Canad. Math. Bull. Vol. **62** (2), 2019 pp. 393–404
http://dx.doi.org/10.4153/S0008439518000061

Vanishing Fourier Transforms and Generalized Differences in $L^2(\mathbb{R})$

Rodney Nillsen

Abstract. Let $\alpha, \beta \in \mathbb{R}$ and $s \in \mathbb{N}$ be given. Let δ_x denote the Dirac measure at $x \in \mathbb{R}$, and let $*$ denote convolution. If μ is a measure, $\mu^\star$ is the measure that assigns to each Borel set A the value $\overline{\mu(-A)}$. If $u \in \mathbb{R}$, we put $\mu_{\alpha,\beta,u} = e^{iu(\alpha-\beta)/2}\delta_0 - e^{iu(\alpha+\beta)/2}\delta_u$. Then we call a function $g \in L^2(\mathbb{R})$ a *generalized* (α,β)*-difference of order* $2s$ if for some $u \in \mathbb{R}$ and $h \in L^2(\mathbb{R})$ we have $g = [\mu_{\alpha,\beta,u} + \mu^\star_{\alpha,\beta,u}]^s * h$. We denote by $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$ the vector space of all functions f in $L^2(\mathbb{R})$ such that f is a finite sum of generalized (α,β)-differences of order $2s$. It is shown that every function in $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$ is a sum of $4s+1$ generalized (α,β)-differences of order $2s$. Letting $\widehat{f}$ denote the Fourier transform of a function $f \in L^2(\mathbb{R})$, it is shown that $f \in \mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$ if and only if $\widehat{f}$ "vanishes" near α and β at a rate comparable with $(x-\alpha)^{2s}(x-\beta)^{2s}$. In fact, $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$ is a Hilbert space where the inner product of functions f and g is $\int_{-\infty}^{\infty}(1+(x-\alpha)^{-2s}(x-\beta)^{-2s})\widehat{f}(x)\overline{\widehat{g}(x)}\,dx$. Letting D denote differentiation, and letting I denote the identity operator, the operator $(D^2 - i(\alpha+\beta)D - \alpha\beta I)^s$ is bounded with multiplier $(-1)^s(x-\alpha)^s(x-\beta)^s$, and the Sobolev subspace of $L^2(\mathbb{R})$ of order $2s$ can be given a norm equivalent to the usual one so that $(D^2 - i(\alpha+\beta)D - \alpha\beta I)^s$ becomes an isometry onto the Hilbert space $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$. So a space $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$ may be regarded as a type of Sobolev space having a negative index.

1 Introduction

Let $\mathbb{R}$ denote the set of real numbers, let $\mathbb{T}$ denote the set of complex numbers of modulus 1, and let G denote either $\mathbb{R}$ or $\mathbb{T}$. Note that in some contexts $\mathbb{T}$ may be identified with the interval $[0, 2\pi)$ under the mapping $t \mapsto e^{it}$ (some comments on this are in [9, p. 1034]). Then G is a group and its identity element we denote by e, so that $e = 0$ when $G = \mathbb{R}$ and $e = 1$ when $G = \mathbb{T}$. Let $\mathbb{N}$ denote the set of natural numbers, $\mathbb{Z}$ the set of integers, and let $s \in \mathbb{N}$. The Fourier transform of $f \in L^2(G)$ is denoted by $\widehat{f}$, and is given by $\widehat{f}(n) = (2\pi)^{-1}\int_0^{2\pi} f(e^{it})e^{-int}\,dt$ for $n \in \mathbb{Z}$ (in the case of $\mathbb{T}$), and by the extension to all of $L^2(\mathbb{R})$ of the transform given by $\widehat{f}(x) = \int_{-\infty}^{\infty} e^{-ixu} f(u)\,du$ for $x \in \mathbb{R}$ (in the case of $\mathbb{R}$). Let $M(G)$ denote the family of bounded Borel measures on G. If $x \in G$ let δ_x denote the Dirac measure at x, and let $*$ denote convolution in $M(G)$.

We call a function $f \in L^2(G)$ a *difference of order* s if there is a function $g \in L^2(G)$ and $u \in G$ such that $f = (\delta_e - \delta_u)^s * g$. The functions in $L^2(G)$ that are a sum of a finite number of differences of order s we denote by $\mathcal{D}_s(G)$. Note that $\mathcal{D}_s(G)$ is a vector subspace of $L^2(G)$. In the case of $\mathbb{T}$ it was shown by Meisters and Schmidt [5]

Received by the editors June 19, 2018; revised September 14, 2018.
Published online on Cambridge Core March 15, 2019.
AMS subject classification: **42A38**, 42A45.
Keywords: Fourier transform, generalized difference, Hilbert space, multiplier.

that

$$\mathcal{D}_1(\mathbb{T}) = \{f : f \in L^2(\mathbb{T}) \text{ and } \widehat{f}(0) = 0\},$$

and that every function in $\mathcal{D}_1(\mathbb{T})$ is a sum of 3 differences of order 1. It was shown in [6] that, for all $s \in \mathbb{N}$,

$$\mathcal{D}_s(\mathbb{T}) = \mathcal{D}_1(\mathbb{T}) = \{f : f \in L^2(\mathbb{T}) \text{ and } \widehat{f}(0) = 0\}, \tag{1.1}$$

and that every function in $\mathcal{D}_s(\mathbb{T})$ is a sum of $2s + 1$ differences of order s. It was also shown in [6] that

$$\mathcal{D}_s(\mathbb{R}) = \left\{ f : f \in L^2(\mathbb{R}) \text{ and } \int_{-\infty}^{\infty} \frac{|\widehat{f}(x)|^2}{|x|^{2s}}\, dx < \infty \right\}, \tag{1.2}$$

and again, that every function in $\mathcal{D}_s(\mathbb{R})$ is a sum of $2s+1$ differences of order s. Further results related to the work of Meisters and Schmidt in [5] may be found in [1–4, 7].

The Sobolev space of order s in $L^2(G)$ is the space of all functions $f \in L^2(G)$ such that $D^s(f) \in L^2(G)$, where D denotes differentiation in the sense of Schwartz distributions. Then D^s is a *multiplier operator* on $W^s(\mathbb{T})$ with *multiplier* $(in)^s$, in the sense that $D^s(f)\widehat{\ }(n) = (in)^s \widehat{f}(n)$ for all $f \in W^s(\mathbb{T})$ and $n \in \mathbb{Z}$. Also, D^s is a multiplier operator on $W^s(\mathbb{R})$ with multiplier $(ix)^s$, in the sense that $D^s(f)\widehat{\ }(x) = (ix)^s \widehat{f}(x)$, for almost all $x \in \mathbb{R}$ for $f \in W^s(\mathbb{R})$. Note that $W^s(\mathbb{T})$ is a Hilbert space where the inner product of $f, g \in W^s(\mathbb{T})$ is $\sum_{n=-\infty}^{\infty}(1 + |n|^{2s})\widehat{f}(n)\overline{\widehat{g}(n)}$. Note also that $W^s(\mathbb{R})$ is a Hilbert space for which the usual inner product is given by

$$\langle f, g\rangle_{W^s} = \int_{-\infty}^{\infty} (1 + |x|^{2s})\widehat{f}(x)\overline{\widehat{g}(x)}\, dx, \quad \text{for } f, g \in W^s(\mathbb{R}). \tag{1.3}$$

Using these observations, together with Plancherel's Theorem, it is easy to verify that

$$D^s\big(W^s(\mathbb{T})\big) = \{f : f \in L^2(\mathbb{T}) \text{ and } \widehat{f}(0) = 0\}, \quad \text{and that} \tag{1.4}$$

$$D^s\big(W^s(\mathbb{R})\big) = \left\{ f : f \in L^2(\mathbb{R}) \text{ and } \int_{-\infty}^{\infty} \frac{|\widehat{f}(x)|^2}{|x|^{2s}}\, dx < \infty \right\}. \tag{1.5}$$

In view of (1.4) and (1.5), (1.1) together with (1.2) can be regarded as describing the ranges of D^s upon $W^s(\mathbb{T})$ and $W^s(\mathbb{R})$ as spaces consisting of finite sums of differences of order s. Corresponding results have been obtained in [8] for operators $\big(D^2 - i(\alpha + \beta)D - \alpha\beta I\big)^s$ acting on $W^{2s}(\mathbb{T})$, where $\alpha, \beta \in \mathbb{Z}$ and I denotes the identity operator. In this paper, the main aim is to derive corresponding results for the operator $\big(D^2 - i(\alpha+\beta)D - \alpha\beta I\big)^s$, where $\alpha, \beta \in \mathbb{R}$, for the non-compact case of $\mathbb{R}$ in place of the compact group $\mathbb{T}$. Note that, in general, the range of a multiplier operator depends upon the behaviour of Fourier transforms at or around the zeros of the multiplier of the operator, as in (1.4) and (1.5). Note also that on $\mathbb{R}$, $\big(D^2 - i(\alpha + \beta)D - \alpha\beta I\big)^s$ is a multiplier operator whose multiplier is $(-1)^s(x - \alpha)^s(x - \beta)^s$, which has zeros at α and β.

Given $\alpha, \beta \in \mathbb{R}$ and $s \in \mathbb{N}$, a *generalized* (α, β)*-difference of order* $2s$ is a function $f \in L^2(\mathbb{R})$ such that for some $g \in L^2(\mathbb{R})$ and $u \in \mathbb{R}$ we have

$$f = \Big[\big(e^{iu(\frac{\alpha-\beta}{2})} + e^{-iu(\frac{\alpha-\beta}{2})}\big)\delta_0 - \big(e^{iu(\frac{\alpha+\beta}{2})}\delta_u + e^{-iu(\frac{\alpha+\beta}{2})}\delta_{-u}\big)\Big]^s * g. \tag{1.6}$$

https://doi.org/10.4153/S0008439518000061 Published online by Cambridge University Press

It may be called also an (α,β)-*difference of order* $2s$, or simply a *generalized difference*. The vector space of functions in $L^2(\mathbb{R})$ that can be expressed as some finite sum of (α,β)-differences of order $2s$ is denoted by $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$. Thus, $f\in\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$ if and only if there are $m\in\mathbb{N}$, $u_1,u_2,\dots,u_m\in\mathbb{R}$ and $f_1,f_2,\dots,f_m\in L^2(\mathbb{R})$ such that

$$f=\sum_{j=1}^{m}\big[\big(e^{iu_j(\frac{\alpha-\beta}{2})}+e^{-iu_j(\frac{\alpha-\beta}{2})}\big)\delta_0-\big(e^{iu_j(\frac{\alpha+\beta}{2})}\delta_{u_j}+e^{-iu_j(\frac{\alpha+\beta}{2})}\delta_{-u_j}\big)\big]^s*f_j.$$

We prove that if $f\in L^2(\mathbb{R})$, $f\in\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$ if and only if $\widehat{f}$ is "vanishing" near α and β in the sense that

$$\int_{-\infty}^{\infty}(x-\alpha)^{-2s}(x-\beta)^{-2s}|\widehat{f}(x)|^2\,dx<\infty,$$

in which case f is a sum of $4s+1$ (α,β)-differences of order $2s$. It follows that $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$ is a Hilbert space where the inner product of $f,g\in\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$ is

$$\int_{-\infty}^{\infty}\big(1+(x-\alpha)^{-2s}(x-\beta)^{-2s}\big)\,\widehat{f}(x)\overline{\widehat{g}(x)}\,dx.$$

In fact, it follows straightforwardly from the above that the usual norm on $W^{2s}(\mathbb{R})$, as derived from (1.3), can be replaced by a natural equivalent norm in which the operator $\big(D^2-i(\alpha+\beta)D-\alpha\beta I\big)^s(\mathbb{R})$ is an isometry from $W^{2s}(\mathbb{R})$ onto $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$. Consequently, the space $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$ may be thought of a "Sobolev-type" space with a negative index, consisting of sums of generalized differences associated with the operator.

2 Preliminaries and Proof of the Main Result

We need the following result, which characterises those functions that are a sum of convolutions of other functions by given measures.

Theorem 2.1 *Let $f\in L^2(\mathbb{R})$ and let $\mu_1,\mu_2,\dots,\mu_r\in M(\mathbb{R})$. Then the following conditions* (i) *and* (ii) *are equivalent.*

(i) *There are $f_1,f_2,\dots,f_r\in L^2(\mathbb{R})$ such that $f=\sum_{j=1}^{r}\mu_j*f_j$.*

(ii)

$$\int_{-\infty}^{\infty}\frac{|\widehat{f}(x)|^2}{\sum_{j=1}^{r}|\widehat{\mu}_j(x)|^2}\,dx<\infty.$$

Proof This is essentially proved in [5, pp. 411–412], but see also [6, pp. 77–88] and [7, p. 23]. ∎

Lemma 2.2 *Let J, K be two closed intervals of positive length such that $J\cap K$ also has positive length. Let $\xi\in J$ and $\eta\in K$ be given. If $\xi\in J\cap K$ put $\tilde{\xi}=\xi$, and if $\xi\notin J\cap K$, let $\tilde{\xi}$ be the end point of $J\cap K$ that is closest to ξ. If $\eta\in J\cap K$ put $\tilde{\eta}=\eta$, and if $\eta\notin J\cap K$ let $\tilde{\eta}$ be the endpoint of $J\cap K$ that is closest to η. Then*

$$|x-\xi|\cdot|x-\eta|\ge|x-\tilde{\xi}|\cdot|x-\tilde{\eta}|\quad\textit{for all } x\in J\cap K.$$

https://doi.org/10.4153/S0008439518000061 Published online by Cambridge University Press

Proof The result is immediate from the observation that for all $x \in J \cap K$, $|x - \xi| \geq |x - \tilde{\xi}|$ and $|x - \eta| \geq |x - \tilde{\eta}|$. ∎

The main aim in this paper is to prove the following. In the proof we will A^c denote the complement of the set A.

Theorem 2.3 *Let $s \in \mathbb{N}$ and let $\alpha, \beta \in \mathbb{R}$. Let $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$ be the vector space of functions in $L^2(\mathbb{R})$ that can be expressed as some finite sum of generalized (α,β)-differences of order $2s$. Then the following conditions* (i)–(iii) *are equivalent for a function $f \in L^2(\mathbb{R})$.*

(i)

$$\int_{-\infty}^{\infty} \frac{|\widehat{f}(x)|^2}{(x-\alpha)^{2s}(x-\beta)^{2s}}\,dx < \infty.$$

(ii) $f \in \mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$.

(iii) *There are $u_1, u_2, \dots, u_{4s+1} \in \mathbb{R}$ and $f_1, f_2, \dots, f_{4s+1} \in L^2(\mathbb{R})$ such that*

$$f = \sum_{j=1}^{4s+1} \Big[\big(e^{iu_j(\frac{\alpha-\beta}{2})} + e^{-iu_j(\frac{\alpha-\beta}{2})} \big)\delta_0 - \big(e^{iu_j(\frac{\alpha+\beta}{2})}\delta_{u_j} + e^{-iu_j(\frac{\alpha+\beta}{2})}\delta_{-u_j} \big) \Big]^s * f_j. \tag{2.1}$$

Furthermore, the following statements (iv), (v), *and* (vi) *hold.*

(iv) *When the conditions* (i)–(iii) *hold for a given function $f \in L^2(\mathbb{R})$, for almost all $(u_1, u_2, \dots, u_{4s+1}) \in \mathbb{R}^{4s+1}$, there are $f_1, f_2, \dots, f_{4s+1} \in L^2(\mathbb{R})$ such that* (2.1) *holds.*

(v) *The vector space $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$ is a Hilbert space with the inner product $\langle \cdot,\cdot \rangle_{\alpha,\beta,s}$ given by*

$$\langle f, g \rangle_{\alpha,\beta,s} = \int_{-\infty}^{\infty} \Big(1 + \frac{1}{(x-\alpha)^{2s}(x-\beta)^{2s}} \Big) \widehat{f}(x)\overline{\widehat{g}(x)}\,dx, \quad \text{for } f, g \in \mathcal{D}_{\alpha,\beta,s}(\mathbb{R}).$$

(vi) *For $f, g \in W^{2s}(\mathbb{R})$, put*

$$\langle f, g \rangle_{W^{2s},\alpha,\beta} = \int_{-\infty}^{\infty} \big(1 + (x-\alpha)^{2s}(x-\beta)^{2s} \big) \widehat{f}(x)\overline{\widehat{g}(x)}\,dx.$$

Then $\langle \cdot,\cdot \rangle_{W^{2s},\alpha,\beta}$ is an inner product on $W^{2s}(\mathbb{R})$ that is equivalent to the usual one on $W^{2s}(\mathbb{R})$ as given in (1.3). *The operator $\big(D^2 - i(\alpha+\beta)D - \alpha\beta I\big)^s$ has the multiplier $(-1)^s(x-\alpha)^s(x-\beta)^s$, and it is an isometry that maps $W^{2s}(\mathbb{R})$ with the inner product $\langle \cdot,\cdot \rangle_{W^{2s},\alpha,\beta}$ onto $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$.*

Proof If (iii) holds, then (ii) holds, by definition.

Let (ii) hold. If $u \in \mathbb{R}$, define $\lambda_u \in M(\mathbb{R})$ by

$$\lambda_u = \frac{1}{2}\big[e^{iu(\frac{\alpha-\beta}{2})} + e^{-iu(\frac{\alpha-\beta}{2})} \big]\delta_0 - \frac{1}{2}\big[e^{iu(\frac{\alpha+\beta}{2})}\delta_u + e^{-iu(\frac{\alpha+\beta}{2})}\delta_{-u} \big]. \tag{2.2}$$

The Fourier transform $\widehat{\lambda}_u$ of λ_u is given for $x \in \mathbb{R}$ by

$$\widehat{\lambda}_u(x) = 2\sin\Big(\frac{u(x-\alpha)}{2} \Big) \sin\Big(\frac{u(x-\beta)}{2} \Big). \tag{2.3}$$

So if $u \in \mathbb{R}$ and $f, g \in L^2(\mathbb{R})$ are such that $f = \lambda_u^s * g$, we have

$$\int_{-\infty}^{\infty} \frac{|\widehat{f}(x)|^2}{(x-\alpha)^{2s}(x-\beta)^{2s}}\,dx = 2^s \int_{-\infty}^{\infty} \frac{\sin^{2s}\big(u(x-\alpha)/2\big)\,\sin^{2s}\big(u(x-\beta)/2\big)}{(x-\alpha)^{2s}(x-\beta)^{2s}} |\widehat{g}(x)|^2\,dx < \infty.$$

Using (2.2), we deduce that (ii) implies (i).

Now we assume that (i) holds, and we will prove that (iii) holds. Let $x \in \mathbb{R}$ be given but with $x \notin \{\alpha, \beta\}$. Note that it may happen that $\alpha = \beta$. For each $k \in \mathbb{Z}$, put

$$a_k = \frac{k\pi}{|x-\alpha|}, \quad b_k = \frac{k\pi}{|x-\beta|}, \quad a'_k = \frac{(k-1/2)\pi}{|x-\alpha|}, \quad \text{and} \quad b'_k = \frac{(k-1/2)\pi}{|x-\beta|}. \tag{2.4}$$

Then put, again for each $k \in \mathbb{Z}$,

$$A_k = [a'_k, a'_{k+1}] \quad \text{and} \quad B_k = [b'_k, b'_{k+1}]. \tag{2.5}$$

Note that a_k is the mid-point of A_k and b_k is the mid-point of B_k. The points a_k are the zeros of $u \mapsto \sin\big(u(x-\alpha)\big)$, while the b_k are the zeros of $u \mapsto \sin\big(u(x-\beta)\big)$. Using (2.4) and (2.5), we see that for each $k \in \mathbb{Z}$,

$$\lambda(A_k) = \frac{\pi}{|x-\alpha|} \quad \text{and} \quad \lambda(B_k) = \frac{\pi}{|x-\beta|}. \tag{2.6}$$

We will use the notation that $d_{\mathbb{Z}}(w)$ denotes the distance from $w \in \mathbb{R}$ to the nearest integer. Note that $d_{\mathbb{Z}}(w) = |w|$ if and only if $-1/2 \le w \le 1/2$. Note also that $|\sin(\pi w)| \ge 2d_{\mathbb{Z}}(w)$ for all $w \in \mathbb{R}$ (for example see [7, p. 89] or [10, p. 233]).

Now

$$u \in A_j \Longrightarrow \frac{(j-1/2)\pi}{|x-\alpha|} \le u \le \frac{(j+1/2)\pi}{|x-\alpha|} \Longrightarrow -1/2 \le |x-\alpha| \Big| \frac{u}{\pi} - \frac{j}{|x-\alpha|} \Big| \le 1/2.$$

So for $u \in A_j$,

$$\begin{aligned} \big|\sin\big(u(x-\alpha)\big)\big| &= \Big|\sin\Big(\pi|x-\alpha| \Big| \frac{u}{\pi} - \frac{j}{|x-\alpha|} \Big|\Big)\Big| \\ &\ge 2d_{\mathbb{Z}}\Big(|x-\alpha| \Big| \frac{u}{\pi} - \frac{j}{|x-\alpha|} \Big|\Big) \\ &= 2|x-\alpha| \Big| \frac{u}{\pi} - \frac{j}{|x-\alpha|} \Big| \\ &= \frac{2}{\pi}|x-\alpha| \Big| u - \frac{j\pi}{|x-\alpha|} \Big|. \end{aligned} \tag{2.7}$$

Similarly, for $u \in B_k$,

$$\big|\sin\big(u(x-\beta)\big)\big| \ge \frac{2}{\pi}|x-\beta| \Big| u - \frac{k\pi}{|x-\beta|} \Big|. \tag{2.8}$$

We see from (2.7) and (2.8) that for all $u \in A_j \cap B_k$ we have

$$\big|\sin\big(u(x-\alpha)\big)\,\sin\big(u(x-\beta)\big)\big| \ge \frac{4}{\pi^2}|(x-\alpha)(x-\beta)|\left|u-\frac{j\pi}{|x-\alpha|}\right|\cdot\left|u-\frac{k\pi}{|x-\beta|}\right|.$$

That is, for $u \in A_j \cap B_k$ we have

$$\big|\sin\big(u(x-\alpha)\big)\,\sin\big(u(x-\beta)\big)\big| \ge \frac{4}{\pi^2}|(x-\alpha)(x-\beta)|\cdot|u-a_j|\cdot|u-b_k|, \tag{2.9}$$

where a_j and b_k are the points as given in (2.4).

Recall that $x \notin \{\alpha,\beta\}$ has been given. Let also $c>0$ be given, and let the intervals A_j such that $\lambda(A_j \cap [-c,c]) > 0$ be $A_{m_1},\dots,A_{m_1+r-1}$, and let the intervals B_k such that $\lambda(B_k \cap [-c,c]) > 0$ be $B_{m_2},\dots,B_{m_2+s-1}$.

Then put

$$\mathcal{P}_1 = \{A_{m_1}, A_{m_1+1},\dots,A_{m_1+r-1}\}, \quad \mathcal{P}_2 = \{B_{m_2}, B_{m_2+1},\dots,B_{m_2+s-1}\}. \tag{2.10}$$

Note that in (2.10), $\mathcal{P}_1$ is a partition of some closed interval into closed subintervals in the sense described in [8, p. 1430]. The same comment applies to $\mathcal{P}_2$. We put

$$\mathcal{A} = \{(j,k) : 0 \le j \le r-1, 0 \le k \le s-1, \lambda(A_{m_1+j} \cap B_{m_2+k}) > 0\}, \tag{2.11}$$

$$\mathcal{P} = \{A_{m_1+j} \cap B_{m_2+k} : (j,k) \in \mathcal{A}\}, \tag{2.12}$$

and we observe that

$$[-c,c] \subseteq \bigcup_{(j,k)\in\mathcal{A}} A_{m_1+j} \cap B_{m_2+k}. \tag{2.13}$$

The family $\mathcal{P}$ of closed intervals in (2.12) is a partition of some closed interval into closed subintervals, and by (2.11) and Lemma 3.2 in [8], we have

$$\begin{aligned}(\text{the number of intervals in } \mathcal{P}) &= (\text{the number of elements of } \mathcal{A})\\ &\le r+s-1.\end{aligned} \tag{2.14}$$

Now from (2.6) we see that all lengths of the r intervals in the closed-interval partition $\mathcal{P}_1$ equal $\pi/|x-\alpha|$, so that $(r-2)\pi/|x-\alpha| < 2c$. Hence,

$$1 \le r < \frac{2c|x-\alpha|}{\pi} + 2 = \frac{2c}{\pi}\Big(1+\frac{\pi}{c|x-\alpha|}\Big)|x-\alpha|. \tag{2.15}$$

Let $0<\delta<1/2$. Then if $|x-\alpha| > \pi\delta/c$, we have from (2.15) that

$$1 \le r < \frac{2c}{\pi}\Big(1+\frac{1}{\delta}\Big)|x-\alpha|. \tag{2.16}$$

On the other hand, if $|x-\alpha| \le \pi\delta/c$, as $0<\delta<1/2$ we have $2c < \pi/|x-\alpha|$, and it follows from (2.6) that $[-c,c] \subseteq A_0$, so that $m_1 = 0$ and

$$r = 1. \tag{2.17}$$

Again let $0<\delta<1/2$. Then, as in the preceding argument, but with β replacing α, if $|x-\beta| > \pi\delta/c$ we have

$$1 \le s < \frac{2c}{\pi}\Big(1+\frac{1}{\delta}\Big)|x-\beta|, \tag{2.18}$$

while if $|x-\beta| \le \pi\delta/c$, we have

$$s = 1. \tag{2.19}$$

Now we again let $0 < \delta < 1/2$. We see now from (2.16), (2.17), (2.18) and (2.19) that if either $|x-\alpha| > \pi\delta/c$ or $|x-\beta| > \pi\delta/c$ (perhaps with both holding), then we have

$$
\begin{aligned}
r+s-1 &< 2\max\{r,s\} \\
&\le 2\max\Big\{\frac{2c}{\pi}\Big(1+\frac{1}{\delta}\Big)|x-\alpha|, \frac{2c}{\pi}\Big(1+\frac{1}{\delta}\Big)|x-\beta|\Big\} \\
&= \frac{4c}{\pi}\Big(1+\frac{1}{\delta}\Big)\max\{|x-\alpha|,|x-\beta|\}.
\end{aligned} \tag{2.20}
$$

Also, observe that if $0 < \delta < 1/2$, $|x-\alpha| \le \pi\delta/c$ and $|x-\beta| \le \pi\delta/c$, we have from (2.17) and (2.19) that

$$r = s = 1.$$

Note that in the above, a_k, b_k, A_k, B_k, and so on, depend upon x and c. Also, r and s depend upon x and c.

We now take $m \in \mathbb{N}$ with $m \ge 4s+1$, and we estimate the integral

$$\int_{[-c,c]^m} \frac{du_1 du_2 \cdots du_m}{\sum_{j=1}^m \sin^{2s} u_j(x-\alpha)\sin^{2s} u_j(x-\beta)},$$

allowing for the different values x may be, but recall that $x \notin \{\alpha,\beta\}$. We let $\mathcal{P}_1$, $\mathcal{P}_2$ be the partitions as given in (2.10) and let $\mathcal{P}$ be the partition as in (2.12). We have, using the definitions and (2.4), (2.9), (2.12) and (2.13),

$$
\begin{aligned}
&\int_{[-c,c]^m} \frac{du_1 du_2 \cdots du_m}{\sum_{j=1}^m \sin^{2s} u_j(x-\alpha)\sin^{2s} u_j(x-\beta)} \\
&\le \sum_{(j_1,k_1),\dots,(j_m,k_m)\in\mathcal{A}} \int_{\prod_{t=1}^m A_{m_1+j_t}\cap B_{m_2+k_t}} \\
&\qquad \times \frac{du_1 du_2 \cdots du_m}{\sum_{j=1}^m \sin^{2s} u_j(x-\alpha)\sin^{2s} u_j(x-\beta)} \\
&\le \Big(\frac{\pi^{4s}}{2^{4s}(x-\alpha)^{2s}(x-\beta)^{2s}}\Big) \\
&\qquad \times \Big(\sum_{(j_1,k_1),\dots,(j_m,k_m)\in\mathcal{A}} \int_{\prod_{t=1}^m A_{m_1+j_t}\cap B_{m_2+k_t}} \\
&\qquad\qquad \times \frac{du_1 du_2 \cdots du_m}{\sum_{j=1}^m (u_j - a_{m_1+j_t})^{2s}(u_j - b_{m_2+k_t})^{2s}}\Big).
\end{aligned} \tag{2.21}
$$

In (2.21) we have $a_{m_1+j_t} \in A_{m_1+j_t}$ and $b_{m_2+k_t} \in B_{m_2+k_t}$, but neither $a_{m_1+j_t}$ nor $b_{m_2+k_t}$ necessarily belongs to $A_{m_1+j_t}\cap B_{m_2+k_t}$. If $a_{m_1+j_t} \in A_{m_1+j_t}\cap B_{m_2+k_t}$ put $\tilde{a}_{m_1+j_t} = a_{m_1+j_t}$; otherwise let $\tilde{a}_{m_1+j_t}$ be the endpoint of $A_{m_1+j_t}\cap B_{m_2+k_t}$ closest to $a_{m_1+j_t}$. If $b_{m_2+k_t} \in A_{m_1+j_t}\cap B_{m_2+k_t}$ put $\tilde{b}_{m_2+k_t} = b_{m_2+k_t}$; otherwise let $\tilde{b}_{m_2+k_t}$ be the endpoint of $A_{m_1+j_t}\cap B_{m_2+k_t}$ closest to $b_{m_2+j_t}$. Then from Lemma 2.2, for all $t \in \{1,2,\dots,m\}$, we have that in (2.21),

$$
|(u - a_{m_1+j_t})(u - b_{m_2+k_t})| \ge |(u - \tilde{a}_{m_1+j_t})(u - \tilde{b}_{m_2+k_t})|, \quad \text{for all } u \in A_{m_1+j_t}\cap B_{m_2+k_t}. \tag{2.22}
$$

https://doi.org/10.4153/S0008439518000061 Published online by Cambridge University Press

Now let $0 < \delta < 1/2$ and assume that we have either $|x - \alpha| > \pi\delta/c$ or $|x - \beta| > \pi\delta/c$. Then from (2.14), the right-hand side of (2.20) gives an upper bound for the number of elements in $\mathcal{P}$. Using (2.21) and (2.22), and then using (2.20), the assumption that $m \geq 4s + 1$, and Lemma 4.1 in [8], we have in this case that

(2.23)

$$
\begin{aligned}
&\int_{[-c,c]^m} \frac{du_1 du_2 \cdots du_m}{\sum_{j=1}^m \sin^{2s} u_j(x-\alpha) \sin^{2s} u_j(x-\beta)} \\
&\quad \leq \frac{\pi^{4s}}{2^{4s}(x-\alpha)^{2s}(x-\beta)^{2s}} \\
&\qquad \times \sum_{(j_1,k_1),\dots,(j_m,k_m)\in\mathcal{A}} \int_{\prod_{t=1}^m A_{m_1+j_t} \cap B_{m_2+k_t}} \\
&\qquad\quad \times \frac{du_1 du_2 \cdots du_m}{\sum_{j=1}^m |u_j - \tilde{a}_{m_1+j_t}|^{2s} |u_j - \tilde{b}_{m_2+k_t}|^{2s}} \\
&\quad \leq \frac{\pi^{4s} M}{2^{4s}(x-\alpha)^{2s}(x-\beta)^{2s}} \\
&\qquad \times \sum_{(j_1,k_1),\dots,(j_m,k_m)\in\mathcal{A}} \big(\max\{\lambda(A_{m_1+j_1} \cap B_{m_2+k_1}), \dots, \lambda(A_{m_1+j_m} \cap B_{m_2+k_m})\}\big)^{m-4s},
\end{aligned}
$$

where $M > 0$ and M depends only upon m and s, as in Lemma 4.1 of [8],

$$
\begin{aligned}
&\quad \leq \frac{\pi^{4s-m}(\delta+1)^m 2^{2m-4s} c^m M}{\delta^m (x-\alpha)^{2s}(x-\beta)^{2s}} \\
&\qquad \times \Big(\max\{|x-\alpha|^m, |x-\beta|^m\} \min\Big\{\frac{\pi^{m-4s}}{|x-\alpha|^{m-4s}}, \frac{\pi^{m-4s}}{|x-\beta|^{m-4s}}\Big\}\Big),
\end{aligned}
$$

where we have used (2.6),

$$
\leq Q \max\Big\{\frac{(x-\alpha)^{2s}}{(x-\beta)^{2s}}, \frac{(x-\beta)^{2s}}{(x-\alpha)^{2s}}\Big\}.
$$

So far, x has been fixed with $x \notin \{\alpha, \beta\}$, but allowing for the possibility that $\alpha = \beta$. The constant Q in (2.23) is independent of x, so we deduce that (2.23) holds for all $x \in \mathbb{R}$ such that either $|x - \alpha| > \pi\delta/c$ or $|x - \beta| > \pi\delta/c$. We now consider the cases where $\alpha \neq \beta$ and $\alpha = \beta$.

Case I: $\alpha \neq \beta$.

In this case, choose δ so that

$$
0 < \delta < \min\Big\{\frac{1}{2}, \frac{c|\alpha - \beta|}{2\pi}\Big\}.
$$

Then define disjoint intervals J, K by putting

$$
J = \Big[\alpha - \frac{\pi\delta}{c}, \alpha + \frac{\pi\delta}{c}\Big] \quad \text{and} \quad K = \Big[\beta - \frac{\pi\delta}{c}, \beta + \frac{\pi\delta}{c}\Big].
$$

https://doi.org/10.4153/S0008439518000061 Published online by Cambridge University Press

Clearly, there is $C_1 > 0$ such that

$$\max\Big\{\frac{(x-\alpha)^{2s}}{(x-\beta)^{2s}}, \frac{(x-\beta)^{2s}}{(x-\alpha)^{2s}}\Big\} \le C_1, \quad \text{for all } x \in (J \cup K)^c. \tag{2.24}$$

As well, $(x-\beta)^{-2s}$ is bounded on J, so we see that there is $C_2 > 0$ such that

$$\max\Big\{\frac{(x-\alpha)^{2s}}{(x-\beta)^{2s}}, \frac{(x-\beta)^{2s}}{(x-\alpha)^{2s}}\Big\}(x-\alpha)^{2s} \le C_2, \quad \text{for all } x \in J \cap \{\alpha\}^c. \tag{2.25}$$

And, as $(x-\alpha)^{-2s}$ is bounded on K, there is $C_3 > 0$ such that

$$\max\Big\{\frac{(x-\alpha)^{2s}}{(x-\beta)^{2s}}, \frac{(x-\beta)^{2s}}{(x-\alpha)^{2s}}\Big\}(x-\beta)^{2s} \le C_3, \quad \text{for all } x \in K \cap \{\beta\}^c. \tag{2.26}$$

We now have from (2.23), (2.24), (2.25) and (2.26), that

$$\begin{aligned} &\int_{-\infty}^{\infty}\Big(\int_{[-c,c]^m} \frac{du_1 du_2 \cdots du_m}{\sum_{j=1}^m \sin^{2s} u_j(x-\alpha)\sin^{2s} u_j(x-\beta)}\Big)|\widehat{f}(x)|^2\,dx \\ &\le C_1 Q\int_{(J\cup K)^c}|\widehat{f}(x)|^2\,dx + C_2 Q\int_J \frac{|\widehat{f}(x)|^2}{(x-\alpha)^{2s}}\,dx + C_3 Q\int_K \frac{|\widehat{f}(x)|^2}{(x-\beta)^{2s}}\,dx \\ &< \infty, \end{aligned} \tag{2.27}$$

as we are assuming that $\int_{-\infty}^{\infty}|\widehat{f}(x)|^2(x-\alpha)^{-2s}(x-\beta)^{-2s}\,dx < \infty$.

Case II. $\alpha = \beta$.

Let's assume that $\alpha \in (-c, c)$ and that

$$\delta < \min\Big\{\frac{1}{2}, \frac{c(c-|\alpha|)}{\pi}\Big\}. \tag{2.28}$$

Put $L = (\alpha - \pi\delta/c, \alpha + \pi\delta/c)$, and observe that because of (2.28), $L \subseteq (-c, c)$. Let $x \in L$ be given. Then $|x-\alpha| < \pi\delta/c$ and as $\delta < 1/2$, it follows that $c < \pi/2|x-\alpha|$. Consequently, using the definitions of A_0 and B_0 as given by (2.4) and (2.5), we see that $(-c, c) \subseteq A_0 = B_0$. Note that although A_0 and B_0 each depends upon x, $(-c, c) \subseteq A_0 = B_0$ occurs regardless of $x \in L$. Putting $j = k = 0$ in (2.9), we now deduce that for all $u \in (-c, c)$ and all $x \in L$,

$$\big|\sin\big(u(x-\alpha)\big)\big| \ge \frac{2}{\pi}|u|\cdot|x-\alpha|. \tag{2.29}$$

Let $C > 0$ be such that

$$\sum_{j=1}^m u_j^{4s} \ge C\Big(\sum_{j=1}^m u_j^2\Big)^{2s}, \quad \text{for all } (u_1, u_2, \ldots, u_m) \in \mathbb{R}^m. \tag{2.30}$$

https://doi.org/10.4153/S0008439518000061 Published online by Cambridge University Press

We now have from (2.29) and (2.30) that if $m \geq 4s+1$ and $x \in L$,

$$\int_{[-c,c]^m} \frac{du_1 du_2 \cdots du_m}{\sum_{j=1}^m \sin^{4s} u_j(x-\alpha)} \leq \frac{\pi^{4s}}{2^{4s}(x-\alpha)^{4s}} \int_{[-c,c]^m} \frac{du_1 du_2 \cdots du_m}{\sum_{j=1}^m u_j^{4s}} \tag{2.31}$$
$$\leq \frac{1}{C} \cdot \frac{\pi^{4s}}{2^{4s}(x-\alpha)^{4s}} \int_{[-c,c]^m} \frac{du_1 du_2 \cdots du_m}{\left(\sum_{j=1}^m u_j^2\right)^{2s}}$$
$$\leq \frac{D}{C} \cdot \frac{\pi^{4s}}{2^{4s}(x-\alpha)^{4s}} \int_0^{c\sqrt{m}} r^{m-4s-1}\, dr,$$
for some $D > 0$, by [10, pp. 394–395],
$$\leq \frac{G}{(x-\alpha)^{4s}},$$

for some $G > 0$ that is independent of $x \in L \cap \{\alpha\}^c$.

On the other hand, if $x \notin L$ we have $|x-\alpha| \geq \pi\delta/c$, so that if we apply (2.23) with $\alpha = \beta$ we have

$$\int_{[-c,c]^m} \frac{du_1 du_2 \cdots du_m}{\sum_{j=1}^m \sin^{4s} u_j(x-\alpha)} \leq Q < \infty. \tag{2.32}$$

Assuming that $|\alpha| < c$, we now have, using (2.31) and (2.32), that

$$\int_{-\infty}^{\infty} \left(\int_{[-c,c]^m} \frac{du_1 du_2 \cdots du_m}{\sum_{j=1}^m \sin^{4s} u_j(x-\alpha)} \right) |\widehat{f}(x)|^2\, dx \tag{2.33}$$
$$\leq G \int_L \frac{|\widehat{f}(x)|^2}{(x-\alpha)^{4s}}\, dx + Q \int_{L^c} |\widehat{f}(x)|^2\, dx$$
$$< \infty,$$

as $\alpha = \beta$ and we are assuming that $\int_{-\infty}^{\infty} |\widehat{f}(x)|^2 (x-\alpha)^{-2s}(x-\beta)^{-2s}\, dx < \infty$.

We have considered both the cases $\alpha \neq \beta$ and $\alpha = \beta$. The dénouement results from using Fubini's Theorem, (2.27), and (2.33). We see that provided $|\alpha| < c$ and $m \geq 4s+1$, in both cases we have

$$\int_{[-c,c]^m} \left(\int_{-\infty}^{\infty} \frac{|\widehat{f}(x)|^2\, dx}{\sum_{j=1}^m \sin^{2s} u_j(x-\alpha) \sin^{2s} u_j(x-\beta)} \right) du_1 du_2 \cdots du_m < \infty.$$

We conclude from this that, for almost all $(u_1, u_2, \ldots, u_m) \in [-c,c]^m$,

$$\int_{-\infty}^{\infty} \frac{|\widehat{f}(x)|^2\, dx}{\sum_{j=1}^m \sin^{2s}\big(u_j(x-\alpha)\big) \sin^{2s}\big(u_j(x-\beta)\big)} < \infty. \tag{2.34}$$

By letting c tend to ∞ through a sequence of values, we deduce that, in fact, the inequality in (2.34) holds for almost all $(u_1, u_2, \ldots, u_m) \in \mathbb{R}^m$. But then, using (2.2), (2.3) and Theorem 2.1, we see that provided $m \geq 4s+1$, for almost all $(u_1, u_2, \ldots, u_m) \in \mathbb{R}^m$ there are $f_1, f_2, \ldots, f_m \in L^2(\mathbb{R})$ such that

$$f = \sum_{j=1}^m \left[\left(e^{iu_j(\frac{\alpha-\beta}{2})} + e^{-iu_j(\frac{\alpha-\beta}{2})}\right)\delta_0 - \left(e^{iu_j(\frac{\alpha+\beta}{2})}\delta_{u_j} + e^{-iu_j(\frac{\alpha+\beta}{2})}\delta_{-u_j}\right)\right]^s * f_j.$$

We deduce that (i) implies (ii) in Theorem 2.3 and, by taking $m = 4s + 1$, we see that (i) implies (iii).

We have now proved that (i), (ii) and (iii) are equivalent. Also, we have proved statement (iv), that (iii) is possible for almost all $(u_1, u_2, \dots, u_{4s+1}) \in \mathbb{R}^{4s+1}$.

The final statements (v) and (vi) now follow in a routine way, using as needed the equivalence of the statements (i), (ii) and (iii). This completes the proof of Theorem 2.3. ∎

Note that in Theorem 2.3, if we take the special case $\alpha = \beta = 0$ we obtain the identity (1.2) for the case $s = 2$, proved originally in [6] and [7].

In the case when $\alpha, \beta \in \mathbb{Z}$, and if we identify $\mathbb{T}$ with $[0, 2\pi)$ in the usual way, we can define a generalized (α, β)-difference of order s in $L^2(\mathbb{T})$ to be a function as given in (1.6), but with $g \in L^2\big([0, 2\pi)\big)$ and $u \in [0, 2\pi)$. Then, by analogy with $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$, define $\mathcal{D}_{\alpha,\beta,s}(\mathbb{T})$ to be the vector subspace of $L^2(\mathbb{T})$ consisting of finite sums of generalized (α, β)-differences of order s in $L^2(\mathbb{T})$. It was proved in [8, Theorem 2.3] that

$$\mathcal{D}_{\alpha,\beta,s}(\mathbb{T}) = \{f : f \in L^2(\mathbb{T}) \text{ and } \widehat{f}(\alpha) = \widehat{f}(\beta) = 0\}. \tag{2.35}$$

There is an obvious similarity between this fact and the result derived from Theorem 2.3 which is that

$$\mathcal{D}_{\alpha,\beta,s}(\mathbb{R}) = \left\{ f : f \in L^2(\mathbb{R}) \text{ and } \int_{-\infty}^{\infty} \frac{|\widehat{f}(x)|^2}{(x-\alpha)^{2s}(x-\beta)^{2s}}\, dx < \infty \right\}. \tag{2.36}$$

However, note in (2.35) that the right-hand side is independent of s whereas in (2.36) the right-hand side depends upon s. At first sight this may seem surprising, but each equality expresses a condition that $\widehat{f}$ "vanishes" at or near α and β. Since the dual $\mathbb{Z}$ of $\mathbb{T}$ is discrete, the only way this can occur in the case of $\mathbb{T}$ is if $\widehat{f}$ *actually vanishes* at α and β, and this forces the independence from s in the right hand side of (2.35). In the case of $\mathbb{R}$, however, because the dual of $\mathbb{R}$ is itself and so is a continuum, there is an infinity of possible behaviours of $\widehat{f}$ near α and β expressing the idea that $\widehat{f}$ "vanishes" near α and β, and we observe a dependence upon s in the right-hand side of (2.36).

Another difference between $\mathcal{D}_{\alpha,\beta,s}(\mathbb{T})$ and $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$ is that the former has finite algebraic codimension in $L^2(\mathbb{T})$ while the latter has infinite algebraic codimension in $L^2(\mathbb{R})$. Note further that when $\alpha, \beta \in \mathbb{Z}$, it has been shown [8, Theorem 2.3] that $\big(D^2 - i(\alpha+\beta)D - \alpha\beta I\big)^s$ maps $W^{2s}(\mathbb{T})$ onto $\mathcal{D}_{\alpha,\beta,s}(\mathbb{T})$ (which is independent of s), while here we have seen that $\big(D^2 - i(\alpha+\beta)D - \alpha\beta I\big)^s$ maps $W^{2s}(\mathbb{R})$ onto $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$.

In [5] Meisters and Schmidt showed that every translation-invariant linear form on $L^2(\mathbb{T})$ is continuous, but in [3] Meisters showed that there are discontinuous translation-invariant linear forms on $L^2(\mathbb{R})$, and this latter result may also be deduced from the identity (1.2) in the case $s = 1$. The following introduces, in the present context, a notion corresponding to translation-invariant linear forms.

Definition 2.4 Let $\alpha, \beta \in \mathbb{R}$ and let $s \in \mathbb{N}$. Then a linear form T on $L^2(\mathbb{R})$ is called (α, β, s)-*invariant* if, for all $f \in L^2(\mathbb{R})$ and $u \in \mathbb{R}$,

$$T\Big(\big[(e^{iu(\frac{\alpha-\beta}{2})} + e^{-iu(\frac{\alpha-\beta}{2})})\delta_0 - (e^{iu(\frac{\alpha+\beta}{2})}\delta_u + e^{-iu(\frac{\alpha+\beta}{2})}\delta_{-u})\big]^s * f\Big) = 0.$$

Equivalently, the linear form T on $L^2(\mathbb{R})$ is (α,β,s)-*invariant* when $T\big(\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})\big) = \{0\}$.

A linear form T on $L^2(\mathbb{R})$ is $(\alpha,-\alpha,1)$-invariant when, for all $f\in L^2(\mathbb{R})$ and $u\in\mathbb{R}$,

$$T\big(2^{-1}(\delta_u+\delta_{-u})*f\big) = \cos\alpha\, T(f),$$

from which we see that if T is a translation-invariant linear form on $L^2(\mathbb{R})$ it is $(0,0,1)$-invariant.

When $\alpha,\beta\in\mathbb{Z}$, we may also introduce the corresponding notion of (α,β,s)-invariant linear forms on $L^2(\mathbb{T})$. It was shown in [8, Theorem 7.1] that an $(\alpha,\beta,1)$-invariant linear form on $L^2(\mathbb{T})$ is continuous and, in fact, any (α,β,s)-invariant linear form on $L^2(\mathbb{T})$ is continuous (proved by the technique used for the case $s=1$ in [8]). However, the following corollary to Theorem 2.3 shows that the situation pertaining to translation-invariant linear forms on $L^2(\mathbb{R})$ is mirrored by that for (α,β,s)-invariant linear forms on $L^2(\mathbb{R})$.

Corollary 2.5 *Let $\alpha,\beta\in\mathbb{R}$ and let $s\in\mathbb{N}$. Then there are discontinuous (α,β,s)-invariant linear forms on $L^2(\mathbb{R})$.*

Proof It is a consequence of Theorem 2.3 that $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$ has infinite algebraic codimension in $L^2(\mathbb{R})$. Consequently there are discontinuous linear forms on $L^2(\mathbb{R})$ that vanish on $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$, and such forms are (α,β,s)-invariant. ∎

Acknowledgement The author thanks the referee for helpful suggestions that have improved the content and presentation of the paper.

References

[1] J. Bourgain, *Translation invariant forms on $L^p(G)$, $1<p<\infty$*. Ann. Inst. Fourier (Grenoble) 36(1986), 97–104.

[2] B. E. Johnson, *A proof of the translation invariant form conjecture for $L^2(G)$*. Bull. Sci. Math. 107(1983), 301–310.

[3] G. Meisters, *Some discontinuous translation-invariant linear forms*. J. Funct. Anal. 12(1973), 199–210.

[4] G. Meisters, *Some problems and results on translation-invariant linear forms*. In: *Lecture Notes in Math., 975*, eds. Bachar J. M. and Bade W. G., *et al.* Springer, New York, 1983, pp. 423–444.

[5] G. Meisters and W. Schmidt, *Translation invariant linear forms on $L^2(G)$ for compact abelian groups G*. J. Funct. Anal. 11(1972), 407–424.

[6] R. Nillsen, *Banach spaces of functions and distributions characterized by singular integrals involving the Fourier transform*. J. Funct. Anal. 110(1992), 73–95.

[7] R. Nillsen, *Difference spaces and Invariant Linear Forms*. Lecture Notes in Math., 1586, Springer, Berlin–Heidelberg–New York, 1994.

[8] R. Nillsen, *Vanishing Fourier coefficients and the expression of functions in $L^2(\mathbb{T})$ as sums of generalized differences*. J. Math. Anal. Appl. 455(2017), 1425–1443.

[9] K. A. Ross, *A trip from classical to abstract Fourier analysis*. Notices Amer. Math. Soc. 61(2014), 1032–1038.

[10] K. R. Stromberg, *An Introduction to Classical Real Analysis*. Wadsworth, Belmont, 1981.

Department of Mathematics and Applied Statistics, University of Wollongong, Wollongong, NSW 2522, Australia
e-mail: nillsen@uow.edu.au