \( \Omega_{K_0} \). For each place \( w \in \Omega_L \) lying above a place \( v \in \Omega_{K_0} \), we let \( e(v \mid v_0) \) be the ramification index, that is, normalising the exponential valuation \( v \) on \( L \) so that its value group is \( \mathbb{Z} \), we set \( e(v \mid v_0) := v(U) \), where \( U \in K_0 \) is a uniformiser for the valuation \( v_0 \) (that is, \( v_0(U) = 1 \)). Also, we let \( n_v := n_{v_0} \cdot f(v \mid v_0) \), where \( f(v \mid v_0) \) is the degree of the residue field extension corresponding to the two places. Then once again we have a product formula:

\[ \sum_{v \in \Omega_L} n_v \cdot v(x) = 0, \tag{2.2} \]

for any nonzero \( x \in L \).

For any positive integer \( N \), the Weil height of a point \( P := (a_1, \ldots, a_N) \in \mathbb{A}^N(L) \) is defined by

\[ h(P) := \frac{1}{[L : K_0]} \sum_{v \in \Omega_L} n_v \cdot \max\{0, -v(a_1), -v(a_2), \ldots, -v(a_N)\}. \tag{2.3} \]

As proven in [7] (see also [3, Ch. 4] for a comprehensive discussion regarding valuations on arbitrary function fields and heights for points on varieties defined over function fields), the Weil height is well defined and it is independent of the choice of particular field \( L \) which contains the coordinates \( a_i \) of the point \( P \). Using the geometric definition of the Weil height as in [7] (see also [6, Proposition 3.2, page 63]), each point \( P \in \mathbb{P}^N(K) \) corresponds to a rational function \( \psi : X \rightarrow \mathbb{P}^N \), where \( X \subset \mathbb{P}^r \) is a projective variety defined over \( k \), regular in codimension 1, whose function field is \( K \);
then
\[ h(P) := \deg(\psi^{-1}(L)) \] (2.4)
for a generic hyperplane \( L \) of \( \mathbb{P}^N \), where the degree of \( \psi^{-1}(L) \) is computed with respect to the embedding of \( X \) into \( \mathbb{P}^r \). The normalisation in our valuations depends on our initial choice of the functions \( t_1, \ldots, t_m \), but obviously, once these functions are fixed, the definition of all valuations and in turn of the corresponding Weil height is uniquely determined. A different choice for the rational functions \( t_i \) would lead to another Weil height \( h_2 \) comparable with respect to the first Weil height, that is, there would be positive constants \( c_1 \) and \( c_2 \) such that \( c_1 h_1(P) \leq h_2(P) \leq c_2 h_1(P) \) for any point \( P \in \mathbb{A}^N(\overline{K}) \).

3. Proof of our main results

**Proposition 3.1.** Let \( k \) be an arbitrary field, let \( K \) be a function field over \( k \), let \( \overline{K} \) be the algebraic closure of \( K \) and \( \overline{k} \) be the algebraic closure of \( k \) inside \( \overline{K} \). Let \( d, N, D \geq 1 \) be integers and let \( f_i \in k[x_1, \ldots, x_N] \), for \( i = 0, \ldots, d \), be polynomials of total degree at most \( D \). Let \( t \in \overline{K} \setminus \overline{k} \) and let \( f \in k[t][x_1, \ldots, x_N] \) be defined as

\[ f(x_1, \ldots, x_N) := \sum_{i=0}^d t^i f_i(x_1, \ldots, x_N). \] (3.1)

We let \( h : \mathbb{A}^N(\overline{K}) \rightarrow \mathbb{Q}_{\geq 0} \) be the Weil height corresponding to the function field \( K/k \). Then for any point \( P \in \mathbb{A}^N(\overline{K}) \) for which \( f(P) = 0 \), either

\[ h(P) \geq \frac{h(t)}{D}, \] (3.2)

or

\[ f_i(P) = 0 \quad \text{for each } i = 0, \ldots, d. \] (3.3)

**Proof.** First we note that \( h(t) > 0 \) since the only elements of \( \overline{K} \) of height equal to 0 are the elements of \( \overline{k} \).

Let \( P := (a_1, \ldots, a_N) \in \mathbb{A}^N(\overline{K}) \) such that \( f(P) = 0 \) and \( L := K(t, a_1, \ldots, a_N) \). As in Section 2, we let \( \Omega_L \) be the set of places of the function field \( L/k \).

We assume that (3.3) does not hold; in particular, this means that the set

\[ I := \{0 \leq i \leq d : f_i(P) \neq 0\} \]

contains at least two such indices. We will prove that \( h(P) \geq h(t)/D \), that is, (3.2) must hold.

Let \( j \) be the largest element of \( I \). We let \( S_\infty \) be the finite set of places of the function field \( L/k \) consisting of all places \( v \in \Omega_L \) with the property that \( v(t) < 0 \).

Let \( v \in S_\infty \). Since \( f(P) = 0 \), there must exist an index \( i \in I \setminus \{j\} \) (depending on \( v \)) such that

\[ v(t^i \cdot f_i(a_1, \ldots, a_N)) \leq v(t^j \cdot f_j(a_1, \ldots, a_N)), \] (3.4)
since otherwise, the ultrametric inequality yields \(|f(P)|_v = |t^j f_j(P)|_v \neq 0\), a contradiction. Since \(i \neq j\) and \(j\) was chosen to be the largest element of \(I\), it follows that \(i < j\). Let

\[ c_v := v(f_j(a_1, \ldots, a_N)). \]  

(3.5)

Using (3.4) and (3.5) (along with the fact that \(j - i \geq 1\) and \(v(t) < 0\)), we find that

\[ v(f_i(a_1, \ldots, a_N)) \leq v(t) + c_v \]  

(3.6)

and thus

\[ -v(f_i(a_1, \ldots, a_N)) \geq -v(t) - c_v. \]  

(3.7)

Since \(f_i\) is a polynomial of degree at most \(D\) in the variables \(x_1, \ldots, x_N\) with coefficients in the constant field \(k\), inequality (3.7) yields, for each \(v \in S_\infty\),

\[ \max\{0, -v(a_1), \ldots, -v(a_N)\} \geq \frac{1}{D} \cdot \max\{0, -v(t) - c_v\}. \]  

(3.8)

But since \(f_i(a_1, \ldots, a_N) \neq 0\), applying the product formula (2.2) for the nonzero element \(f_j(a_1, \ldots, a_N)\) of \(L\),

\[ \sum_{w \in \Omega_L} n_w \cdot \max\{0, -w(f_j(a_1, \ldots, a_N))\} = \sum_{w \in \Omega_L} n_w \cdot \max\{0, w(f_j(a_1, \ldots, a_N))\} \]

and so, using (3.5),

\[ \sum_{w \in \Omega_L} n_w \cdot \max\{0, -w(f_j(a_1, \ldots, a_N))\} \geq \sum_{v \in S_\infty} n_v \cdot \max\{0, c_v\}. \]  

(3.9)

Furthermore, (3.5) and (3.9) yield

\[ \sum_{w \in \Omega_L \setminus S_\infty} n_w \cdot \max\{0, -w(f_j(a_1, \ldots, a_N))\} \geq \sum_{v \in S_\infty} n_v \cdot (\max\{0, c_v\} - \max\{0, -c_v\}). \]

(3.10)

Using the fact that \(f_j \in k[x_1, \ldots, x_N]\) is a polynomial of degree at most \(D\), for each place \(w \in \Omega_L \setminus S_\infty\),

\[ \max\{0, -w(a_1), \ldots, -w(a_N)\} \geq \frac{\max\{0, -w(f_j(a_1, \ldots, a_N))\}}{D} \]

and thus inequality (3.10) yields

\[ \sum_{w \in \Omega_L \setminus S_\infty} n_w \cdot \max\{0, -w(a_1), \ldots, -w(a_N)\} \geq \frac{1}{D} \cdot \left( \sum_{v \in S_\infty} n_v \cdot \max\{0, c_v\} - \max\{0, -c_v\} \right). \]

(3.11)

Using the formula for the Weil height of the point \(P \in \mathbb{A}^N\), and also using the notation from Section 2 for the rational function field \(K_0 \subseteq K\) with respect to which the Weil height is defined, we arrive at

\[ h(P) = \frac{1}{[L : K_0]} \sum_{w \in \Omega_L} n_w \cdot \max\{0, -w(a_1), \ldots, -w(a_N)\}. \]
Then combining inequalities (3.8) and (3.11),
\[ h(P) \geq \frac{1}{D} \cdot \frac{1}{[L : K_0]} \cdot \sum_{v \in S_{\infty}} n_v \cdot (\max\{0, -v(t) - c_v\} + \max\{0, c_v\} - \max\{0, -c_v\}). \quad (3.12) \]

Now, for any real numbers \( \alpha \) and \( \beta \) with \( \alpha > 0 \),
\[ \max\{0, \alpha - \beta\} + \max\{0, \beta\} - \max\{0, -\beta\} \geq \alpha. \]
Thus, for each \( v \in S_{\infty} \),
\[ \max\{0, -v(t) - c_v\} + \max\{0, c_v\} - \max\{0, -c_v\} \geq -v(t). \quad (3.13) \]

Finally, since
\[ h(t) = \frac{1}{[L : K_0]} \cdot \sum_{v \in S_{\infty}} -v(t), \]

inequalities (3.12) and (3.13) deliver the desired inequality from (3.2). This concludes our proof of Proposition 3.1.

\[ \square \]

**Proof of Theorem 1.3.** We let \( P := (a_1, \ldots, a_N) \in V(K) \). We assume that
\[ h(P) < \frac{1}{D} \]
and we will prove that \( P \in W(K) \), as claimed in the conclusion of Theorem 1.3.

Let \( f \in k(t_1, \ldots, t_m)[x_1, \ldots, x_N] \) be one of the finitely many generators of the vanishing ideal of \( V \); in particular, according to our hypothesis, the total degree of \( f \) as a polynomial in \( x_1, \ldots, x_N \) is at most equal to \( D \geq 1 \). At the expense of multiplying \( f \) by a suitable nonzero polynomial in \( k[t_1, \ldots, t_m] \), we may assume from now on that \( f \in k[t_1, \ldots, t_m][x_1, \ldots, x_n] \). We write
\[ f(x_1, \ldots, x_N) := \sum_{i=0}^{d_1} t_1^i \cdot f_i(x_1, \ldots, x_N), \quad (3.15) \]
where \( d_1 \geq 0 \) is an integer and each \( f_i \in k[t_2, \ldots, t_m][x_1, \ldots, x_n] \); in other words, \( d_1 \) is the largest power of \( t_1 \) appearing in any coefficient of \( f \) (each such coefficient being itself a polynomial in \( k[t_1, \ldots, t_m] \)).

We now let \( K_1 := k(t_2, \ldots, t_m) \) and let \( K_1 \) be its algebraic closure inside \( \bar{K} \). We let \( h_1 : \mathbb{A}^N(K_1) \to \mathbb{Q}_{\geq 0} \) be the Weil height associated to the function field \( K/K_1 \). Since \( h_1 \) counts only the degree in \( t_1 \) of any rational function in \( K \) (see also (2.4)),
\[ h(Q) \geq h_1(Q) \quad \text{for each } Q \in \mathbb{A}^N(K_1). \quad (3.16) \]

By Proposition 3.1 applied to the polynomial \( f \) from (3.15), either \( h_1(P) \geq 1/D \) (note that \( h_1(t_1) = 1 \)) and thus, using (3.16),
\[ h(P) \geq \frac{1}{D}, \]
or \( f_i(P) = 0 \) for each \( i = 0, \ldots, d_1 \). Since we assumed the opposite inequality for the height of \( P \) as in (3.14), we must indeed have \( f_i(P) = 0 \) for each \( i = 0, \ldots, d_1 \).
Now, for each \( i = 0, \ldots, d_1 \), we write

\[
 f_i(x_1, \ldots, x_N) := \sum_{j=0}^{d_2} t_{ij}^j \cdot f_{i,j}(x_1, \ldots, x_N),
\]  

(3.18)

where each polynomial \( f_{i,j} \in k[t_3, \ldots, t_m][x_1, \ldots, x_N] \). By our original assumption on the total degree of the polynomial \( f \), each \( f_{i,j} \) has total degree in \( x_1, \ldots, x_N \) at most equal to \( D \).

Then we let \( K_2 := k(t_3, \ldots, t_m) \) and let \( \overline{K}_2 \) be its algebraic closure inside \( \overline{K} \). We let \( h_2 : \mathbb{A}^N(\overline{K}) \to \mathbb{Q}_{\geq 0} \) be the Weil height corresponding to the function field \( K/K_2 \); once again, similar to inequality (3.16), since the height \( h_2 \) picks up the total degree in \( t_1 \) and \( t_2 \) of any rational function in \( K = k(t_1, \ldots, t_m) \), as opposed to the total degree in all \( m \) variables as for the Weil height \( h : \mathbb{A}^N(\overline{K}) \to \mathbb{Q}_{\geq 0} \) which corresponds to the function field \( K/k \), we have that

\[
 h_2(Q) \leq h(Q) \quad \text{for each } Q \in \mathbb{A}^N(\overline{K}).
\]  

(3.19)

Now, applying Proposition 3.1 to each polynomial \( f_i \) from (3.18), we conclude that either \( h_2(P) \geq 1/D \) (note that \( h_2(t_2) = 1 \)), which in turn (due to inequality (3.19)) yields

\[
 h(P) \geq \frac{1}{D},
\]  

(3.20)

or \( f_{i,j}(P) = 0 \) for each \( j = 0, \ldots, d_2 \). Since we assumed that the opposite inequality (3.14) holds (which contradicts (3.20)), we must have \( f_{i,j}(P) = 0 \) for each \( i = 0, \ldots, d_1 \) and each \( j = 0, \ldots, d_2 \).

We continue the above process, this time applying Proposition 3.1 to each polynomial \( f_{i,j} \) which we write in terms of the powers of \( t_3 \) appearing in its coefficients. For example, at step \( \ell \) in our process (for some \( \ell = 1, \ldots, m \)), we deal with polynomials of the form \( f_{i_1, \ldots, i_{\ell-1}, j} \in k[t_{\ell+1}, \ldots, t_m][x_1, \ldots, x_N] \), for some \( i_j \in \{0, \ldots, d_j\} \) for each \( j = 1, \ldots, \ell - 1 \) (where the \( d_j \) are the maximum degrees in \( t_j \) of the coefficients of the original polynomial \( f \in k[t_1, \ldots, t_m][x_1, \ldots, x_N] \)). Then we write each such polynomial as

\[
 f_{i_1, \ldots, i_{\ell-1}, j}(x_1, \ldots, x_N) := \sum_{j=0}^{d_{\ell}} t_j^j \cdot f_{i_1, \ldots, i_{\ell-1}, j}(x_1, \ldots, x_N),
\]  

(3.21)

where each \( f_{i_1, \ldots, i_{\ell-1}, j} \in k[t_{\ell+1}, \ldots, t_m][x_1, \ldots, x_N] \) has total degree at most \( D \) in the variables \( x_1, \ldots, x_N \). Then letting \( K_\ell := k(t_{\ell+1}, \ldots, t_m) \) and \( \overline{K}_\ell \) be its algebraic closure inside \( \overline{K} \), we let \( h_\ell : \mathbb{A}^N(\overline{K}) \to \mathbb{Q}_{\geq 0} \) be the Weil height associated to the function field \( K/K_\ell \) (with the choice \( t_1, \ldots, t_\ell \) for the algebraically independent functions generating the function field, as in Section 2). As before (see inequalities (3.16) and (3.19)), since \( h_\ell \) counts only the total degree in the variables \( t_1, \ldots, t_\ell \),

\[
 h_\ell(Q) \leq h(Q) \quad \text{for each } Q \in \mathbb{A}^N(\overline{K}).
\]  

(3.22)
Proposition 3.1 applied to each polynomial $f_{i_1,\ldots,i_{r-1}}$ as in (3.21) shows that either $h_t(P) \geq 1/D$ (note that $h_t(t_\ell) = 1$), which would contradict inequality (3.14) according to (3.22), or we must have

$$f_{i_1,\ldots,i_{r-1},j}(P) = 0 \quad \text{for each } j = 0, \ldots, d_t,$$

which allows us to continue our process. After $m$ steps, we conclude that we can write the original polynomial $f \in k[t_1, \ldots, t_m][x_1, \ldots, x_N]$ as

$$f(x_1, \ldots, x_N) = \sum_{0 \leq i_j \leq d_j} \left( \prod_{j=1}^{m} t_j^{i_j} \right) \cdot f_{i_1,\ldots,i_m}(x_1, \ldots, x_N),$$

where each $f_{i_1,\ldots,i_m} \in k[x_1, \ldots, x_N]$ (while $d_j$ is the maximum degree of $t_j$ appearing in the coefficients of $f$, which are themselves polynomials in $k[t_1, \ldots, t_m]$). Furthermore, repeated applications of Proposition 3.1 (as explained above), coupled with our assumption from (3.14), yield

$$f_{i_1,\ldots,i_m}(P) = 0 \quad \text{for each } i_j = 0, \ldots, d_j \text{ and for each } j = 1, \ldots, m. \quad (3.23)$$

Equations (3.23) show that indeed $P \in W(K)$, where $W$ is the largest subvariety of $V$ defined over $k$. This concludes our proof of Theorem 1.3. \hfill \Box

**Proof of Theorem 1.1.** First of all, at the expense of replacing $k$ by $\overline{k}$ and also replacing $K$ by $\overline{K} \cdot K$, we may assume from now on that $k$ is algebraically closed (and $K$ is a function field over $k$).

Since $K$ is a function field over $k$ of transcendence degree $m$, we may pick algebraically independent functions $t_1, \ldots, t_m$ in the function field $K/k$ such that $K$ is algebraic over $k(t_1, \ldots, t_m)$. We assume that the Weil height for the points in $\mathbb{A}^N(\overline{K})$ was constructed with respect to the places of the function field $k(t_1, \ldots, t_m)/k$ (see Section 2).

In the case where $k$ has characteristic $p > 0$, at the expense of replacing each $t_i$ by $t_i^{1/p^c}$ for a suitable integer $\ell \geq 0$ (and also adjoining each $t_i^{1/p^c}$ to $K$), we may assume that $K$ is separable over $k(t_1, \ldots, t_m)$ (where we prefer to keep the notation for $t_i$ rather than formally replacing $t_i$ by $t_i^{1/p^c}$). Also, note that constructing the Weil height for the points in $\mathbb{A}^N(\overline{K})$ using the normalisation for the places of the function field $k(t_1^{1/p^c}, \ldots, t_m^{1/p^c})/k$ simply introduces a factor of $p^c$ (and thus would change the absolute constant $c_0$ from the conclusion in Theorem 1.1 only by a factor of $1/p^c$). Therefore, from now on, we assume $K/k(t_1, \ldots, t_m)$ is a finite, separable extension and that the Weil height of the points in $\mathbb{A}^N(\overline{K})$ was constructed with respect to the places of the function field $k(t_1, \ldots, t_m)/k$.

We let $K_0 := k(t_1, \ldots, t_m)$ and, replacing $K$ by a finite extension, we may assume that $K/K_0$ is a Galois extension. Then we let $X \subseteq \mathbb{A}^N$ be the union of all Galois conjugates...
of $V$ over $K_0$, that is,

$$X := \bigcup_{\sigma \in \text{Gal}(K/K_0)} V^\sigma.$$ 

Then $X$ is an affine variety defined over $K_0 = k(t_1, \ldots, t_m)$. Furthermore, the subvariety $W \subseteq V$ being defined over $k$ is invariant under $\text{Gal}(K/K_0)$, and therefore $W$ is also the largest subvariety of $X$ defined over $k$ (since each point in $X(k)$ is actually a point in $V(k)$). By Theorem 1.3, there exists a positive constant $c_0$ (simply depending on the maximum total degree of the polynomials from a minimal generating set for the vanishing ideal of $X$) such that for each point $P \in X(K)$, either $h(P) \geq c_0$ or $P \in W(K)$. Since $V \subseteq X$, we obtain the desired conclusion in Theorem 1.1. $\square$

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References


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