ASYMPTOTIC VARIATIONAL FORMULAE FOR EIGENVALUES

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1. Introduction. The eigenvalues of a second order self-adjoint elliptic differential operator on Riemannian n-space R will be considered. Our purpose is to obtain asymptotic variational formulae for the eigenvalues under the topological deformations of (i) removing an ε-cell (and adjoining an additional boundary condition on the boundary component thereby introduced); and (ii) attaching an ε-handle, valid on a half-open interval $0 < \varepsilon \leq \varepsilon_o$. In particular the formulae will exhibit the non-analytic nature of the variation. Similar variational problems for singular ordinary differential operators have been considered by the writer in [3] and [4].

The variation of harmonic Green's functions and other domain functionals on finite Riemann 2-surfaces has been considered at length by M. Schiffer and D. C. Spencer in their book [7]. This elegant theory depends on analytic function theory and most of the results are written in complex form. Our treatment depends on the theory of elliptic differential equations [2] and functional analysis, and has the advantage that the results are obtained for $n \geq 2$ and for differential equations more general than Laplace's equation. Even in the case of the Laplacian operator on finite 2-surfaces, our results are not readily available in the literature.

The first theorem gives a general asymptotic variational formula, which in particular can be applied to deformations of the type (i) and (ii) above. This formula is in effect a reformulation of Green's symmetric identity. To apply it

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to the cases (i) and (ii) we shall use some uniform asymptotic estimates for eigenfunctions which were obtained in [1]. The main results are given in theorems 3 and 5.

2. Preliminaries. Let M be an open, connected domain with compact closure in R whose boundary B consists of smooth (n-1)-dimensional closed manifolds. The latter are supposed to be homeomorphic images of the unit (n-1)-sphere in Euclidean space, with continuous unit normal vectors. We do not exclude the possibility that M is a closed Riemannian space, that is, B is void. Let Δ denote the Laplacian operator on M and let $a: p \to a(p)$ denote a continuous, positive-valued function on M. Eigenvalue problems will be considered for the formally self-adjoint elliptic differential operator L defined by

$$(Lf)(p) = -(\Delta f)(p) + a(p)f(p), \quad p \in M, \quad f \in C^2[M].$$

The basic domain D is defined to be the set of all complex-valued functions on $\bar{M}$ which are of class $C^2[M]$, continuous on $\bar{M}$, and zero on B, (the last condition being deleted in the case that B is void). The basic eigenvalue problem for L is

$$(2.1) \qquad Lx = \lambda x, \quad x \in D.$$

Our purpose is to derive asymptotic variational formulae for the eigenvalues λ of L when the domain D is perturbed to a "slightly different" domain D_ε (or D_ε^*) by the deformation of removing an ε-cell (or attaching an ε-handle) to M.

Let $s(p, q)$ denote the geodesic distance in M from p to q, uniquely determined for q in some neighbourhood of p [2]. Let q_j ($j = 1, 2, \ldots J$) be fixed but arbitrary points in M. The specific ε-cells to be considered are the open balls $N_{\varepsilon j}$ defined by

$$N_{\varepsilon j} = \{ p \in R: s(p, q_j) < \varepsilon \}, \quad 0 < \varepsilon \leq \varepsilon_o; \ j = 1, 2, \ldots J.$$

It will be supposed that the positive number ε_o has been selected so that (i) $N_{\varepsilon j} \subset M$, and (ii) the boundary $\gamma_{\varepsilon j}$ of $N_{\varepsilon j}$ is a smooth homeomorphic image of the unit (n-1)-sphere in Euclidean space, whenever $0 < \varepsilon \leq \varepsilon_o$ (j = 1, 2, ... J). The parameter ε measures the smallness of $N_{\varepsilon j}$, and as $\varepsilon \to 0$, $N_{\varepsilon j}$ shrinks to the point q_j.

The notations

$$\gamma_\varepsilon = \bigcup_{j=1}^{J} \gamma_{\varepsilon j}, \quad N_\varepsilon = \bigcup_{j=1}^{J} N_{\varepsilon j}, \quad M_\varepsilon = M - \bar{N}_\varepsilon$$

will be used. The domain $\overset{o}{D}_\varepsilon$ is defined to be the set of all complex-valued functions on $\bar{M}_\varepsilon$ which are of class $C^2[M_\varepsilon]$, continuous on $\bar{M}_\varepsilon$, and zero on $B \cup N_\varepsilon$. The notations (,) and || || will designate the inner product and norm in the Hilbert space $L^2[M]$.

The following lemma is an easy consequence of Green's symmetric identity for L on M_ε [2]. The unit positive normal $\underline{n}$ to γ_ε is supposed to point toward the outside of γ_ε (inside of M_ε).

LEMMA. *If* $u \in \overset{o}{D}_\varepsilon$, $v \in \overset{o}{D}_\varepsilon$, *then*

(2.2) $$(u, Lv) - (Lu, v) = I_\varepsilon[u, v],$$ *where*[1]

$$I_\varepsilon[u, v] = \int_{\gamma_\varepsilon} (u\nabla\bar{v} - \bar{v}\nabla u) \cdot \underline{n}\, dS$$

3. The main variational formula. The asymptotic variational formula (3.1) below is to be applied in the sequel

1 A bar over a lower case letter denotes the complex conjugate.

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to the non-analytic surface deformations referred to in the introduction. The form of (3.1) is somewhat similar to Hadamard's classical formula [7, p. 274]. The latter is essentially an analytic formula, however, and is not pertinent to situations in which the basic and perturbed regions are of different topological types.

THEOREM 1. *Let* λ *be an eigenvalue of the basic problem and let* x *be an arbitrary eigenfunction associated with* λ. *Let* μ *be a complex number such that there exists a non-zero* $y \in \overset{o}{D}_\varepsilon$ *satisfying* $Ly = \mu y$ *and* $\|y - x\| \le \delta \|x\|$, *where* $0 < \delta \le \delta_o < 1$. *Then*

$$(3.1) \qquad \bar{\mu} - \lambda = \|x\|^{-2} I_\varepsilon [x, y][1 + 0(\delta)] .$$

Proof. Let u be the function with support $\bar{M}_\varepsilon$ that coincides with x on $\bar{M}_\varepsilon$. Since $x \in D$, it follows that $u \in \overset{o}{D}_\varepsilon$. We can then apply the lemma to u and y to obtain

$$\bar{\mu}(u, y) - \lambda(u, y) = (u, Ly) - (Lu, y) = I_\varepsilon [u, y] .$$

However, $(u, y) = (x, y)$ and $u(p) = x(p)$ for $p \in \gamma_\varepsilon$. Then

$$(3.2) \qquad (\bar{\mu} - \lambda)(x, y) = I_\varepsilon [x, y] .$$

By hypothesis, $|(y, x) - (x, x)| = |(y-x, x)| \le \delta \|x\|^2$, and $|(y, x)| \ge \|x\|^2 - |(y-x, x)| \ge \|x\|^2 - \delta\|x\|^2 = (1-\delta)\|x\|^2$.

Hence (3.2) yields

$$|(\bar{\mu} - \lambda) - \|x\|^{-2} I_\varepsilon [x, y]| = \left| \frac{(y, x) - (x, x)}{(y, x)(x, x)} I_\varepsilon [x, y] \right|$$

$$\le \frac{\delta}{1-\delta} \|x\|^{-2} \, |I_\varepsilon [x, y]| .$$

4. Asymptotic variation under cell removal. In this section N_ε will be specialized to a single open ball $N_{\varepsilon 1}$ with centre $q_1 \in M$ and boundary γ_ε. We define the perturbed

domain D_ε to be the set of all $f \in D_\varepsilon^o$ which vanish on γ_ε, and consider the perturbed eigenvalue problem

$$(4.1) \qquad Ly = \mu y, \quad y \in D_\varepsilon .$$

The eigenvalues will be denoted by μ_i $(0 < \mu_1 \leq \mu_2 \leq \ldots)$ and a corresponding orthonormal set of eigenfunctions by y_i $(i = 1, 2, \ldots)$.

An L-measure for M_ε with respect to the boundary components B and γ_ε is defined to be the uniquely-determined solution h of the Dirichlet problem [2]

$$(4.2) \qquad (Lh)(p) = 0, \ p \in M_\varepsilon; \ h(p) = 0, \ p \in B; \ h(p) = 1, \ p \in \gamma_\varepsilon .$$

Let φ be the positive-valued function on $0 < \varepsilon \leq \varepsilon_o$ defined as follows:

$$\begin{aligned} \varphi(\varepsilon) &= -1/\log \varepsilon && \text{if } n = 2 \\ &= \varepsilon^{n-2} && \text{if } n \geq 3 \end{aligned}$$

Except for a multiplicative constant, φ is the reciprocal of the parametrix [2]. Estimates of the type stated in the following theorem were obtained in [1].

THEOREM 2. <u>Corresponding to each eigenvalue</u> λ <u>of the basic problem (2.1), of multiplicity</u> m, <u>there are positive constants</u> ε_1 <u>and</u> c <u>(independent of</u> ε) <u>such that exactly</u> m <u>eigenvalues</u> μ_i <u>of (4.1) lie in the interval</u> $[\lambda, \lambda + c\varphi(\varepsilon)]$ <u>provided</u> $0 < \varepsilon \leq \varepsilon_1$. <u>If</u> $y_1, y_2, \ldots$ <u>are orthonormal eigenfunctions associated with</u> $\mu_1, \mu_2, \ldots,$ <u>there exists an orthonormal set</u> $x_1, x_2, \ldots, x_m$ <u>in the eigenspace of</u> λ <u>such that the uniform estimates</u>

$$(4.3) \qquad y_i(p) = x_i(p) - x_i(q_1)h(p) + 0(\psi)$$

$$p \in M_\varepsilon, \quad 0 < \varepsilon \leq \varepsilon_1, \qquad i = 1, 2, \ldots, m$$

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are valid where $\psi(\varepsilon) = \varphi(\varepsilon)$ if $n = 2$ and $\psi(\varepsilon) = \varepsilon$ if $n \geq 3$.

It is not our purpose to reproduce the entire proof here. To indicate some of the arguments, we shall deduce the first part of the theorem in the cases $n = 2, 3$ rather directly from some spectral estimation theory given by the writer in [6]. Let A, A_ε be the linear integral operators whose kernels are the respective Green's functions $G(p,q)$, $G_\varepsilon(p,q)$ associated with M, M_ε. The eigenvalues α, α_ε are known to be reciprocals of λ, μ respectively. Let X_α be the eigenspace corresponding to the m-fold degenerate eigenvalue α, and let $X_{\alpha\varepsilon} = P_\varepsilon X_\alpha$ where P_ε is the projection mapping from $L^2[M]$ onto $L^2[M_\varepsilon]$. Clearly ε_o can be chosen so that $\dim X_{\alpha\varepsilon} = \dim X_\alpha$ for $0 < \varepsilon \leq \varepsilon_o$.

For $u \in X_{\alpha\varepsilon}$, the function $f = A_\varepsilon u - \alpha u$ is a solution of $Lf = 0$ in M_ε satisfying $f = -\alpha u$ on γ_ε. Let functions g and F be defined in M_ε by the equations

$$g(p) = \omega\varphi(\varepsilon)G(p, q_1), \qquad F(p) = [2 \max_{\gamma_\varepsilon} |f|]g(p) - f(p)$$

where $\omega = 2\pi$ or 4π according as $n = 2$ or 3. There is no loss of generality in supposing ε_o has been selected so that $g(p) \geq 1/2$ for all $p \in \gamma_\varepsilon$ whenever $0 < \varepsilon \leq \varepsilon_o$, because of the singularity of $G(p, q_1)$ at $p = q_1$.

Since $LF = 0$ in M_ε, $F = 0$ on B, and $F \geq 0$ on γ_ε, it follows from the maximum principle for elliptic differential equations [2, p. 102] that $F(p) \geq 0$ throughout M_ε. Then $f(p) \leq 2[\max_{\gamma_\varepsilon}|f|]g(p)$. A lower bound for $f(p)$ is established similarly, and we then obtain $|f(p)| \leq 2\alpha[\max_{\gamma_\varepsilon}|u|]g(p)$, $p \in M_\varepsilon$.

Then $||A_\varepsilon u - \alpha u|| = ||f|| \leq c\varphi(\varepsilon)||u||$ for all $u \in X_{\alpha\varepsilon}$. Since

A_ε is a symmetric and completely continuous linear transformation on $L^2[M_\varepsilon]$, a known spectral estimation theorem [6, p. 35] shows that at least m eigenvalues $\alpha_{\varepsilon i}$ of A_ε lie in the interval $[\alpha - c\varphi(\varepsilon), \alpha]$. It is well-known from the minimum-maximum principle for eigenvalues that $M_\varepsilon \subset M$ implies $\alpha_n \geq \alpha_{\varepsilon n}$ $(n = 1, 2, \ldots)$. Then an easy induction proof establishes that there are exactly m eigenvalues in $[\alpha - c\varphi(\varepsilon), \alpha]$. This is equivalent to the first statement of theorem 2. The arguments used to prove the second part are similar to those used in [5] and will not be given here.

Theorem 2 will now be used to obtain the following special case of theorem 1.

THEOREM 3. <u>If</u> λ, μ_i <u>are eigenvalues of (2.1), (4.1) and</u> x_i, y_i <u>are corresponding normalized eigenfunctions, as described in theorem 2, then the following asymptotic variational formulae are valid:</u>

$$\mu_i - \lambda = [-|x_i(q_1)|^2 + 0(\psi)] \int_{\gamma_\varepsilon} \nabla h \cdot \underline{n}\, dS \tag{4.4}$$

as $\varepsilon \to 0$, $i = 1, 2, \ldots, m$.

<u>Proof</u>. Since the L-measure has the property $||h|| = 0(\psi)$, it follows from (4.3) that $||y_i - x_i|| \leq \delta(\varepsilon)||x_i||$, where $\delta(\varepsilon) = c\psi(\varepsilon)$, $0 < \varepsilon \leq \varepsilon_1$. Theorem 1 can then be applied provided ε is on a positive interval $(0, \varepsilon_0]$ such that $0 < \delta(\varepsilon) \leq \delta_0 < 1$. Since μ_i is real and y_i vanishes on γ_ε, (3.1) reduces to

$$\mu_i - \lambda = \int_{\gamma_\varepsilon} x_i \nabla \bar{y}_i \cdot \underline{n}\, dS\, [1 + 0(\psi)] \tag{4.5}$$

We apply (2.2) to h, y_i and h, x_i in turn to obtain

$$\mu_i(h, y_i) = \int_{\gamma_\varepsilon} \nabla \bar{y}_i \cdot \underline{n}\, dS\ , \tag{4.6}$$

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(4.7) $$\lambda(h, x_i) = \int_{\gamma_\varepsilon} (h\nabla \overline{x}_i - \overline{x}_i \nabla h)\cdot \underline{n}\, dS$$

Use of (4.3), (4.6), and (4.7) yields

$$\int_{\gamma_\varepsilon} \nabla \overline{y}_i \cdot \underline{n}\, dS = [\lambda + 0(\varphi)][(h, x_i) - (h,h)x_i(q_1) + (h,1)0(\psi)]$$

$$= \lambda(h, x_i) + 0(\psi^2)$$

(4.8) $$= -\overline{x}_i(q_1)[1 + 0(\varepsilon)] \int_{\gamma_\varepsilon} \nabla h\cdot \underline{n}\, dS + 0(\psi^2)\,.$$

The result (4.4) then follows from (4.5) and (4.8).

As an example, consider the elliptic operator $L = I - \Delta$, where I is the identity operator, on the unit 2-sphere. The metric is $ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$, where θ, φ are the usual spherical polar angles. We select for q_1 the north pole $\theta = 0$. Then γ_ε is the closed curve $\theta = \varepsilon$, $0 \le \varphi \le 2\pi$ about q_1. The eigenvalues of the basic problem (2.1) are $\lambda_m = m^2 - m + 1$, $m = 1, 2, \ldots,$ which are (2m-1)-degenerate. The corresponding normalized eigenfunctions are $x_{mi} = S^{i-1}_{m-1} / \| S^{i-1}_{m-1} \|$, where S^{i-1}_{m-1} are the spherical harmonics. It will be sufficient to consider the values $i = 1, 2, \ldots, m$. Thus $x^2_{mi}(q_1) = (2m-1)\delta_{i1}/4\pi$ from the properties of Legendre functions, where δ_{i1} is the Kronecker symbol, and (4.4) yields

$$\mu_{mi} = m^2 - m + 1 + \frac{1}{2}(2m-1)\delta_{i1}\left(\log\frac{1}{\varepsilon}\right)^{-1} + 0\left[\left(\log\frac{1}{\varepsilon}\right)^{-2}\right].$$

If $i = 2, 3, \ldots, m$, the leading variational term vanishes (i.e. x_{mi} has a zero at q_1). The variables in the partial differential equation are separable in this example, and consideration of the associated Legendre ordinary differential equation by the methods of [3] or [4] leads to an asymptotic

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variation of order ε^{2i-2} if $i \geq 2$. It is left as an open question to decide if this is the general situation when x has a zero of order $i-1$ at q_1.

5. <u>Asymptotic variation under handle attachment.</u> In this section, N_ε will be specialized to two open balls $N_{\varepsilon 1}$, $N_{\varepsilon 2}$, with centres $q_1 \in M$, $q_2 \in M$ and boundaries $\gamma_{\varepsilon 1}$, $\gamma_{\varepsilon 2}$ respectively. For a fixed homeomorphism h of $\gamma_{\varepsilon 1}$ into $\gamma_{\varepsilon 2}$, let points $p_1 \in \gamma_{\varepsilon 1}$ and $p_2 \in \gamma_{\varepsilon 2}$ be identified whenever $p_2 = h(p_1)$. The corresponding perturbed region M_ε^* consists of all points in $M_\varepsilon = M - \bar{N}_\varepsilon$ with the boundaries $\gamma_{\varepsilon 1}$, $\gamma_{\varepsilon 2}$ identified according to the rule $p_2 = h(p_1)$. We assume that h is an orientation-preserving homeomorphism. Thus M_ε^* is orientable along with M, and $\gamma_{\varepsilon 1}$, $\gamma_{\varepsilon 2}$ are oppositely oriented with respect to the common domain M_ε.

The perturbed domain D_ε^* is defined to be the set of all continuous complex-valued functions on the closure of M_ε^* which are of class $C^2[M_\varepsilon^*]$ and zero on B. The perturbed eigenvalue problem for this domain is

$$(5.1) \qquad Ly = \mu y, \qquad y \in D_\varepsilon^* .$$

Instead of (4.2), the L-measure h to be used in this section is the solution of the Dirichlet problem

$$(5.2) \qquad (Lh)(p) = 0, \quad p \in M_\varepsilon ; \qquad h(p) = 0, \quad p \in B;$$

$$h(p) = (-1)^j, \quad p \in \gamma_{\varepsilon j} \qquad (j = 1, 2).$$

The following analogue of theorem 2 has been obtained by the writer by a proof similar to that in [1].

THEOREM 4. <u>The assertions of theorem 2 remain valid if (4.1) is replaced by (5.1) and (4.3) is replaced by</u>

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(5.3) $$y_i(p) = x_i(p) - \frac{1}{2}[x_i(q_2) - x_i(q_1)]h(p) + 0(\psi).$$

The following is then obtained as the analogue of theorem 3.

THEOREM 5. *If* λ, μ_i *are eigenvalues of (2.1), (5.1) and* x_i, y_i *are corresponding normalized eigenfunctions, as described in theorem 4, then*

(5.4) $$\mu_i - \lambda = [\frac{1}{2}|x_i(q_2) - x_i(q_1)|^2 + 0(\psi)] \int_{\gamma_{\varepsilon 1}} \nabla h \cdot \underline{n}\, dS$$

as $\varepsilon \to 0$, $i = 1, 2, \ldots, m$.

Proof. With (5.1) instead of (4.1), (4.5) is replaced by

$$\mu_i - \lambda = \int_{\gamma_\varepsilon} (x_i \nabla \bar{y}_i - \bar{y}_i \nabla x_i) \cdot n\, dS\, [1 + 0(\psi)].$$

Hence

$$\mu_i - \lambda = [x_i(q_1) + 0(\psi)] \int_{\gamma_{\varepsilon 1}} \nabla \bar{y}_i \cdot \underline{n}\, dS$$

$$+ [x_i(q_2) + 0(\psi)] \int_{\gamma_{\varepsilon 2}} \nabla \bar{y}_i \cdot \underline{n}\, dS + 0(\varepsilon^{n-1}).$$

The result (5.4) would follow if we knew that the order relation (5.3) could be differentiated. The actual proof of (5.4) is similar to that of theorem 3 and will be omitted.

In the example at the end of section 4, if we take q_1 and q_2 to be the north and south poles respectively, then $x^2_{m1}(q_1) = (2m-1)/4\pi$, $x_{m1}(q_2) = (-1)^{m-1} x_{m1}(q_1)$, and (5.4) gives in particular

$$\mu_{m1} = m^2 - m + 1 + \frac{1}{2}[1 + (-1)^m](2m-1)\left(\log\frac{1}{\varepsilon}\right)^{-1} + 0\left[\left(\log\frac{1}{\varepsilon}\right)^{-2}\right],$$

$m = 1, 2, \ldots.$

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