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Subnormal structure in some classes of infinite groups

D.J. McCaughan

Let p be a prime and G a group with a p-reduced nilpotent normal subgroup N such that G/N is a nilpotent p-group. It is shown that if G has the subnormal intersection property and if G/N is finite or N is p-torsion-free, then G is nilpotent. This result is used to prove that an abelian-by-finite group has the subnormal intersection property if and only if it has a bound for the subnormal indices of its subnormal subgroups.

1. Introduction

The purpose of this paper is to prove and apply some results similar to those which were used in [5] to investigate metanilpotent groups with bounded subnormal indices. For general background, notation and terminology we refer the reader to [5].

In studies of this nature a typical result is one which states that under certain conditions, involving the subnormal structure to varying degrees, a group is necessarily nilpotent. (For example 4.4 and 5.4 of [6], and Lemma 4 of [8].) We hope to indicate how further results of this type can be used to describe in some detail the structure of some simple instances of groups with restricted subnormal structure, and also to determine conditions under which the subnormal intersection property implies the (more manageable) property of having bounded subnormal indices.

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In §2, following the pattern of [5], we present a series of preliminary lemmas enabling us to investigate abelian-by-cyclic groups with the subnormal intersection property. In §3 we apply the results of §2 to prove the central theorem of the paper, Theorem A, which deals with metanilpotent groups. Theorems B and C clarify to some extent the link between the subnormal intersection property and the property of having bounded subnormal indices, for metanilpotent groups. In Theorem D (§4) we show that for abelian-by-finite groups the two properties coincide.

2. Preliminary results

The results in the early part of this section concern automorphisms of abelian groups, and may be already well known. In the interests of brevity we fix some notation. We will denote by A an arbitrary abelian group and by H a group acting on A, in the sense that there is given a specific (but not always explicit) homomorphism from H into the group of automorphisms of A. For elements a of A and h of H we write a^h for the image of a under the automorphism which is the image of h, and $[a, h]$ for the element $a^{-1}a^h$. The subgroups $\langle [a, h] : a \in A \rangle$, $\langle [a, h] : a \in A, h \in H \rangle$ and $\langle a : a^h = a$ for all h in $H \rangle$ will be denoted by $[A, h]$, $[A, H]$ and C or $C_A(H)$ respectively. If B is an H-invariant subgroup of A, the actions of H on B and A/B are defined in the natural way.

Our first lemma is merely a statement of some obvious facts.

LEMMA 1. *(a) The group* A/C *may be embedded in the cartesian product* $[A, H]^H$ *by the monomorphism* α *defined by:*

$$(aC)\alpha(h) = [a, h] .$$

(b) If $a \in A - C$ *and* $(aC)^h = aC$ *for all* h *in* H *then the map* θ *defined by* $h\theta = [a, h]$ *is a non-trivial homomorphism from* H *into* C .

LEMMA 2. *If* H *is finite,* B *is any* H*-invariant subgroup of* A *and* $a_1 \in A$ *has the property that* $(a_1B)^h = a_1B$ *for all* h *in* H *, then*

$a_1^{|H|} \in BC$.

Proof. The map β defined for each a in A by

$$a\beta = \prod_{h \in H} a^h$$

is clearly a homomorphism from A into C . Now

$$\begin{aligned} a_1\beta &= \prod_{h \in H} (a_1[a_1, h]) \\ &= a_1^{|H|} \prod_{h \in H} [a_1, h] \end{aligned}$$

and the result follows.

REMARK. This lemma shows incidentally that if the action of H on A is fixed-point-free (that is, $C = 1$) and A/B has no non-trivial elements of order dividing $|H|$, then the action of H on A/B remains fixed-point-free. If A is finite, fixed-point-free action is transferred to the factor group modulo any H-invariant subgroup (see [*1*], p. 335). A moment's contemplation of the infinite dihedral group should indicate that this is not true in general if A is infinite.

Our next lemma yields an analogue of a result for finite groups which appears on p. 172 of [*11*].

LEMMA 3. *If* H *is finite then*

(a) $(C \cap [A, H])^{|H|} = 1$,

(b) $A^{|H|} \le [A, H]C$,

(c) $[A^{|H|}, H] \le [[A, H], H]$.

Proof. The homomorphism β defined in the proof of Lemma 2 has the properties that $(a^h)\beta = a\beta$ for each $h \in H$ and $a \in A$, and $c\beta = c^{|H|}$ if $c \in C$. Then

$$\begin{aligned} (C \cap [A, H])^{|H|} &= (C \cap [A, H])\beta \\ &= 1 \text{ , proving } (a). \end{aligned}$$

Statement *(b)* follows from Lemma 2 on taking $B = [A, H]$, and *(c)* is a trivial consequence of *(b)*.

We will require the following more detailed version of Lemma 3.3 of [5].

LEMMA 4. *Let* x *be an automorphism of order* p^r *of a* p*-reduced abelian group* A , *where* p *is a prime and* r *a non-negative integer. Define* A_i *for each non-negative integer* i *as follows:* $A_0 = A$, $A_{i+1} = [A_i, x]$. *If* $A_k = A_{k+1}$ *for some* k *then* $A_k = 1$, *and if in addition* A *is torsion-free,* $A_1 = 1$.

Proof. The first part of this result appears as Lemma 3.3 of [5]. To prove the second, it will suffice to show that if $A_{m+1} \leq C = C_A(x)$ for some non-negative integer m , then $A_m \leq C$ also. But if $A_{m+1} \leq C$ the subgroup A_mC/C is elementwise fixed under the action of $\langle x \rangle$. On the other hand, since A is torsion-free, Lemma 1 *(b)* implies that the action of $\langle x \rangle$ on A/C is fixed-point-free. Thus $A_m \leq C$, as required.

The central lemma of this work is a variant of Lemma 3.5 of [5]. We include a proof for completeness.

LEMMA 5. *Let* G *be a group with a normal abelian subgroup* A *such that* $G = \langle x, A \rangle$ *and* $[x^k, A] = 1$ *for some integer* k . *If* G *has the subnormal intersection property and* A *is* p*-reduced for each prime* p *dividing* k , *then* G *is nilpotent; if, in addition,* A *is torsion-free then* G *is abelian.*

Proof. There is no loss of generality in assuming that k is a prime power. If we write M for the subgroup $\langle x \rangle \cap A$, G/M inherits the subnormal intersection property and so the standard series in G/M of the subgroup $\langle xM \rangle$ must become stationary after finitely many terms. But this standard series is just $\{\langle x \rangle A_i/M : i \geq 0\}$ where the A_i are defined as in Lemma 4. Thus for some positive integer m , $\langle x \rangle A_{m-1} = \langle x \rangle A_m$. Then

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$$\begin{aligned} A_{m+1} &= [A_m, x] = [\langle x\rangle A_m, x] \\ &= [\langle x\rangle A_{m-1}, x] = [A_{m-1}, x] \\ &= A_m . \end{aligned}$$

From Lemma 4 we deduce that $A_m = 1$, and if A is torsion-free, that $A_1 = 1$. The obvious fact that, for each i , A_i coincides with the $(i+1)$th term $\gamma_{i+1}G$ of the lower central series of G now implies the result.

Example (a). The wreath product G of the additive group of rational numbers with any finite cyclic group has the property that every subnormal subgroup of G is normal in G . But G is not nilpotent, showing that the p-reducedness condition in Lemma 5 is essential.

Example (b). Since, for each prime p , the wreath product of a finite abelian p-group with a p-cycle may have arbitrarily large nilpotent class, the torsion-freeness condition in Lemma 5 cannot be removed without loss.

It is possible to obtain fairly detailed information about the structure of abelian-by-(finite cyclic) groups with the subnormal intersection property, provided we retain some torsion-freeness requirement.

LEMMA 6. *Let $G = \langle A, x\rangle$ where A is an abelian normal subgroup of G and is p-torsion-free for some prime p . Suppose that $\left[x^{p^r}, A\right] = 1$ for some non-negative integer r , and denote by P the maximal p-radicable subgroup of A . Then if G has the subnormal intersection property,*

(i) $P = G'C_P(x)$ *and* $G' \cap C_P(x) = 1$;

(ii) $G' = \gamma_3 G$;

(iii) $A = G'C_A(x)$ *and* $G' \cap C_A(x) = 1$;

(iv) $G' \cap \zeta_1(G) = 1$;

(v) a subnormal subgroup of G which contains an element xa ,

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$a \in A$, contains G' and is normal in G.

Proof. By Lemma 3, P is the direct product of the subgroups $C_P(x)$ and $[P, x]$. Clearly *(i)* will be established if we show that $G' \leq [P, x]$, that is, $[A, x] \leq [P, x]$.

Let $a \in A$. By Lemma 5, G/P is abelian since A/P remains p-torsion-free. Thus $[a, x] \in P$ and x commutes with $[a, x]$ modulo $[P, x]$. It follows that

$$[a, x]^{p^r} = \left[a, x^{p^r}\right]y = y$$

for some $y \in [P, x]$. But $P/[P, x]$ is p-torsion-free, hence $[a, x] \in [P, x]$ and we have shown $[A, x] \leq [P, x]$ as required.

Since $G' = [P, x]$ and $\left[[P, x], x\right] = [P, x]$ by Lemma 3, *(ii)* follows immediately.

To prove *(iii)* suppose for the moment that $C_A(x) = 1$. Since A/P is p-torsion-free, the action of $\langle x \rangle$ on A/P remains fixed-point-free by Lemma 2. But G/P is abelian, so that $A = P$ is the only possibility, and by Lemma 3, $A = [A, x]$.

To deal with the general case we note that by Lemma 1, $A/C_A(x)$ is p-torsion-free and acted on fixed-point-freely by $\langle x \rangle$. The argument above then shows that

$$A = [A, x]C_A(x) = G'C_A(x) ,$$

and since $G' \cap C_A(x) = G' \cap C_P(x) = 1$, *(iii)* is established.

Statement *(iv)* is now clear, for $G' \cap \zeta_1(G) = G' \cap C_A(x)$. The fact that

$$[G', xa] = [G', x] = G'$$

proves *(v)*.

REMARKS. To indicate the drastic effect of removing p-torsion-freeness from the hypotheses of Lemma 6, we point out that for any non-abelian group of the type mentioned in Example (b) above, all of the conclusions *(i)*-*(v)* fail to hold.

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Robinson ([9], Theorem E) has shown that the unrestricted standard wreath product of any abelian group with an infinite cyclic group has the property that every subnormal subgroup has subnormal index at most 2 . This indicates that in the context of Lemmas 5 and 6 a restriction of the form $[x^k, A] = 1$ is necessary to make possible a useful description of abelian-by-cyclic groups with the subnormal intersection property. Under these conditions, as we shall see (Theorem D) such groups have bounded subnormal indices; for abelian-by-cyclic groups in general the subnormal intersection property does not necessarily imply the existence of a bound for the subnormal indices (see Example 5.2 of [4]).

3. The main theorems

In this section we obtain some results similar to those of [5] in the context of metanilpotent groups with the subnormal intersection property. We also attempt to give conditions under which such groups have bounded subnormal indices.

THEOREM A. *Let G be a group with a nilpotent normal subgroup N such that G/N is a nilpotent π-group, for a given non-empty set of primes π, and N is p-reduced for each prime p in π. If G has the subnormal intersection property and if*

(i) $\pi = \{p\}$ *and* G/N *is finite; or*

(ii) $\pi = \{p\}$ *and* N *is p-torsion-free; or*

(iii) $\{p, q\} \subseteq \pi$ *with* $p \neq q$,

then G is nilpotent, and in cases (ii) and (iii) the nilpotent class of G is at most the larger of the classes of N and G/N.

Proof. To prove the result in case *(i)* we proceed by induction on $n = |G/N|$, noting that the case $G = N$ is trivial. Suppose that $n > 1$ and the result holds for groups in which the order of the relevant quotient group is less than n . Since G/N is finite and nilpotent, there is a proper normal subgroup H of G , containing N , such that G/H is cyclic. By the induction hypothesis H is nilpotent since it inherits all the properties of G ; moreover since N and H/N are p-reduced, H is also p-reduced. Applying Lemma 3.5 of [5], we deduce that G is nilpotent and our inductive proof is complete.

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In case *(ii)* we use induction on the nilpotent class of N. If $N = 1$ the result is immediate, so we assume $N > 1$ and put $A = \zeta_1(N)$. Then N/A is p-reduced by Lemma 3.4 of [5] and by a well-known result of Mal'cev N/A is p-torsion-free. Since G/A inherits the subnormal intersection property we may assume that G/A is nilpotent.

Now let x be any element of G. Since G/A is nilpotent, the subgroup $\langle x, A\rangle$ is subnormal in G and thus inherits the subnormal intersection property. Moreover since for some integer k, $x^{p^k} \in N$, we have $\left[x^{p^k}, A\right] = 1$. Hence by Lemma 5, recalling that A is p-torsion-free, $\langle x, A\rangle$ must be abelian. This shows that A is central in G and that G is nilpotent. To bound the nilpotent class of G we may assume that the p-torsion subgroup $T(p)$ of G is trivial, for $N \cap T(p) = 1$ and G is a subdirect product of G/N and $G/T(p)$. Under this additional assumption, we point out that by the above argument it is easy to show $\zeta_i(N) \le \zeta_i(G)$ for each non-negative integer i. If c is the nilpotent class of N, $G/\zeta_c(G)$ is thus a p-group. But because G is now assumed p-torsion-free, this forces $G = \zeta_c(G)$ and proves the claim.

In case *(iii)* the hypotheses imply that N is torsion-free. The proof runs essentially as in *(ii)* and will be omitted.

REMARKS. (a) If there is a prime p in π for which N is not p-reduced, Example (a) of §2 shows that G need not be nilpotent. Similar examples show that the weaker condition that N should be π-reduced is not sufficient.

(b) The group C_p wr C_{p^∞} shows that the case $\pi = \{p\}$ is special, and that the additional restrictions on N and G/N are needed to give nilpotency.

(c) If, in the statement of Theorem A, we insist that G should have a bound on its subnormal indices, then the result will hold, by Theorem A of [5], without the extra conditions on G/N and N in *(i)* and *(ii)*, but the estimate of nilpotent class will still depend on a torsion-freeness condition (see Example (b) of §2).

In view of Remark (c), the question arises as to whether, for a nilpotent-by-(periodic nilpotent) group with the subnormal intersection property, the conditions in *(i)* and *(ii)* of Theorem A are enough to imply the existence of a bound for the subnormal indices. Such is indeed the case, as we will now deduce from Theorem A.

THEOREM B. *Let G be a group with a nilpotent normal subgroup N such that G/N is finite and nilpotent. Then G has the subnormal intersection property if and only if G has a bound on its subnormal indices.*

Proof. The implication in the reverse direction is immediate. We prove the non-trivial half of the theorem by induction on $n = |G/N|$, noting that the case $G = N$ is trivial. Suppose then that $n > 1$. If G/N cannot be expressed as the product of two proper normal subgroups it must be a cyclic p-group for some prime p. Then, denoting by P the maximal p-radicable subgroup of N, Theorem A implies that G/P is nilpotent, say of class d. If the nilpotent class of N is c, and S is any subnormal subgroup of G, we may apply Lemma 3.7 of [5] to deduce that $s(SP : S) \leq c$. Hence

$$s(G : S) \leq s(G : SP) + s(SP : S) \leq d + c ,$$

and in this case the subnormal indices are certainly bounded.

We may therefore assume that G/N can be expressed as the product of two proper normal subgroups. Now if S is a subnormal subgroup of G with $SN \neq G$, then SN coincides with one of the finitely many proper subnormal subgroups of G which contain N. Each of these subnormal subgroups satisfies the conditions of the theorem and so, by the obvious inductive assumption, must have a bound for the subnormal indices of its subnormal subgroups. Thus there is an integer k, independent of S, such that $s(SN : S) \leq k$. If the nilpotent class of G/N is m, we then have $s(G : S) \leq k + m$. On the other hand, if S is a subnormal subgroup of G with $SN = G$, then by our assumption on G/N, S can be expressed as the product of two subgroups S_1 and S_2, each normal in S, with S_1N/N and S_2N/N proper normal subgroups of G/N. Then S_1 and S_2 are subnormal in G with subnormal indices at most $k + m$, as

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above. Hence by Lemma 2.2 of [7], $s(G : S) \le (k+m)^2$.

To sum up, any subnormal subgroup S of G will have subnormal index at most $(k+m)^2$; this integer is independent of the choice of S , so our inductive proof is complete.

THEOREM C. *Let* π *be a non-empty set of primes and* G *a group with a* π-*torsion-free nilpotent normal subgroup* N *such that* G/N *is a nilpotent* π-*group. Then* G *has the subnormal intersection property if and only if* G *has a bound on its subnormal indices.*

Proof. To prove the non-trivial half of the theorem we argue by induction on the nilpotent class n of N . We prove in fact that a bound for the subnormal indices is given by $f(n) = r + \frac{n}{2}(n+2r+5)$ where $r = \max(m, n)$, m being the nilpotent class of G/N . We begin by remarking that this formula is valid for $n = 0$, that is, when $N = 1$. Writing A for $\zeta_1(N)$ we may assume, since N/A is π-torsion-free, that G/A has its subnormal indices bounded by $r' + \left(\frac{n-1}{2}\right)(n+2r'+4)$ where $r' = \max(n-1, m)$. Since $r' \le r$ we may replace this bound by $r + \left(\frac{n-1}{2}\right)(n+2r+4)$.

For each prime p in π let $G(p)/N$ denote the Sylow p-subgroup of G/N and $Q(p)$ the maximal p-radicable subgroup of N . If S is any subnormal subgroup of G , let $S(p)/S\cap N$ denote the Sylow p-subgroup of $S/S\cap N$. Then $S(p)$ is a subnormal subgroup of $G(p)$, and by Lemma 3.7 of [5] we have $s\big(S(p)Q(p) : S(p)\big) \le n$. Now by Theorem A *(ii)*, $G(p)/Q(p)$ is nilpotent of class at most r , since $N/Q(p)$ is torsion-free. It follows that $s\big(G : S(p)\big) \le n + r + 1$ for each p in π .

Now the group $SA/S\cap N$ is the join of an abelian normal subgroup $A(S\cap N)/S\cap N$ and a nilpotent subnormal subgroup $S/S\cap N$. By Lemma 4.5 of [7] this means that $SA/S\cap N$ is nilpotent. Then $S/S\cap N$ lies in the torsion subgroup $T/S\cap N$ of $SA/S\cap N$; moreover each term (after the first) of the standard series of $S/S\cap N$ in $T/S\cap N$ is just the direct product of the corresponding terms of the standard series of the $S(p)/S\cap N$ as p ranges over π . Hence

$$s(T/S\cap N : S/S\cap N) \le n + r + 1 .$$

We then have

$$s(G:S) \le s(G:SA) + s(SA:T) + s(T:S)$$
$$\le r + \left[\frac{n-1}{2}\right](n+2r+4) + 1 + n + r + 1$$
$$\le r + \frac{n}{2}(n+2r+5) = f(n) .$$

4. An application

In this section we show, using our previous results, that any abelian-by-finite group which has the subnormal intersection property has a bound on the subnormal indices of its subnormal subgroups.

THEOREM D. *Let G be a group with an abelian normal subgroup A such that G/A is finite. Then G has the subnormal intersection property if and only if G has a bound on its subnormal indices.*

Proof. The implication in one direction is immediate. To prove the converse implication, we argue by contradiction as follows. Suppose that there are counterexamples to this implication, that is, abelian-by-finite groups with the subnormal intersection property but having unbounded subnormal indices. Let G be a counterexample, with an abelian normal subgroup A such that the factor group G/A has the least possible order (clearly $A \neq G$). We show that we can make the following assumptions about G :

(i) $\bigcap\{\gamma AG^i : i \ge 0\} = 1$,

(ii) $A = C_G(A)$.

To establish (i) we note that each member H of the finite set of proper subnormal subgroups of G containing A will satisfy the hypotheses of the theorem, with $|H/A| < |G/A|$. By the minimality of $|G/A|$ it is then possible to find a positive integer k , independent of H , which will bound the subnormal indices (in H) of subnormal subgroups of H . If d is the composition length of G/A , then for any subnormal subgroup S of G such that SA is a proper subgroup of G we must have

$$s(G:S) \le s(G:SA) + s(SA:S) \le d + k .$$

If we now denote by Σ the set of subnormal subgroups S of G with

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$SA = G$, the remarks above indicate that the subnormal indices of the subgroups in Σ must be unbounded, since G is a counterexample. But if $S \in \Sigma$ then for some positive integer m , depending on S , we have

$$\gamma AG^m = \gamma A(SA)^m = \gamma AS^m \leq S .$$

Hence S certainly contains $B = \bigcap\{\gamma AG^i : i \geq 0\}$ and it is clear that G/B must also be a counterexample to the theorem. Thus, by replacing G by G/B if necessary, we may assume that G satisfies (i).

If (ii) is not satisfied, we have $A < C = C_G(A)$. Then C and its derived group C' are normal subgroups of G , and G/C' has an abelian normal subgroup C/C' , of index $|G/C| < |G/A|$. Since G/C' satisfies the hypotheses of the theorem, the minimality of $|G/A|$ implies that G/C' has a bound on its subnormal indices. But A is central in C , so that, by a well-known result of Schur (see [2], Theorem 8.1), C' is finite. Then by Lemma 1 of [8], G has a bound on its subnormal indices, contradicting our choice of G . We have thus shown that $A = C_G(A)$ and we may assume that G satisfies both (i) and (ii).

Now if we define $A_i = \gamma AG^i$ for each $i \geq 0$, and put $A_\omega = 1 = \bigcap\{A_i : i \geq 0\}$, using (i), we see that the chain $\{A_i : 0 \leq i \leq \omega\}$ is an invariant descending series of A , in the sense of [3], 1.2. Moreover, by (ii), the group G/A is isomorphic to a subgroup of the stability group of this series, again using the terminology of [3]. But by Lemma 16 of [3] a finite group which is embeddable in the stability group of an invariant descending series of a group is necessarily abelian-by-nilpotent. Thus there is a normal subgroup M of G , containing A , such that M/A is abelian and G/M nilpotent.

Now since M inherits the subnormal intersection property, M has a bound on its subnormal indices by Theorem B. By a result of Robinson (Corollary to Theorem 1 of [10]) the nilpotent class of a nilpotent group with a bound k on its subnormal indices can be at most $R(k)$, for some function R . It easily follows that the lower central series of any group with bounded subnormal indices must become stationary after finitely many terms. Thus for some normal subgroup M_1 of M , M/M_1 is nilpotent and $[M_1, M] = M_1$. Then M_1 is clearly contained in A ; indeed a simple inductive argument shows that for any non-negative integer i, $M_1 \leq \gamma AM^i \leq \gamma AG^i$. It then follows from (i) that $M_1 = 1$ and thus M is nilpotent. But now since G/M is nilpotent we may apply Theorem B to deduce that G must have a bound on its subnormal indices. With this contradiction to our choice of G the theorem is proved.

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Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
Canberra, ACT.