for the pulsation mode, while the taut wire mode, in which the z-axis vibrates like a stretched wire, is described by

$$u = (u_i \cos k_i x + i u_i \sin k_i x) \exp i(\omega t - k_i z). \quad (4)$$

The continuity of the transverse velocity at all points on the interface entails that the angular frequency \(\omega\) and the wave number in the direction of the field \(k_i\) be the same in both the magnetic and the non-magnetic regions. However, the transverse wave number \(k_i\) need not be the same and in the non-magnetic region is represented by \(k_i'\).

For the flux tube the transverse variations in the perturbation amplitudes must be written in terms of Bessel functions. Thus perturbations have the form

$$u = u(k, r, m) \exp i(\omega t - m\psi - k_i z), \quad (5)$$

where

$$u(k, r, m) = iJ_m(k, r)Y + J_m(k, r)/r + J_m(k, r)\tilde{k},$$

$$m > 0,$$

$$= iJ_0(k, r)Y + J_0(k, r)/r + J_0(k, r)\tilde{k},$$

$$m = 0.$$

Within the magnetic region these expressions are consistent with the usual dispersion relation for a compressible plasma, i.e.

$$\omega^4/k^4 - \omega^2/k^2(A^2 + S_i^2) + A^2S_i^2k_3^2/k^4 = 0. \quad (6)$$

This may also be written in the form

$$k_3^2 = (V_i^2 - A^2)(V_i^2 - S_i^2)/(A^2 + S_i^2) V_i^2 - A^2S_i^2, \quad (7)$$

where \(V_i = \omega/k_i\) is the phase velocity in the direction of the field. In the non-magnetic region we have

$$\text{im} \frac{k_1^2}{k_3^2} = \frac{V_i^2 - S_i^2}{S_i^2}. \quad (8)$$

The conditions at the boundary of course provide a further constraint on the permitted wave numbers and phase velocities. If the flux sheath is effectively thin so that \(\sin \kappa \approx k_i x\), etc., the continuity of velocity and total pressure entail for the pulsation mode

$$V_i^4 - 2V_i[\sqrt{S_i^2 + S_i^2/2 + (S_i^2 - S_i^2)S_i^2/A^2}] + 2S_i^2S_i^2 = 0, \quad (9)$$

and for the taut wire mode

$$V_i^2 = S_i^2A^2/(S_i^2 - S_i^2 - A^2/2). \quad (10)$$

It is interesting to note that the corresponding approximation for a thin flux tube, i.e. \(J_i(k, r) \approx k_ir\), etc., yields the same two equations for the \(m = 0\) and \(m = 1\) modes respectively. Thus the flux sheath geometry, which entails the simpler mathematical formulation, may well represent the behaviour of the more realistic flux tube model in these two modes.

For the taut wire mode the phase velocity along the tube \(V_i\) tends to the Alfvén speed \(A\) as \(S_i \to \infty\). However, when \(S_i^2 \to S_i^2 + A^2/2\) from above \(V_i \to \infty\) and if \(S_i^2 \to S_i^2 + A^2/2\) \(V_i\) is imaginary. Imaginary values of \(V_i\) entail that either \(\omega\) or \(k_i\) is imaginary and thus the perturbations may grow either in time or with distance along the sheath unless the boundary conditions are such that the positive exponential terms are excluded. Thus, unless the temperature in the external region is significantly greater than that within the sheath, the perturbations are potentially unstable.

The pulsation modes obtained from the solution of equation (9) are more complex. In general, the solutions for \(V_i\) are real and positive when \(S_i > S_i\) for any value of \(A\) representing the superposition of either slow or fast mode waves within the flux sheath. For given values of \(A\) and \(S_i\), the interaction with the non-magnetic region selects one value of \(k_i\) for the fast mode and one for the slow mode appropriate to the particular value of \(S_i\). Evanescent modes, for which \(k_i < 0\), are also permitted. In the non-magnetic region the vibrations may represent the superposition of two plane acoustic modes if \(k_i > 0\) or evanescent modes if \(k_i < 0\).

When \(S_i < S_i\) the solutions may be real or complex depending on the ratio \(A^2/S_i^2\). For example, if \(S_i^2 = S_i^2/2\), \(V_i^2\) is complex when \(A^2/S_i^2 < \infty\) and again the perturbations are potentially unstable.

Thus for both the pulsation and taut wire modes unstable modes are likely to occur when the sound speed within the magnetic element is comparable with or greater than that in the non-magnetic regions; i.e. cool flux sheaths are likely to propagate hydromagnetic waves efficiently; hot flux tubes tend to be dynamically unstable.


Entropy and the Spontaneous Emission of Plasma Waves

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Historical Background

The emission of plasma waves by beams of electrons travelling in a plasma is a phenomenon of critical importance in applied plasma physics (for instance in problems directly related to the achievement of controlled nuclear fusion) and also astrophysical research (e.g. in the theory of solar radio bursts). In principle, the mechanisms involved are all contained in the Boltzmann-Vlasov equation, where the field is the self-consistent electromagnetic field produced by the interaction between beam and plasma. Unfortunately this celebrated equation cannot be solved directly, because both the analytical and numerical methods that can deal with this equation are plagued by secular terms which restrict the time domain of
validity of the solutions to a few thousand plasma periods. In all applications of interest this domain is far too small; indeed in all astrophysical cases it is quite negligible compared with the duration of the observed phenomena (it is even much shorter than the time resolution of present-day equipment, such as dynamic spectrographs).

This situation is very similar to that encountered in the long-term study of sustained non-linear oscillations, where perturbation and numerical analyses could only be used to cover a few periods; beyond that the secular terms masked the actual evolution. The problem was solved by using the famous 'van der Pol method' (see e.g. Minorsky 1962) or, more generally, by introducing the multiple time-scale formalism of Frieman and Sandri (see e.g. Davidson 1972). These methods aimed at representing the evolution of 'envelopes' of the oscillations. The random phase approximation fulfils a similar role in the Boltzmann-Vlasov equation and in this spirit a first theory was presented simultaneously by American and Russian plasma physicists (Drummond and Pines 1962; Vedenov et al. 1962). This theory became very popular under the (rather unsuitable) name of 'quasi-linear' theory; it was however severely criticized by mathematicians and theoretical physicists alike on the ground of mathematical inconsistency (Dolph 1963) and — a more serious criticism — because the original equations could not deal with wave absorption and were inconsistent with thermodynamics.

A quantum mechanical formalism, which might seem somewhat exotic in the context of a non-degenerate plasma, was suggested by Pines and Schrieffer (1962) and fully developed by Harris (1969). This introduced new terms, called 'spontaneous emission terms'; indeed they could be formally derived from Einstein's coefficient of spontaneous emission and the emission rates obtained in the original theory (see e.g. Tsytovich 1970). The spontaneous emission terms were however considered entirely negligible and often ignored. Subsequently, these terms were retrieved in a more critical reappraisal of the 'classical' theory by many authors (Register and Oberman 1968; Kaufman 1972).

After correcting some serious misunderstandings, Fukai and Harris (1972) proved that the quasi-linear equations were conservative when particles and waves were considered. Indeed, the plasma waves represent the reaction of the background plasma which gains the momentum and energy lost by the beam. Moreover, in the correct interpretation given by Fukai and Harris the theory was perfectly capable of handling wave absorption without running into the paradoxical phenomena of anti-diffusion.

It has so far not been realized that an entropy theorem also applies to the system, and that the spontaneous emission terms are in fact essential. This is not of purely academic interest, since the absence of these terms renders the whole quasi-linear approximation fully inconsistent (both mathematically and physically) and any so-called 'solution' of the set of quasi-linear equations without these terms is highly suspect (as pointed out in Grognard 1975). For instance, any seemingly 'reasonable' behaviour of numerical solutions of such an inconsistent set can be traced back to spurious 'spontaneous' terms introduced implicitly by the algorithms used in the calculations (as in Appert et al. 1976; and, partly, Magelssen 1976). In the particular case of the solar type III burst theory, it was indeed suggested by Melrose (1974) that the principal mechanism was the spontaneous emission of plasma waves by the streaming electrons, followed by their amplification by induced processes, while the induced emission acting on the plasma background was, at the very least, one order of magnitude below the main mechanism. This view was entirely confirmed by numerical analysis (Grognard 1975; Magelssen 1976) and now seems to be accepted even by its original detractors (Smith 1977).

However, spontaneous emission is still frequently neglected. For instance, in a recent substantial treatise Akhiezer et al. (1975) present (section 9.1.2) an entropy theorem supposedly applicable to the (inconsistent) set of quasi-linear equations, in which in fact no account whatsoever is taken of the plasma waves (a persistent error in this theory, as pointed out in Grognard (1975)). It is the aim of the second part of the present paper to correct this error.

The Entropy Theorem
To establish a valid entropy theorem for the beam-plasma system, it is necessary to start by rewriting the system of quasi-linear equations in a more rigorous formalism than usual.

Let \( \mathbf{f} = f(x,v,t) \) be the distribution of electrons at any time \( t \geq 0 \) at any point \( x = (x_1,x_2,x_3) \) in some subspace \( \mathbf{X} \subset \mathbf{E}^3 \) and for any velocity \( v = (v_1,v_2,v_3) \) in some velocity subspace \( \mathbf{V} \subset \mathbf{E}^3 \). Let \( P = P(x,k,t) \) be the corresponding density of spectral energy (normalized by \( KT \)), in the random wave approximation, for plasma waves with wave vectors \( k = (k_1,k_2,k_3) \in K \subset E^3 \), where \( K \) is the subset of permissible wave vectors \( |k| \neq 0 \) and \( |k| \leq k_0 \), the wave vector beyond which plasma waves are fully absorbed by Landau damping in the background plasma; for a Maxwellian plasma, \( k_0 \sim 2\pi \), in the normalized units.

The equations of evolution of the beam-plasma system are the coupled partial differential equations

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \left( D \cdot \frac{\partial f}{\partial v} + A f \right),
\]

\[
\frac{\partial P}{\partial t} = \gamma P + S,
\]

where the coefficients \( D \) (a tensor), \( A \) (a vector), \( \gamma \) (a scalar) and the source term \( S \) (a scalar) are given by

\[
D_{ij} = \int \frac{k_i k_j}{k^2} P \delta(k \cdot v - 1) \, dk \quad \text{(a linear functional in } P); \]

*We use the following normalizations: (1) unit of velocity: \( v_{in} = \sqrt{KT} / m \); (2) unit of (macroscopic) time: \( t_0 = (2\pi)^2 / \pi \xi \omega_0 \) \((\xi = 1 / N \omega_0 \lambda_0^3)\) is the plasma parameter; (3) unit of (microscopic) length (to measure the wave vectors): \( \lambda_0 \approx v_{in} / \omega_0 \) (Debye length). The distribution function \( f \) is the number of electrons per cell of volume \((2\pi \lambda_0 v_{in})^3 \) in the space \( \mathbf{X} \times \mathbf{V} \).
\[ A_s = \int f \left( k \cdot v - 1 \right) \frac{dk}{|k|^2} = \frac{2\pi}{|v|^3} t_n(k_0, |v|) v, \]

if \( v \in V = V - \{v : k_0, |v| < 1\} \)

\[ = 0, \text{ if } v \notin V; \]

\[ \gamma = \frac{k}{|k|^2} \int \frac{\partial f}{\partial v} \left( k \cdot v - 1 \right) dv \text{ (linear functional in } f); \]

\[ S = \frac{1}{|k|^2} \int f \left( k \cdot v - 1 \right) dv \text{ (linear functional in } f). \]

All these integrals are understood in the sense of Radon (see Gelfand et al. (1966), whose Chapter 1 is a prerequisite to a full understanding of the present work).

The initial and boundary conditions are

\[ f(x, v, t = 0) = f_0(x, v) \geq 0 \text{ for all } x \in X \text{ and all } v \in V; \]

\[ P(x, v, t = 0) = P_0(x, k) \geq 0 \text{ for all } x \in X \text{ and all } k \in K; \]

and if \( \partial X \) is the boundary of \( X \) and \( \partial V \) the boundary of \( V \), the function \( f(x \in \partial X, v \in \partial V) \) is fixed for all \( t \geq 0 \). In practice we assume that \( f \) vanishes on \( \partial X \times \partial V \) for all time \( t \geq 0 \), while \( V \) extends to the whole of velocity space except the domain \( |v^*| = 1/k_0 \), over which the background plasma is defined; i.e. we stop following the evolution before the beam and background electrons begin to merge.

Proofs of the existence and uniqueness of the solutions in a functional space appropriate for the pair \( (f, P) \) do not yet exist. Indeed, even the positive definite character of both \( f \) and \( P \) for all \( t > 0 \) seems to be an open question, unless one simply resorts to postulating either \( f \) or \( P \) to be positive definite, which does, of course, look physically natural but which could lead to a mathematical inconsistency. In the following, the positive definiteness of \( f \) and \( P \) are taken for granted. If this assumption is shown to lead to a contradiction suitable modifications to the formulation of the problem will be needed (for instance, restrictions on the initial conditions might be necessary to avoid pathological cases).

Let us introduce the entropy functions

\[ S_0 = - \int_0^v f \ln f dv, \quad (7) \]

\[ S_c = + \int_0^v \ln P dk. \quad (8) \]

While (7) is the classical definition, (8) can be formally obtained from the corresponding quantum mechanical definition in Harris (1969; equation 4.25). We now prove that

\[ \frac{d}{dt}(S_0 + S_c) \geq 0 \text{ for all } t \geq 0. \quad (9) \]

In order to show the relative importance of each of the four terms in (1) and (2), their contributions to \( dS/dt \) have been calculated separately; we have the following results.

\[ \left( \frac{dS}{dt} \right)_{\text{diffusion}} = \int_0^v \int_k \frac{P(k \cdot \partial f/\partial v)^2}{|k|^2} \delta(k \cdot v - 1) dk dv \geq 0 \]

(the only result considered by Akhiezer et al. (1975)).

\[ \left( \frac{dS}{dt} \right)_{\text{friction}} = \int_0^v \int_k \frac{k \cdot \partial f/\partial v}{|k|^2} \delta(k \cdot v - 1) dk dv \]

\[ = - \int_0^v \frac{\partial}{\partial v} A dv \leq 0. \]

A word of comment is needed here. The fact that the 'friction' term: \( \partial/\partial v \cdot (Af) \) (a friction in velocity space), decreases the entropy arises directly from the inequality: \( \partial/\partial v \cdot A \geq 0 \) (easily proved even without explicitly calculating \( A \), if one makes use of the results of Gelfand and Shilor (1964) and Gelfand et al. (1966)). Such an unusual friction is however still dissipative, as can be easily checked by proving the inequality

\[ \left( \frac{d}{dt} \int_0^v \frac{1}{2} |v|^2 f dv \right)_{\text{friction}} \leq 0. \]

\[ \left( \frac{dS}{dt} \right)_{\text{induced emission}} = \left( \frac{dS}{dt} \right)_{\text{friction}}. \]

This identity arises directly from the rather unexpected equality

\[ \int_k v dk = \int_v A \frac{\partial f}{\partial v} dv \quad (\leq 0). \]

\[ \left( \frac{dS}{dt} \right)_{\text{spontaneous emission}} = \int_0^v \int_k \frac{1}{P |k|^2} \delta(k \cdot v - 1) dk dv \geq 0. \]

Collecting all the results, one finds that

\[ \frac{dS}{dt} = \int_0^v \int_k [G(k, v)/P |k|^2] \delta(k \cdot v - 1) dk dv \geq 0, \]

where \( G(k, v) = Pk \cdot \frac{\partial f}{\partial v} + f. \)

Therefore the quasi-linear theory description of the evolution of the beam-plasma system is unconditionally consistent with the second law of thermodynamics. If the spontaneous emission terms were left out, the time derivative of the entropy would consist of two terms of opposite sign given by (a) and (b), instead of the integral of a perfect square, and nothing could be said a priori concerning the validity of the second principle of thermodynamics for such a (truncated) description. However, it is relatively easy to construct initial conditions for which the initial value of \( dS/dt \) would be negative if the spontaneous emission terms were left out; such distribution functions can even be physically meaningful. For instance, one such class is given by electron distributions which can only absorb waves \( f \) and \( df/dv \cdot |v| < 0 \), injected into a plasma which is initially weakly excited; more general cases can be obtained along the samelines as the one-dimensional study in Appendix B(2) of Grognard (1975).
One notes that with \( P = 1 \) and \( f = A \exp[-\frac{1}{2} |v|^2] \) (Rayleigh-Jeans and Maxwell’s distributions respectively), \( \frac{dS}{dt} = 0 \). Hence, thermodynamic equilibrium is also an equilibrium state for the quasi-linear equations (1) and (2) (it is not an equilibrium state if the spontaneous emission terms are missing). However, the equation \( G(k,v) \equiv P k \cdot \partial f / \partial v + f = 0 \) is not sufficient to establish that this is the only equilibrium or, indeed, that it could be reached. (We recall that in the one-dimensional model the existence of an integral of motion prevents the system reaching the thermodynamical equilibrium except for a particular class of initial conditions.)

**Conclusion**

A correct entropy theorem has been established for the beam-plasma system whose interactions are described by the set of quasi-linear equations including spontaneous emission terms. The role and interplay of the induced and spontaneous emission of plasma waves and their effects on the streaming electrons, i.e., diffusion and ‘friction’ in velocity space, are thereby greatly elucidated (besides establishing consistency with the second law of thermodynamics, one of the foremost principles of theoretical physics). The present analysis shows that the spontaneous terms truly cannot be neglected, a fact that was previously suggested (Grognard 1975) on the basis of less rigorous analysis and numerical work. It must be stressed, however, that any claim on the basis of numerical analysis has to be treated with extreme caution: because any algorithm used to represent the quasi-linear equations on a finite grid is doomed to introduce spurious terms which have the same qualitative effect as actual spontaneous emission terms.

According to Dyson (1960), Malthusian pressures may have led extra-terrestrial civilizations to utilize significant fractions of the energy output from their stars or the total amount of matter in their planetary systems in their search for living space. This would have been achieved by constructing from a large number of independently orbiting colonies, an artificial biosphere surrounding their star. Biospheres of this nature are known as Dyson spheres. If enough matter is available to construct an optically thick Dyson sphere, the result of such astroengineering activity, as far as observations from the earth are concerned, would be a point source of infra-red radiation which peaks in the 10 micron range. If not enough matter is available to completely block the stars’ light the result would be anomalous infra-red emission accompanying the visible radiation (Dyson 1960).

Although small amounts of infra-red emission have been detected from a few stars, and some compact infra-red sources have been detected in 

We try to ascertain here which of the two basic resources a civilization is likely to most completely utilize: the energy output from its star, or the matter in its planetary system. We shall do this by examining the types of Dyson spheres that we could build with the matter available in our solar system. Although it will be necessary to make several simplifying assumptions and ultimately to adopt a specific model to obtain definite numerical estimates, the results will be general enough to give some guide to what may have happened in other systems.

The biosphere must consist of independently orbiting colonies because of the physical impossibility of a solid shell completely surrounding the Sun. We consider first a general model in which each colony has the same mass \( m \). If the total mass available for building is \( M \), then \( n = M/m \) colonies can be built. Let these colonies be arranged around the Sun in a series of densely packed circular rings, each ring at a slightly different distance from the Sun. This particular spatial configuration is the one with the maximum number of colonies on non-intersecting orbits at a given distance from the Sun. If more than one ring was placed at the same distance, orbits of individual colonies could intersect, a situation which is not particularly desirable.

Let the total number of rings be \( N_r \) and number them outwards from the Sun. Now assume that each colony in the \( r \)-th ring presents a surface area \( A \), perpendicular to the Sun. Since the intensity of sunlight decreases with distance \( r \) as \( r^{-2} \), we assume...