BULL. AUSTRAL. MATH. SOC. 34G20
VOL. 33 (1986), 419-434.

# KNESER'S THEOREM FOR DIFFERENTIAL EQUATIONS IN BANACH SPACES

NIKOLAOS S. PAPAGEORGIOU

We consider the Cauchy problem $\dot{x}(t) = f(t,x(t))$ , $x(0) = x_0$ defined in a nonreflexive Banach space and with the vector field $f : T \times X \to X$ being weakly uniformly continuous. Using a compactness hypothesis that involves the weak measure of noncompactness, we prove that the solution set of the above Cauchy problem is nonempty, connected and compact in $C_{X_w}(T)$ .

## 1. Introduction

It is well-known by now that Peano's theorem on the existence of solutions for ordinary differential equations does not hold when the underlying Banach space is infinite dimensional. This was illustrated with concrete examples constructed by Dieudonné [10] (for $X = c_0$) and Yorke [21] (for $X = \ell_2$) . Then came Cellina [5], who showed that given any nonreflexive infinite dimensional Banach space, using James' theorem we can set up a Cauchy problem that does not have a solution. Finally it was Godunov [13], who proved that Peano's theorem characterizes finite dimensional Banach spaces. Since then ordinary differential equations in infinite dimensional Banach spaces, with strongly continuous vector fields, have been studied extensively and several interesting results have appeared

Received 23 August 1985. Research supported by N.S.F. Grant D.M.S.-8403135.





https://doi.org/10.1017/S0004972700004007 Published online by Cambridge University Press

(see Deimling [8], Martin [17] and references therein).

However the study of the Cauchy problem in a Banach space relative to the weak topology is lagging behind. Recently Faulkner [12] showed that the weak continuity of the vector field $f(\cdot,\cdot)$ is insufficient for the existence of weak solutions. Before that there were a few papers dealing with this problem, but several of them had wrong proofs. One of the first results was due to Browder [4], but it was shown by Knight [14] to be incorrect. Unfortunately the proof of [14] also contains a flaw. Namely at a certain point there is an erroneous application of the dominated convergence, which as we know it is always valid for sequences but in general it is not valid for nets, unless the net is countably determined, that is it possesses a countable cofinal subset. To illustrate that we include the following counterexample taken from Bourbaki [3]. Let $F$ denote the family of finite subsets of $[0,1]$ ordered by inclusion and let $\chi_k(\cdot)$ be the characteristic function of $k \in F$. Then $\chi_k \to 1$ but $\int_I \chi_k(s)ds = 0$ for all $k \in F$. The mistake of Knight can be easily remedied by proving a version of the Schauder-Tichonoff fixed point theorem which is valid for weakly sequentially continuous maps. Also it should be noted that [14] contains another inaccurate statement. Namely in Corollary 6 he claims that if $f(\cdot,\cdot)$ is a vector field weakly continuous in $x$ and $X$ has the Radon-Nikodym property then every weak solution is a strong solution. This is not correct and we can easily construct counter-examples in the nonseparable Hilbert space $\ell_2[0,1]$. What is needed for Corollary 6 of [14] to be correct is either that $X^*$ is separable (in particular $X$ to be separable reflexive) or that $f(\cdot,\cdot)$ is strong-to-weak continuous (in particular weakly continuous) in both variables. Note by the way that separable dual or reflexive spaces are only a subclass of the family of Banach spaces possessing the Radon-Nikodym property (Dunford-Pettis theorem and Phillips' theorem, see Diestel-Uhl [9], 79-82). Since then, there appeared also the works of Szep [19], Boudourides [2] (whose proof also has a flaw as was pointed out in [18]) and the more general result of Cramer-Lakshmikantham-Mitchell [6]. All those results were extended by the very recent theorem of the author [18], which appears to be the most general result in this direction.

The purpose of the present note is to complete the work initiated in [18] by proving a Kneser's theorem concerning the solution set of a Cauchy problem with a weakly continuous vector field in a nonreflexive Banach space. Earlier results in that direction were obtained by Szufla [20] and Kubiaczyk [15]. Our theorem goes beyond the above mentioned papers.

## 2. Kneser's theorem for weak solutions

Let $X$ be a Banach space, with $X^*$ its topological dual. Let $T = [0,b]$, a bounded closed interval in $\mathbb{R}_+$. By $w$ we will denote the weak topology on $X$.

In [7] DeBlasi introduced the following measure of noncompactness in the weak topology. Let $A$ be a nonempty bounded subset of $X$. We define

$$\beta(A) = \inf\{t > 0 : \exists (K \in P_{wk}(X))\,(A \subseteq K + tB_1)\}$$

where $P_{wk}(X) = \{B \subseteq X : B \neq \phi$ and $B$ is weakly compact$\}$ and $B_1$ is the unit ball in $X$.

The properties of $\beta(\cdot)$ are outlined in Lemmata 2.1 and 2.2 of [18]. Those properties show that $\beta(\cdot)$ is what Banas-Goebel [1] call a "sub-linear measure of noncompactness".

Given a vector field $f : T \times X \to X$, we consider the following Cauchy problem:

$$\left\{\begin{array}{l} \dot{x}(t) = f(t,x(t)) \\ x(0) = x_0 \end{array}\right\}. \qquad (*)$$

By a solution of (*) we understand a function $x(\cdot) \in C_X(T)$ = continuous functions from $T$ into $X$, which is once weakly differentiable and satisfies (*) with $\dot{x}(\cdot)$ denoting the weak derivative. If $f(\cdot,\cdot)$ is continuous from $T \times X_w$ into $X_w$ (that is $f(\cdot,\cdot)$ is weakly continuous), then from what we said in Section 1, we get that every weak solution $x(\cdot)$ is almost everywhere differentiable and satisfies (*) almost everywhere with $x(\cdot)$ denoting the strong derivative (that is $x(\cdot)$ is a strong solution).

In [18] we proved the following existence theorem concerning (*).

https://doi.org/10.1017/S0004972700004007 Published online by Cambridge University Press

THEOREM 2.1 [18]. *If* $f: T \times X \to X$ *is a function such that*

(1) $f(\cdot,\cdot)$ *is continuous from* $T \times X_w$ *into* $X_w$ *(that is* $f(\cdot,\cdot)$ *is weakly continuous),*

(2) *for all* $(t,x) \in T \times X$, $\|f(t,x)\| \le M$,

(3) *for all* $A \subseteq X$ *bounded we have that*

$$\lim_{r\to 0^+} \beta(f(T_{t,r} \times A)) \le w(t,\beta(A)) \text{ almost everywhere}$$

*where* $T_{t,r} = [t,t+r]$ *and* $w(\cdot,\cdot)$ *is a Kamke function,*

*then* (*) *admits a solution.*

Now we are going to examine the topological properties of the solution set $S_0$ of (*). We have the following Kneser's theorem concerning this set.

THEOREM 2.2. *If* $f: T \times X \to X$ *is a function such that*

(1) $f(\cdot,\cdot)$ *is weakly uniformly continuous,*

(2) *for all* $(t,x) \in T \times X$, $\|f(t,x)\| \le M$,

(3) *for all* $A \subseteq X$ *bounded we have that*

$$\lim_{r\to 0^+} \beta(f(T_{t,r} \times A)) \le w(t,\beta(A)) \text{ almost everywhere}$$

*where* $T_{t,r} = [t,t+r]$ *and* $w(\cdot,\cdot)$ *is a Kamke function,*

*then* $S_0$ *is a nonempty, compact and connected subset of* $C_{X_w}(T)$.

Proof. That $S_0$ is nonempty follows from Theorem 2.1.

For $\lambda > 0$ we define the set $S_\lambda$ of all $\lambda$-approximate solutions of the Cauchy problem (*) to be all $x: T \to X$ continuous such that

(i) $x(0) = x_0$, $\|x(t')-x(t)\| \le M|t'-t|$ for $t'$, $t \in T$, strong derivative exists almost everywhere,

(ii) $\left\|x(t') - x(t) - \int_t^{t'} f(s,x(s))ds\right\| < \lambda|t'-t|$ for all $0 \le t \le t' \le b$,

(iii) $\sup_{t\in T}\left\|x(t) - x_0 - \int_0^t f(s,x(s))ds\right\| < \lambda$.

Next let $\varepsilon \in [0,b]$ and define the family $E = \{x_\varepsilon(\cdot)\}$ of $\varepsilon$-Euler polygons by

https://doi.org/10.1017/S0004972700004007 Published online by Cambridge University Press

$$x_\varepsilon(t) = x_0 \qquad \text{for} \quad 0 \le t \le \varepsilon$$
$$= x_\varepsilon(t_k) + (t-t_k)f(t_k, x_\varepsilon(t_k)) \quad \text{for} \quad t \in [t_k, t_{k+1}]$$

where $t_k = k\varepsilon$ with $k \in \{1,2,\ldots,n_\varepsilon\}$, $n_\varepsilon = [\frac{b}{\varepsilon}]$ and $t_{n_\varepsilon+1} = b$.

Clearly $\dot{x}(\cdot)$ exists almost everywhere. Let $t'$, $t \in T$. Assume $t \in [t_i, t_{i+1}]$ and $t' \in [t_j, t_{j+1}]$ with $i \le j$. Then we have:

$$\|x_\varepsilon(t') - x_\varepsilon(t)\| = \|x_0 + \sum_{k=1}^{j-1} (t_{k+1}-t_k)f(t_k, x_\varepsilon(t_k)) + (t'-t_j)f(t_j, x_\varepsilon(t_j))$$
$$- x_0 - \sum_{k=1}^{i-1} (t_{k+1}-t_k)f(t_k, x_\varepsilon(t_k)) - (t-t_i)f(t_i, x_\varepsilon(t_i))\|$$
$$\le |t-t_{i+1}|\|f(t_i, x_\varepsilon(t_i))\|$$
$$+ \sum_{k=i+1}^{j-1} (t_{k+1}-t_k)\|f(t_k, x_\varepsilon(t_k))\| + (t'-t_j)\|f(t_j, x_\varepsilon(t_j))\|$$
$$\le M[t'-t].$$

Next we will show that given any $\lambda > 0$ we can find $\varepsilon(\lambda) > 0$ such that for $\varepsilon \le \varepsilon(\lambda)$ we have that $x_\varepsilon(\cdot) \in S_\lambda$.

We have already verified condition (i) in the definition of $S_\lambda$. Next let us verify condition (iii).

Let $t \in [t_i, t_{i+1}]$. From the Hahn-Banach theorem we know that we can find $x^* \in X^*$ such that $\|x^*\| = 1$ and

$$|(x^*, x_\varepsilon(t) - x_0 - \int_0^t f(s, x_\varepsilon(s))ds)| = \|x_\varepsilon(t) - x_0 - \int_0^t f(s, x_\varepsilon(s))ds\| .$$

Also let $\mu > 0$ and $\delta' > 0$ be such that $\mu M + \delta' b < \lambda$. Because $f(\cdot,\cdot)$ is by hypothesis weakly uniformly continuous we can find $\varepsilon(\lambda) \le \mu$ such that

$$|(x^*, f(s, x_\varepsilon(s)) - f(t_i, x_\varepsilon(t_i))| < \delta'$$

for $t_i \le s \le t_{i+1}$ and $\varepsilon \le \varepsilon(\lambda)$. Hence we can write that:

https://doi.org/10.1017/S0004972700004007 Published online by Cambridge University Press

$$|(x^*, x\ (t) - x_0 - \int_0^t f(s, x_\varepsilon(s))ds)| \le \int_0^{t_1} |(x^*, f(s, x_\varepsilon(s)))|ds$$

$$+ \sum_{k=1}^{i-1} \int_{t_k}^{t_{k+1}} |(x^*, f(t_k, x_\varepsilon(t_k)) - f(s, x_\varepsilon(s))|ds$$

$$+ \int_{t_i}^{t} |(x^*, f(t_i, x_\varepsilon(t_i)) - f(s, x(s)))|ds$$

$$\le \|x^*\| M\mu + \delta' b < \lambda$$

which implies

$$\|x_\varepsilon(t) - x_0 - \int_0^t f(s, x_\varepsilon(s))ds\| < \lambda$$

and so we have verified condition (iii) in the definition of $S_\lambda$ .

Finally we will verify condition (ii). Let $t'$ , $t \in T$ , with $t \in [t_i, t_{i+1}]$ and $t' \in [t_j, t_{j+1}]$ , $i \le j$ . Again by the Hahn-Banach theorem we can find $x^* \in X^*$ with $\|x^*\| = 1$ such that

$$\|x_\varepsilon(t') - x_\varepsilon(t) - \int_t^{t'} f(s, x_\varepsilon(s))ds\| = |(x^*, x_\varepsilon(t') - x_\varepsilon(t) - \int_t^{t'} f(s, x_\varepsilon(s))ds)| .$$

Then we have:

$$|(x^*, x_\varepsilon(t') - x_\varepsilon(t) - \int_t^{t'} f(s, x_\varepsilon(s))ds)|$$

$$\le \int_t^{t_{i+1}} |(x^*, f(t_i, x_\varepsilon(t_i)) - f(s, x_\varepsilon(s))|ds$$

$$+ \sum_{k=1}^{j-i-1} \int_{t_{i+k}}^{t_{i+k+1}} |(x^*, f(t_{i+k}, x_\varepsilon(t_{i+k})) - f(s, x_\varepsilon(s)))|ds$$

$$+ \int_{t_j}^{t} |(x^*, f(t_j, x_\varepsilon(t_j)) - f(s, x_\varepsilon(s))|ds$$

$$< \lambda |t' - t| .$$

Therefore we conclude that $x\ (\cdot) \in S_\lambda$ as claimed.

https://doi.org/10.1017/S0004972700004007 Published online by Cambridge University Press

Now we will show that $\varepsilon \to x_\varepsilon(\cdot)$ is continuous from $[0,\varepsilon(\lambda)]$ into $C_{X_w}(T)$ . To see that we proceed as follows:

Let $t \in [t_i, t_{i+1}]$, $t_i = i\varepsilon$ and $t \in [t_j, t_{j+1}]$ , $t_j = j\delta$ . Also let $\delta \to \varepsilon$ . Then clearly $t_j \to t_i$ and so $x_\delta(t_j) \overset{s}{\to} x_\varepsilon(t_i)$ .

Take $x^* \in X^*\backslash\{0\}$ . We have:

$$\begin{aligned}
&|(x^*, x_\delta(t) - x_\varepsilon(t))| \\
&\quad = |(x^*, x_\delta(t_j) - x_\varepsilon(t_i))| + |(t-t_j)(x^*, f(t_j, x_\delta(t_j))) \\
&\qquad\qquad - (t-t_i)(x^*, f(t_i, x_\varepsilon(t_i)))| \\
&\quad = |(x^*, x_\delta(t_j) - x_\varepsilon(t_i))| + (t-t_j)(x^*, f(t_j, x\ (t_j))) \\
&\qquad\qquad - t_i(x^*, f(t_j, x_\delta(t_j))) + t_i(x^*, f(t_j, x_\delta(t_j))) \\
&\qquad\qquad - (t-t_i)(x^*, f(t_i, x_\varepsilon(t_i)))| \\
&\quad \le |(x^*, x_\delta(t_j) - x_\varepsilon(t_i))| + |(t_i-t_j)(x^*, f(t_j, x_\delta(t_j))| \\
&\qquad\qquad + (t-t_i)|(x^*, f(t_j, x_\delta(t_j)) - f(t_i, x_\varepsilon(t_i)))| \\
&\quad \le |(x^*, x_\delta(t_j) - x_\varepsilon(t_i))| + |t_i - t_j|\ \|x^*\|\ M + (t-t_i)|(x^*, f(t_j, x_\delta(t_j)) \\
&\qquad\qquad - f(t_i, x_\varepsilon(t_i)))| .
\end{aligned}$$

Recall that $f(\cdot,\cdot)$ is a weakly uniformly continuous vector field. So passing to the limit as $\delta \to \varepsilon$ we get that

$$|(x^*, x_\delta(t) - x_\varepsilon(t))| \to 0$$

uniformly in $t$ . So $(x^*, x_\delta(\cdot)) \xrightarrow{C_R} (x^*, x_\varepsilon(\cdot))$ as $\delta \to \varepsilon$ . But the topology of weak uniform convergence on $C_{X_w}(T)$ is determined by the basis

$$U_x(x_1^* \ldots x_m^*; \varepsilon') = \bigcap_{\ell=1}^{m} \{z(\cdot) \in C_{X_w}(T) : \sup_{t \in T} |(x_\ell^*, z(t) - x(t))| < \varepsilon$$

https://doi.org/10.1017/S0004972700004007 Published online by Cambridge University Press

where $x(\cdot) \in C_{X_w}(T)$ , $x_1^* \dots x_m^* \in X^*$ , $\varepsilon > 0$ , $m = 1,2,\dots$ .

Knowing that we can now say that $\varepsilon \to x_\varepsilon(\cdot)$ is continuous from $[0,\varepsilon(\lambda)]$ into $C_{X_w}(T)$ . Hence if $A = \{x_\varepsilon(\cdot)\}_{\varepsilon\in[0,\varepsilon(\lambda)]}$ , then this set is connected in $C_{X_w}(T)$ , because it is the image under the continuous map $\varepsilon \to x_\varepsilon(\cdot)$ of the connected set $[0,\varepsilon(\lambda)]$ .

Next let $p \in T$ and $y(\cdot) \in S_\lambda$ . We define the $(\varepsilon,p,y(\cdot))$-Euler polygon as follows. Fix $\varepsilon < \mu$ .

$$\begin{aligned} y_p(t) &= y(t) && \text{for } 0 \le t \le p \\ &= y(p) && \text{for } 0 \le t \le ([\tfrac{p}{\varepsilon}]+1)\varepsilon = t_p \\ &= y_p(t_k) + (t-t_k)f(t_k,y_p(t_k)) && \text{for } t \in [t_k,t_{k+1}] \end{aligned}$$

for $k = [\frac{p}{\varepsilon}]+1,[\frac{p}{\varepsilon}]+2,\dots,[\frac{b}{\varepsilon}] = n_\varepsilon$ , $t_{n_\varepsilon+1} = b$ . We let $r = [\frac{p}{\varepsilon}]+1$ .

We are going to show that $y_p(\cdot) \in S_\lambda$ . Since $y(\cdot) \in S_\lambda$ , we can easily see that $\dot{y}_p(\cdot)$ exists almost everywhere and

$$\|y(t) - x_0 - \int_0^t f(s,y(s))ds\| < \lambda$$

for all $t \in T$ . Fix $t \in T$ and as before pick $x^* \in X^*$ with $\|x^*\| = 1$ such that

$$|(x^*,y(t) - x_0 - \int_0^t f(s,y(s))ds)| = \|y(t) - x_0 - \int_0^t f(s,y(s))ds\| .$$

Let $\delta > 0$ and $U(x^*,\delta) = \{z\in X : |(x^*,z)| < \delta\} \in N_w(0)$ = filter of weak neighbourhoods of the origin in $X$ . Then due to weak uniform continuity of $f(\cdot,\cdot)$ we can find $V \in N_w(0)$ such that $|(x^*,f(t,x) - f(t,z))| < \delta$ for all $x - z \in V$ . Let $0 < \varepsilon < \mu$ be such that for $t,s \in T$, $|t-s| < \varepsilon$ $y_p(t) - y_p(s) \in V$ and for all $t \in T$

$$\|y(t) - x_0 - \int_0^t f(s,y(s))ds\| + M\varepsilon + \delta b < \lambda .$$

https://doi.org/10.1017/S0004972700004007 Published online by Cambridge University Press

Now we examine separately what happens when $t$ is in one of the intervals involved in the definition of $y_p(\cdot)$.

Firstly, for $0 \le t \le p$ :

$$\|y_p(t) - x_0 - \int_0^t f(s,y_p(s))ds\| = \|y(t) - x_0 - \int_0^t f(s,y(s))ds\| < \lambda .$$

Secondly, for $p \le t \le p + \varepsilon$ :

$$\|y_p(t) - x_0 - \int_0^t f(s,y_p(s))ds\|$$

$$= \|y(p) - x_0 - \int_0^p f(s,y(s))ds - \int_p^t f(s,y(s))ds\|$$

$$\le \|y(p) - x_0 - \int_0^p f(s,y(s))ds\| + \|\int_p^t f(s,y(s))ds\|$$

$$\le \|y(p) - x_0 - \int_0^p f(s,y(s))ds\| + M\varepsilon < \lambda .$$

Thirdly, for $p + \varepsilon \le t \le b$ :

Assume that $t \in [t_i,t_{i+1}]$ . We have

$$\|y_p(t) - x_0 - \int_0^t f(s,y_p(s))ds\|$$

$$= |(x^*,y_p(t) - x_0 - \int_0^t f(s,y_p(s))ds)|$$

$$= |(x^*,y(p) + \sum_{k=r}^{i-1} (t_{k+1}-t_k)f(t_k,y_p(t_k))$$

$$+ (t-t_i)f(t_i,y_p(t_i)) - x_0 - \int_0^t f(s,y_p(s))ds|$$

$$\le \|y(p) - x_0 - \int_0^p f(s,y_p(s))ds\| + \|\int_p^{t_r} f(s,y_p(s))ds\|$$

$$+ \sum_{k=r}^{i-1} \int_{t_k}^{t_{k+1}} |(x^*,f(t_k,y_p(t_k)) - f(s,y_p(s)))|$$

https://doi.org/10.1017/S0004972700004007 Published online by Cambridge University Press

$$+ \int \left|(x^*, f(t_i, y_p(t_i)) - f(s, y_p(s))\right| ds$$

$$\leq \left\| y(p) - x_0 - \int_0^p f(s, y_p(s))ds \right\| + M\varepsilon + \delta b < \lambda \ .$$

Thus we can finally say that $y_p(\cdot) \in S_\lambda$ .

Next we will show that $p \to y_p(\cdot)$ is continuous into $C_{X_w}(T)$ . So let $q \to p$ in $T$ . We will prove that $y_q(\cdot) \to y_p(\cdot)$ in $C_{X_w}(T)$ .

For this purpose let $0 \leq p \leq q \leq b$ . For $t \leq p$ , $\|y_q(t) - y_p(t)\| = \|y(t) - y(t)\| = 0$ .

If $p < t$ and since $q \to p$ , we can take $q$ such that $p \leq q < t$ and $[\frac{p}{\varepsilon}] = [\frac{q}{\varepsilon}]$ . Let $t \in [t_i, t_{i+1}]$ and $x^* \in X^*\backslash\{0\}$ . We have:

$$\left|(x^*, y_q(t) - y_p(t))\right|$$

$$= \Big|(x^*, y(q) + \sum_{k=r}^{i-1} (t_{k+1}-t_k) f(t_k, y_q(t_k)) + (t-t_i) f(t_i, y_q(t_i)))$$

$$- (x^*, y(p) + \sum_{k=r}^{i-1} (t_{k+1}-t_k) f(t_k, y_p(t_k)) + (t-t_i) f(t_i, y_p(t_i)))\Big|$$

$$\left|(x^*, y(q) - y(p))\right| + \sum_{k=r}^{i-1} (t_{k+1}-t_k) \left|(x^*, f(t_k, y_q(t_k)) - f(t_k, y_p(t_k)))\right|$$

$$+ (t-t_i) + \left|(x^*, f(t_i, y_q(t_i)) - f(t_i, y_p(t_i)))\right| \ .$$

Passing to the limit as $q \to p$ and exploiting the fact that $f(\cdot,\cdot)$ is weakly uniformly continuous we get that

$$\left|(x^*, y_q(t) - y_p(t))\right| \to 0$$

uniformly in $t \in T$ . Hence we have shown that $p \to y_p(\cdot)$ is continuous from $T$ into $C_{X_w}(T)$ . Set $B_y = \{y_p(\cdot)\}_{p \in T}$ . Then $B_y$ is connected in $C_{X_w}(T)$ .

https://doi.org/10.1017/S0004972700004007 Published online by Cambridge University Press

Now note that $y_0(\cdot)$ is an $\varepsilon$-Euler polygon for (*) that is, $y_0(\cdot) \in A$. Therefore $A \cap B_y \neq \phi$. So $A \cup B_y$ is connected in $C_{X_w}(T)$ and since this is true for all $y(\cdot) \in S_\lambda$ we get that $R_\lambda = \bigcup_{y(\cdot)\in S} [A \cup B_y]$ is connected in $C_{X_w}(T)$ (see Dugundji [11].) Clearly $S_\lambda \subseteq R_\lambda$. On the other hand by what we proved earlier $A \subseteq S_\lambda$ and for all $y(\cdot) \in S_\lambda$, $B_y \subseteq S_\lambda$. Hence we deduce that $R_\lambda \subseteq S_\lambda$. Therefore we can finally say that $R_\lambda = S_\lambda$ and so $S_\lambda$ is connected in $C_{X_w}(T)$.

Next let $S_\lambda(t) = \{y(t) : y(\cdot) \in S_\lambda\}$. Set $q(t) = \beta(S_\lambda(t))$. Using the fact that $\beta(S_\lambda(t)) \leq \operatorname{diam}(S_\lambda(t))$ and property (i) in the definition of $S_\lambda$ we get that

$$|q(t') - q(t)| \leq M|t'-t|$$

for $t',t \in T$. This implies that $q(\cdot)$ is absolutely continuous and so differentiable at all points $t \in T\backslash N_1$, $\lambda(N_1) = 0$. Let $N_2$ be the exceptional set of measure zero originating from hypothesis 3 of Theorems 2.1 and 2.2. Set $N = N_1 \cup N_2$ and fix $t \in T\backslash N$, $\varepsilon > 0$. We know that we can find $\delta > 0$ such that

$$|w(t,q(t)) - w(t,z)| < \varepsilon \quad \text{where} \quad |q(t) - z| < \delta .$$

Also choose $\gamma > 0$ such that $M\gamma < \delta$ and $t + \gamma \leq b$.

From hypothesis 3 we know that we can find $r > 0$ such that

$$\beta(f(T_{t,r} \times L)) \leq w(t,\beta(L)) + \varepsilon$$

where $L = \{x(s) : x(\cdot) \in S_\lambda, t \leq s \leq t+\gamma\}$, $T_{t,r} = [t,t+r]$, $0 < r < \gamma$.

Observe that by definition $S_\lambda$ is an equicontinuous, bounded family. So we can apply Lemma 2.2 of [18] and get that

$$\beta(L) = \sup\{\beta(L(s)) : s \in [t,t+\gamma]\} = q(\hat{t})$$

for some $\hat{t} \in [t,t+\gamma]$. Then we have:

$$0 \leq \beta(L) - q(t) = q(\hat{t}) - q(t) \leq M|\hat{t}-t| < \delta .$$

This, then, from the choice of $\delta > 0$, implies that

$$|w(t,\beta(L)) - w(t,p(t))| < \varepsilon .$$

Next let $0 < r' \leq r$. Then for any $x(\cdot) \in S_\lambda$ we have that

$$x(t+r') = x(t) + \int_t^{t+r'} (\dot{x}(s) - f(s,x(s)))ds + \int_t^{t+r'} f(s,x(s))ds$$

and so

$$S_\lambda(t+r') \subseteq S_\lambda(t) + Q_\lambda(t,r') + \int_t^{t+r'} f(s,S_\lambda(s))ds$$

where

$$Q_\lambda(t,r') = \{ \int_t^{t+r'} (\dot{x}(s) - f(s,x(s)))ds : x(\cdot) \in S_\lambda\} .$$

Note that $\sup_{\|x\| \in Q_\lambda(t,r')} \|x\| \leq \lambda r'$. Thus $\beta(Q_\lambda(t,r') \leq \text{diam}\, Q_\lambda(t,r') \leq 2\lambda r'$

and this then gives us that

$$q(t+r') \leq q(t) + 2\lambda r' + \beta\left(\int_t^{t+r'} f(s,S_\lambda(s))ds\right) .$$

But recall that since $f(\cdot,\cdot)$ is weakly continuous, we have

$$\int_t^{t+r'} f(s,S_\lambda(s))ds \subseteq r' \,\overline{\text{conv}}\, f(T_{t,r} \times L)$$

and so

$$\beta\left(\int_t^{t+r'} f(s,S_\lambda(s))ds\right) \leq \beta(r' \,\overline{\text{conv}}\, f(T_{t,r} \times L))$$

$$= r'\beta(\overline{\text{conv}}\, f(T_{t,r} \times L)) = r'\beta(f(T_{t,r} \times L)) .$$

Therefore we get that

$$q(t+r') \leq q(t) + 2\lambda r' + r'\beta(f(T_{t,r} \times L))$$

$$\leq q(t) + 2\lambda r' + r'w(t,\beta(L)) + r'\varepsilon$$

$$\leq q(t) + 2\lambda r' + 2r'\varepsilon + r'w(t,q(t)) .$$

Thus

$$\frac{q(t+r') - q(t)}{r'} \leq 2(\lambda+\varepsilon) + w(t,q(t))$$

and so

$$\dot{q}(t) \leq 2(\lambda+\varepsilon) + w(t,q(t)) .$$

Letting $\varepsilon \downarrow 0$ we get that

$$\dot{q}(t) \leq 2\lambda + w(t,q(t)) .$$

Consider $\dot{z}(t) = w(t,z(t)) + 2\lambda$ and let $z_\lambda(\cdot)$ be its maximal solution. Then from Theorem 1.4.1 of Lakshmikantham-Leela [16] we deduce that $q(t) \leq z_\lambda(t)$ for all $t \in T$. Note that $z_\lambda(\cdot) \xrightarrow{C_{\mathbb{R}}} 0$ as $\lambda \downarrow 0$. Since by Lemma 2.2 of [18] we know that $\beta(S_\lambda) = \sup_{t\in T} \beta(S_\lambda(t)) \leq \|z_\lambda\|_\infty$, we get that $\lim_{\lambda\to 0^+} \beta(S_\lambda) = \lim\|z_\lambda\|_\infty = 0$. But $S_0 \subseteq S_\lambda$ for all $\lambda > 0$. So $\beta(S_0) \leq \beta(S_\lambda) \Rightarrow \beta(S_0) = \lim_{\lambda\to 0^+} \beta(S_\lambda) = 0 \Rightarrow S_0$ is compact in $C_{X_w}(T)$.

It remains to show that $S_0$ is connected in $C_{X_w}(T)$. To see this note that

$$S_0 = \bigcap_{n\geq 1} S_{1/n}$$

and we already saw in an earlier part of the proof that each $S_{1/n}$ is connected in $C_{X_w}(T)$. Suppose $S_0$ was not connected in $C_{X_w}(T)$. Then we can find nonempty, disjoint sets $A$, $B$ closed in $C_{X_w}(T)$ such that $S_0 = A \cup B$. Let $U$ be open in $C_{X_w}(T)$ such that $A \subseteq U$ and $\bar{U} \cap B = \phi$. Because each $S_{1/n}$ is connected we have that

$$S_{1/n} \cap U \neq \phi \quad \text{and} \quad S_{1/n}\backslash U \neq \phi$$

which imply that

$$S_{1/n} \cap \partial U \neq \phi ,$$

$\partial U$ being the boundary of $U$. Let $D_{1/n} = S_{1/n} \cap \partial U$. Then

https://doi.org/10.1017/S0004972700004007 Published online by Cambridge University Press

$$\beta(D_{1/n}) \le \beta(S_{1/n}) \to 0 \ .$$

Invoking Theorem 3 of DeBlasi [7] we get that

$$\bigcap_{n\ge 1} D_{1/n} \ne \phi$$

and so

$$\left(\bigcap_{n\ge 1} S_{1/n}\right) \cap \partial U \ne \phi \ ,$$

a contradiction to the choice of $U$. Therefore $S_0$ is also connected in $C_{X_w}(T)$ and this completes the proof.

Remarks.

(1) The above proof gives us also the existence of a weak solution for (*). Just note that $S_0 = \bigcap_{n\ge 1} S_{1/n}$ and $\lim_{n\to\infty} \beta(S_{1/n}) = 0$. Then apply Theorem 3 of DeBlasi [7]. So we do not need to call upon Theorem 2.1 to guarantee existence of solutions of (*).

(2) If $X$ is reflexive, then hypothesis (3) in our theorem is automatically satisfied since every bounded set is relatively weakly compact. So we recover as a corollary to our theorem the result of Szufla [20].

(3) If instead of hypothesis (3) we have the following stronger one:

"$\beta(f(T\times A)) \le k\beta(A)$ for $A \subseteq X$ bounded and $0 \le b, k \le 1$"

then we get as a corollary Theorem 3 of Kubiaczyk [15], which was presented there without a proof.

## References


[1] J. Banas and K. Goebel, *Measures of noncompactness in Banach spaces* (Marcel Dekker Inc., New York, 1980).

[2] M. Boudourides, "An existence theorem for ordinary differential equations in Banach spaces", *Bull. Austral. Math. Soc.* 22 (1980), 457-463.

[3] N. Bourbaki, *Integration* (Chapitre IV, Hermann, Paris, 1968).

[4] F. Browder, "Nonlinear equations of evolution", *Ann. of Math.* 80 (1964), 485-523.

[5] A. Cellina, "On the nonexistence of solutions of ordinary differential equations in Banach spaces", *Bull. Amer. Math Soc.* 78 (1973), 1069-1072.

[6] E. Cramer, V. Lakshmikantham and A. Mitchell, "On the existence of weak solutions of differential equations in nonreflexive Banach spaces", *Nonlinear Anal.* 2 (1978), 169-177.

[7] F. DeBlasi, "On a property of the unit sphere in a Banach space", *Bull. Math. Soc. Sci. Math. R.S. Roumanie* 21 (1977), 259-262.

[8] K. Deimling, *Ordinary differential equations in Banach spaces* (Lecture Notes in Mathematics, 596. Springer, Berlin, 1977).

[9] J. Diestel and J. Uhl, *Vector measures* (Math. Surveys. 15 Amer. Math. Soc. Providence, 1977).

[10] J. Dieudonné, "Deux examples singuliers d'equations differentielles", *Acta Sci. Math. (Szeged)* 12 (1950), 38-40.

[11] J. Dugundji, *Topology* (Allyn and Bacon Inc., Boston, 1966).

[12] G.D. Faulkner, "On the nonexistence of weak solutions to abstract differential equations in nonreflexive spaces", *Nonlinear Anal.* 2 (1978), 505-508.

[13] A.N. Godunov, "Peano's theorem in Banach spaces", *Functional Anal. Appl.* 9 (1975), 53-55.

[14] W.J. Knight, "Solutions of differential equations in Banach spaces", *Duke Math. J.* 41 (1974), 437-442.

[15] I. Kubiaczyk, "Kneser type tyeorems for ordinary differential equations in Banach spaces", *J. Differential Equations* 45 (1982), 139-146.

[16] V. Lakshmikantham and S. Leela, *Differential and integral inequalities*, 1. (Academic Press, New York, 1969).

[17] R. Martin, *Nonlinear operators and differential equations in Banach spaces* (Wiley, New York 1976).

[18] N.S. Papageorgiou, "Weak solutions of differential equations in Banach spaces", *Bull. Austral. Math. Soc.* 33 (1986), 407-418.

[19] A. Szep, "Existence theorem for weak solutions of ordinary differential equations in reflexive Banach spaces", *Studia Sc. Math. Hungar*, 6 (1971), 197-202.

[20] S. Szufla, "Kneser's theorem for weak solutions of ordinary differential equations in reflexive Banach spaces", *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* 26 (1978), 407-413.

[21] J. Yorke, "A continuous differential equation in Hilbert space without existence", *Funkcial. Ekvac.* 13 (1970), 19-22.

Department of Mathematics,
University of Illinois,
1409 W. Green St.,
Urbana, Illinois 61801,
U.S.A.